Composed convex programming: duality and Farkas-type results

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Abstract. It is not hard to prove that many convex optimization problems which are already studied in the literature can be rewritten as a particular instance of the following problem: minimize the sum of a convex function and the composition of a convex and K-increasing function with a K-convex one when the variable varies on a given set. Using a conjugate duality approach we construct the Fenchel-Lagrange dual of this general problem. Moreover, using the connections between the optimal objective values of the primal and the dual problem, a Farkas-type result is proved. It is also shown that some recently obtained Farkas-type results are rediscovered as special cases of our statement.

1 Introduction

Since during the last decades the problems generated by the practical needs turned out to be more and more complex, one of the main problems in optimization is to find some methods and conditions which assure the existence of optimal solution for more and more general problems which encompass as special cases the already studied ones.

The problem treated within this paper consists in minimizing the sum of a convex function and the composition of a convex and K-increasing function with a K-convex one when the variable varies on a given set (K is a closed convex cone). Many optimization problems already treated can be derived as special cases of this general optimization problem; among these special cases we would like to mention only the usual problem of minimizing a convex function regarding geometrical and convex inequality constraints. Because of its generality, the problem had recently drawn the attention of many mathematicians and some new results are to be found in the literature ([1], [8], [10]).

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In order to provide duality assertions for the problem we treat, we use the same approach as in [2] and [3]. Thus, using an auxiliary variable, to the primal problem we associate an equivalent one, but whose dual can be easier established. In order to determine its dual, to the new optimization problem the classical Lagrange dual problem is attached. Moreover, as the inner infimum of the Lagrange problem can be considered itself as an optimization problem, its Fenchel dual problem is also determined. The construction of the dual, which is actually what we call the Fenchel-Lagrange dual problem, is in detail described and a constraint qualification which ensures strong duality between the primal problem and its dual is also given. Regarding the Fenchel-Lagrange dual problem, let us mention that more about this type of dual problem can be found in [4], [5], [6], [7], [13].

In [6] and [7] Boţ and Wanka have presented some Farkas-type results for inequality systems involving finitely many convex functions using an approach based on the theory of conjugate duality for convex optimization problems. Within the present paper, using weak and strong duality assertions developed for the problem we treat, these results are extended to a more general one. Moreover, it is shown that some results in the literature arise as special cases of the problem we treat.

The paper is organized as follows. Within the second section some definitions and results needed later are presented. A dual for the optimization problem with composed convex functions and the weak and strong duality assertions are established in the third section. Section 4 contains the main result of the paper. The duality acquired in Section 3 allows us to give a Farkas-type theorem. The last section contains Farkas-type results for some particular instances of the initial one and some recent results are rediscovered as special cases.

2 Notations and preliminaries

For the sake of the completeness some well-known definitions and results are presented in the following. As usual, by $\mathbb{R}^k$ is denoted the k-dimensional real space for any nonnegative integer k. All vectors are considered as column vectors. Any column vector can be transposed to a row vector by an upper index ^{T}. By $x^Ty = \sum_{i=1}^k x_iy_i$ is denoted the usual inner product of two vectors $x = (x_1, ..., x_k)^T$ and $y = (y_1, ..., y_k)^T$ in $\mathbb{R}^k$. Considering an arbitrary non-empty closed convex cone $K \subseteq \mathbb{R}^k$, the partial ordering induced by the cone is defined by

$$x \leqq_K y \Leftrightarrow y - x \in K, \quad \forall x, y \in \mathbb{R}^k.$$

Let $\mathbb{R}^k$ to be extended by an element ∞ such that for all $x \in \mathbb{R}^k$ it holds $x \leqq_K \infty$. Regarding the partial ordering induced by the cone K over the set

$\mathbb{R}^k$, it is not hard to see that it can be naturally extended to the set $\mathbb{R}^k \cup \{\infty\}$ by taking

$$x \leq_K \infty, \quad \forall x \in \mathbb{R}^k \cup \{\infty\}.$$

Moreover, the addition and the multiplication with a scalar are also natural extended setting

$$\infty + x = x + \infty = \infty \text{ and } t\infty = \infty,$$

for any $x \in \mathbb{R}^k \cup \{\infty\}$ and $t \geq 0$.

To the cone K we can associate its dual cone defined by

$$K^* = \big\{\beta \in \mathbb{R}^k : \beta^T x \geq 0, \forall x \in K\big\}.$$

As any $\beta \in K^*$ is actually a real-valued linear functional $\beta : \mathbb{R}^k \to \mathbb{R}$, we consider its natural extension

$$\beta : \mathbb{R}^k \cup \{\infty\} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}, \quad \beta(x) = \begin{cases} \beta^T x, \; x \in \mathbb{R}^k, \\ +\infty, \; x = \infty. \end{cases}$$

Let us consider an arbitrary set $X \subseteq \mathbb{R}^n$. By $\mathrm{ri}(X)$, $\mathrm{co}(X)$ and $\mathrm{cl}(X)$ are denoted the *relative interior*, the *convex hull* and the *closure* of the set X, respectively. Furthermore, the *cone* and the *convex cone* generated by the set X are denoted by $\mathrm{cone}(X) = \bigcup_{\lambda \geq 0} \lambda X$ and, respectively, $\mathrm{coneco}(X) = \bigcup_{\lambda \geq 0} \lambda \, \mathrm{co}(X)$. By $v(P)$ we denote the optimal objective value of an optimization problem (P).

If $X \subseteq \mathbb{R}^n$ is given, we consider the following two functions, the *indicator function*

$$\delta_X : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad \delta_X(x) = \begin{cases} 0, & x \in X, \\ +\infty, & \text{otherwise}, \end{cases}$$

and the *support function*

$$\sigma_X : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad \sigma_X(u) = \sup_{x \in X} u^T x,$$

respectively.

For a given function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, we denote by $\mathrm{dom}(f) = \big\{x \in \mathbb{R}^n : f(x) < +\infty\big\}$ its *effective domain*, by $\mathrm{epi}(f) = \big\{(x, r) : x \in \mathbb{R}^n, r \in \mathbb{R}, f(x) \leq r\big\}$ its *epigraph*, respectively. The function f is called *proper* if its effective domain is a nonempty set and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$.

We consider also the linear operator

$$\mathcal{T} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}^n, \quad \mathcal{T}\big(x, r\big) = (r, x).$$

When X is a nonempty subset of $\mathbb{R}^n$ we define for the function f the *conjugate relative to the set* X by

$$f_X^* : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad f_X^*(p) = \sup_{x \in X} \big\{p^T x - f(x)\big\}.$$

It is easy to observe that for $X = \mathbb{R}^n$ the conjugate relative to the set X is actually the *(Fenchel-Moreau) conjugate function* of f denoted by f^*. Even more, it is trivial to prove that

$$f_X^* = (f + \delta_X)^* \text{ and } \delta_X^* = \sigma_X.$$

Definition 2.1 The function $g : \mathbb{R}^k \to \overline{\mathbb{R}}$ is called *K-increasing* if for all x and y in $\mathbb{R}^k$ such that $x \leqq_K y$ it holds $g(x) \leq g(y)$.

Definition 2.2 Let the function $h : \mathbb{R}^n \to \mathbb{R}^k \cup \{\infty\}$ be given. The function is called *K-convex* if for all $x, y \in \mathbb{R}^n$ and for all $t \in [0, 1]$ one has

$$h\big(tx + (1-t)y\big) \leqq_K th(x) + (1-t)h(y).$$

Definition 2.3 Given the functions $f_1, ..., f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$, we call their *infimal convolution* the function

$$f_1 \Box ... \Box f_m : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad (f_1 \Box ... \Box f_m)(x) = \inf \left\{ \sum_{i=1}^m f_i(x_i) : x = \sum_{i=1}^m x_i \right\}.$$

The following statements close this preliminary section.

Theorem 2.1 (cf. [12]) Let $f_1, ..., f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions. If the set $\bigcap_{i=1}^m \operatorname{ri}(\operatorname{dom}(f_i))$ is nonempty, then

$$\left(\sum_{i=1}^m f_i \right)^* (p) = (f_1^* \Box ... \Box f_m^*)(p) = \inf \left\{ \sum_{i=1}^m f_i^*(p_i) : p = \sum_{i=1}^m p_i \right\},$$

and for each $p \in \mathbb{R}^n$ the infimum is attained.

Corollary 2.2 (cf. [3]) Let $f_1, ..., f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions. If the set $\bigcap_{i=1}^m \operatorname{ri}(\operatorname{dom}(f_i))$ is nonempty, then

$$\operatorname{epi} \left(\left(\sum_{i=1}^m f_i \right)^* \right) = \sum_{i=1}^m \operatorname{epi}(f_i^*).$$

Proposition 2.3 (cf. [3]) Let $f : \mathbb{R}^k \to \overline{\mathbb{R}}$ be a proper function and $\alpha > 0$ a real number. One has

$$\operatorname{epi}\big((\alpha f)^*\big) = \alpha \operatorname{epi}\big(f^*\big).$$

3 Duality for the general problem

Let $X \subseteq \mathbb{R}^n$ be a nonempty convex set and $K \subseteq \mathbb{R}^k$ a nonempty closed convex cone. Consider the functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $g : \mathbb{R}^k \to \overline{\mathbb{R}}$ and $h : \mathbb{R}^n \to \mathbb{R}^k \cup \{\infty\}$,

$h = \big(h_1, ..., h_k\big)^T$, such that f is proper and convex, g is proper, convex and K-increasing and h is K-convex. The function g is extended to the space $\mathbb{R}^k \cup \{\infty\}$ by defining $g(\infty) = +\infty$. Moreover, throughout this section two conditions are imposed. First of all, we assume that

$$X \cap \mathrm{dom}(f) \cap h^{-1}\big(\mathrm{dom}(g)\big) \neq \emptyset, \tag{1}$$

where $h^{-1}\big(\mathrm{dom}(g)\big) = \{x \in \mathbb{R}^n : h(x) \in \mathrm{dom}(g)\}$. The second condition we consider is

$$\mathrm{ri}\,\big(X \cap h^{-1}(\mathbb{R}^k)\big) \cap \mathrm{ri}\,\big(\mathrm{dom}(f)\big) \neq \emptyset. \tag{2}$$

As a remark, let us mention that these conditions are independent, although at a first look we are tempted to believe that, if the second relation is fulfilled, then the first relation is fulfilled, too.

The problem we work with is

$$(P) \qquad \inf_{x \in X} \big(f(x) + (g \circ h)(x)\big).$$

Regarding this problem, since the relation (1) is fulfilled, it is trivial to see that that the optimal objective value of the problem (P) fulfills $v(P) < +\infty$. Even more, as the function $g \circ h$ is convex, the problem we treat is actually a convex optimization problem with geometric constraints. In order to give a dual problem for (P) we consider the following convex optimization problem

$$(P') \qquad \inf_{\substack{x \in X, y \in \mathrm{dom}(g), \\ h(x) - y \leqq_K 0}} \big(f(x) + g(y)\big).$$

The connection between (P) and (P') is made by the following result.

Theorem 3.1 For the optimal objective values of (P) and (P') we have $v(P) = v(P')$.

Proof. Consider an arbitrary $x \in X$.

If $x \notin \mathrm{dom}(f) \cap h^{-1}\big(\mathrm{dom}(g)\big)$, either $f(x) = +\infty$ or $(g \circ h)(x) = +\infty$ or both, so that $f(x) + (g \circ h)(x) = +\infty \geq v(P')$.

If $x \in \mathrm{dom}(f) \cap h^{-1}\big(\mathrm{dom}(g)\big)$, take $y = h(x) \in \mathrm{dom}(g)$. Then $y - h(x) = 0 \in K$ and the pair (x, y) is obviously feasible to (P'). Even more, as $f(x)+(g\circ h)(x) = f(x)+g(y)$, this equality is enough to secure $f(x)+(g\circ h)(x) \geq v(P')$.

Taking into consideration the inequalities obtained in the two cases considered above, the inequality

$$v(P) \geq v(P')$$

arises as a simple consequence.

In order to prove the reverse inequality, let us consider an arbitrary pair (x, y) feasible to (P').

Let us assume first that $h(x) = \infty$. This would mean that y must be also equal to ∞ and thus $g(y) = +\infty$. But this contradicts the assumption $y \in \mathrm{dom}(g)$ and therefore $h(x) \in \mathbb{R}^k$.

As $h(x) \leqq_K y$ we have that $g(h(x)) \leq g(y)$, so the inequality $f(x) + g(h(x)) \leq f(x) + g(y)$ is also fulfilled. Even more, we get $v(P) \leq f(x) + g(y)$ and, since this inequality is true for an arbitrary pair (x, y) feasible to (P'), the inequality

$$v(P) \leq v(P')$$

follows at hand. This completes the proof. □

This result allows us to affirm that any dual problem of (P') is automatically a dual problem of (P).

To (P') we associate its Lagrange dual problem with $\beta \in K^*$ as dual variable

$$(D) \qquad \sup_{\beta \in K^*} \inf_{\substack{x \in X, \\ y \in \mathrm{dom}(g)}} \Big\{ f(x) + g(y) + \beta^T \big(h(x) - y \big) \Big\}.$$

Using the definition of the conjugate relative to a set, the inner infimum becomes

$$\begin{aligned}
&\inf_{\substack{x \in X, \\ y \in \mathrm{dom}(g)}} \big\{ f(x) + g(y) + \beta^T \big(h(x) - y \big) \big\} \\
&= \inf_{x \in X} \big\{ f(x) + \beta^T h(x) \big\} + \inf_{y \in \mathrm{dom}(g)} \big\{ g(y) - \beta^T y \big\} \\
&= - \sup_{x \in X} \big\{ - f(x) - \beta^T h(x) \big\} - \sup_{y \in \mathrm{dom}(g)} \big\{ \beta^T y - g(y) \big\} \\
&= - \big(f + \beta^T h \big)^*_X (0) - g^*(\beta) \\
&= - g^*(\beta) - \inf_{p \in \mathbb{R}^n} \big\{ f^*(p) + \big(\beta^T h \big)^*_X (-p) \big\},
\end{aligned}$$

and, as relation (2) is accomplished, Theorem 2.1 yields that the last infimum is attained.

Remark. Since $\beta(\infty) = +\infty$ for all $\beta \in K^*$, whenever $h(x) = \infty$ we get $(\beta^T h)(x) = \infty$, for all $\beta \in K^*$, and it is not hard to see that with this condition satisfied we get

$$\mathrm{dom} \big(\beta^T h + \delta_X \big) = X \cap h^{-1}(\mathbb{R}^k), \quad \forall \beta \in K^*.$$

Thus we obtain the following formula for the dual problem to (P') and also (P)

$$(D) \qquad \sup_{\substack{p\in\mathbb{R}^n,\\ \beta\in K^*}} \Big\{ -g^*(\beta) - f^*(p) - \big(\beta^T h\big)^*_X(-p)\Big\}.$$

As a direct consequence of our construction of (D) we get the following weak duality result.

Theorem 3.2 Between the primal problem (P) and the dual (D) weak duality is always satisfied, i.e. $v(P) \geq v(D)$.

The existent literature contains some examples which prove that strong duality is not always fulfilled (see, for example, [13]). Nevertheless, such a situation can be avoided if we consider the following constraint qualification

$$(CQ) \quad \exists x' \in \operatorname{ri}\big(X \cap h^{-1}(\mathbb{R}^k)\big) \cap \operatorname{ri}\big(\operatorname{dom}(f)\big) : h(x') \in \operatorname{ri}\big(\operatorname{dom}(g)\big) - \operatorname{ri}(K).$$

Theorem 3.3 Assume that $v(P)$ is finite. If (CQ) is fulfilled, then between (P) and (D) strong duality holds, i.e. $v(P) = v(D)$ and the dual problem has an optimal solution.

Proof. We actually prove that strong duality holds between the problems (P') and (D). Using Theorem 3.1 the desired result arises as a direct consequence.

To the problem (P') we associate its Lagrange dual

$$(D) \qquad \sup_{\beta\in K^*} \inf_{\substack{x\in X,\\ y\in\operatorname{dom}(g)}} \Big\{ f(x) + g(y) + \beta^T\big(h(x) - y\big)\Big\}.$$

As the condition (CQ) is fulfilled and all the involved functions are convex, is is well-known from the existing literature ([1], [12]) that between (P') and (D') strong duality holds, i.e. $v(P') = v(D')$ and there exists a $\overline{\beta} \in K^*$ such that

$$v(P') = \inf_{\substack{x\in X,\\ y\in\operatorname{dom}(g)}} \Big\{ f(x) + g(y) + \overline{\beta}^T\big(h(x) - y\big)\Big\}.$$

As (CQ) is fulfilled we get using the above calculation

$$\inf_{\substack{x\in X,\\ y\in\operatorname{dom}(g)}} \Big\{ f(x)+g(y)+\overline{\beta}^T\big(h(x)-y\big)\Big\} = -g^*(\overline{\beta}) - \inf_{p\in\mathbb{R}^n} \Big\{ f^*(p) + \big(\overline{\beta}^T h\big)^*_X(-p)\Big\}$$

and the infimum in the right-hand side is attained. Therefore there exist $\overline{p} \in \mathbb{R}^n$ and $\overline{\beta} \in K^*$ such that

$$v(P') = -g^*(\overline{\beta}) - f^*(\overline{p}) - \big(\overline{\beta}^T h\big)^*_X(-\overline{p}).$$

Using Theorem 3.1 we obtain $v(P) = v(D)$ and $(\overline{p}, \overline{\beta})$ is an optimal solution for (D). $\square$

4 Farkas-type results via weak and strong duality

Using the results presented within the previous section, the following Farkas-type result can be easily proved.

Theorem 4.1 Suppose that (CQ) holds. Then the following assertions are equivalent:

(i) $x \in X \Rightarrow f(x) + (g \circ h)(x) \geq 0$;
(ii) there exist $p \in \mathbb{R}^n$ and $\beta \in K^*$ such that

$$g^*(\beta) + f^*(p) + \left(\beta^T h\right)_X^*(-p) \leq 0. \tag{3}$$

Proof. "$(i) \Rightarrow (ii)$" The statement (i) implies $v(P) \geq 0$ and, since the assumptions of Theorem 3.3 are fulfilled, strong duality holds, i.e. $v(D) = v(P) \geq 0$ and the dual (D) has an optimal solution. Thus there exist $p \in \mathbb{R}^n$ and $\beta \in K^*$ fulfilling (3).

"$(ii) \Rightarrow (i)$" As we can find some $p \in \mathbb{R}^n$ and $\beta \in K^*$ fulfilling (3), it follows right away that

$$v(D) \geq -g^*(\beta) - f^*(p) - \left(\beta^T h\right)_X^*(-p) \geq 0.$$

Weak duality between (P) and (D) always holds and thus we obtain $v(P) \geq 0$, i.e. (i) is true. $\square$

The previous statement can be reformulated as a theorem of the alternative.

Corollary 4.2 Assume that the hypothesis of Theorem 4.1 is fulfilled. Then either the inequality system

$$(I) \quad x \in X, f(x) + (g \circ h)(x) < 0$$

has a solution or the system

$$(II) \; g^*(\beta) + f^*(p) + \left(\beta^T h\right)_X^*(-p) \leq 0, \quad p \in \mathbb{R}^n, \beta \in K^*$$

has a solution, but never both.

Theorem 4.3 The statement (ii) in Theorem 4.1 is equivalent to

$$(0,0,0) \in \{0\} \times \mathcal{T}\big(\operatorname{epi}(g^*)\big) + \operatorname{epi}(f^*) \times \{0\} + \bigcup_{\beta \in K^*} \Big(\operatorname{epi}\big((\beta^T h)^*_X\big) \times \{-\beta\}\Big).$$

Proof. "$\Rightarrow$" Since the statement (ii) holds, there exist $p \in \mathbb{R}^n$ and $\beta \in K^*$ such that

$$g^*(\beta) + f^*(p) + \big(\beta^T h\big)^*_X(-p) \leq 0.$$

As $g^*(\beta)$ and $(\beta^T h)^*_X(-p)$ have both finite real values, by definition follows

$$\big(\beta, g^*(\beta)\big) \in \operatorname{epi}(g^*)$$

and

$$\big(-p, (\beta^T h)^*_X(-p)\big) \in \operatorname{epi}\big((\beta^T h)^*_X\big).$$

Thus

$$\big(-p, (\beta^T h)^*_X(-p), -\beta\big) \in \operatorname{epi}\big((\beta^T h)^*_X\big) \times \{-\beta\}$$

and it follows

$$\big(-p, \big(\beta^T h\big)^*(-p), -\beta\big) \in \bigcup_{\beta \in K^*} \Big(\operatorname{epi}\big((\beta^T h)^*_X\big) \times \{-\beta\}\Big). \tag{4}$$

Taking into consideration the definition of the operator $\mathcal{T}$ introduced in the first section of the paper, the relation

$$\big(0, g^*(\beta), \beta\big) \in \{0\} \times \mathcal{T}(\operatorname{epi}(g^*)) \tag{5}$$

follows at once.

On the other hand the inequality

$$f^*(p) \leq -g^*(\beta) - \big(\beta^T h\big)^*(-p)$$

is also fulfilled, and, as the value in the right-hand side is finite, it holds

$$\big(p, -g^*(\beta) - \big(\beta^T h\big)^*_X(-p)\big) \in \operatorname{epi}\big(f^*\big).$$

This implies

$$\big(p, -g^*(\beta) - \big(\beta^T h\big)^*_X(-p), 0\big) \in \operatorname{epi}\big(f^*\big) \times \{0\}. \tag{6}$$

Combining relations (4), (5) and (6) we get

$$(0,0,0) = \big(0, g^*(\beta), \beta\big) + \big(p, -g^*(\beta) - \big(\beta^T h\big)^*_X(-p), 0\big) + \big(-p, \big(\beta^T h\big)^*_X(-p), -\beta\big)$$

$$\in \{0\} \times \mathcal{T}\big(\operatorname{epi}(g^*)\big) + \operatorname{epi}(f^*) \times \{0\} + \bigcup_{\beta \in K^*} \Big(\operatorname{epi}\big((\beta^T h)^*_X\big) \times \{-\beta\}\Big).$$

"$\Leftarrow$" Since

$$(0,0,0) \in \{0\} \times \mathcal{T}\big(\operatorname{epi}(g^*)\big) + \operatorname{epi}(f^*) \times \{0\} + \bigcup_{\beta \in K^*} \Big(\operatorname{epi}\big((\beta^T h)_X^*\big) \times \{-\beta\} \Big),$$

we can find some $p \in \mathbb{R}^n$ and $r \in \mathbb{R}$ such that

$$(p, r, 0) \in \operatorname{epi}(f^*) \times \{0\} \tag{7}$$

and

$$(-p, -r, 0) \in \{0\} \times \mathcal{T}(\operatorname{epi}(g^*)) + \bigcup_{\beta \in K^*} \operatorname{epi}\big((\beta^T h)_X^*\big) \times \{-\beta\}. \tag{8}$$

Using the definition of the epigraph of a function, from relation (7) we acquire directly

$$f^*(p) \leq r. \tag{9}$$

By relation (8), there exists a $\beta \in K^*$ such that

$$(-p, -r, 0) \in \{0\} \times \mathcal{T}(\operatorname{epi}(g^*)) + \operatorname{epi}\big((\beta^T h)_X^*\big) \times \{-\beta\}.$$

The definition of the operator $\mathcal{T}$ and the previous relation imply that there exist two real numbers r_1 and r_2 such that $-r = r_1 + r_2$, while the pairs (β, r_1) and $(-p, r_2)$ are in $\operatorname{epi}(g^*)$ and $\operatorname{epi}\big((\beta^T h)_X^*\big)$, respectively. Thus

$$g^*(\beta) + (\beta^T h)_X^*(-p) \leq r_1 + r_2 = -r. \tag{10}$$

Combining relations (9) and (10), the desired result is straightforward. □

5 The ordinary problem as a particular case

Let $X \subseteq \mathbb{R}^n$ be a nonempty convex set and $K \subseteq \mathbb{R}^k$ a nonempty closed convex cone. Consider the functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $h : \mathbb{R}^n \to \mathbb{R}^k \cup \{\infty\}$, $h = \big(h_1, ..., h_k\big)^T$, such that f is proper and convex and h is K-convex.

Take the problem

$$(P_1) \qquad \inf_{\substack{x \in X, \\ h(x) \leq_K 0}} f(x)$$

and assume that

$$X \cap \operatorname{dom}(f) \cap h^{-1}(-K) \neq \emptyset.$$

It is not hard to remark that for all $x \in \mathbb{R}^n$ we have

$$h(x) \leq_K 0 \Leftrightarrow \delta_{-K}\big(h(x)\big) = 0 \Leftrightarrow (\delta_{-K} \circ h)(x) = 0.$$

Thus we get

$$v(P_1) = \inf_{x\in X} \big(f(x) + (\delta_{-K} \circ h)(x)\big)$$

and, so, (P_1) can be further written as

$$(P_1) \qquad \inf_{x\in X} \big(f(x) + (\delta_{-K} \circ h)(x)\big).$$

Taking into consideration the results obtained in the previous section (to prove that the function δ_{-K} is K-increasing is trivial), to the problem (P_1) we can associate the following dual problem

$$(D_1) \qquad \sup_{\substack{p\in\mathbb{R}^n,\\ \beta\in K^*}} \Big\{ -(\delta_{-K})^*(\beta) - f^*(p) - \big(\beta^T h\big)^*_X(-p) \Big\}.$$

Even more, it is easy to prove that

$$(\delta_{-K})^*(\beta) = \begin{cases} 0, & \beta \in K^*, \\ +\infty, & \text{otherwise,} \end{cases}$$

so that the dual (D_1) becomes

$$(D_1) \qquad \sup_{\substack{p\in\mathbb{R}^n,\\ \beta\in K^*}} \Big\{ - f^*(p) - \big(\beta^T h\big)^*_X(-p) \Big\}.$$

In order to get strong duality between the problems (P_1) and (D_1), the fulfilling of the following constraint qualification is required

$$(CQ_1)\ \exists x' \in \operatorname{ri}\big(X \cap h^{-1}(\mathbb{R}^k)\big) \cap \operatorname{ri}\big(\operatorname{dom}(f)\big) : h(x') \in \operatorname{ri}\big(\operatorname{dom}(\delta_{-K})\big) - \operatorname{ri}(K).$$

But

$$\operatorname{ri}\big(\operatorname{dom}(\delta_{-K})\big) - \operatorname{ri}(K) = \operatorname{ri}(-K) - \operatorname{ri}(K) = -\operatorname{ri}(K) - \operatorname{ri}(K) = -\operatorname{ri}(K),$$

therefore we acquire

$$(CQ_1) \quad \exists x' \in \operatorname{ri}\big(X \cap h^{-1}(\mathbb{R}^k)\big) \cap \operatorname{ri}\big(\operatorname{dom}(f)\big) : h(x') \in -\operatorname{ri}(K).$$

The following outcomes are easy consequences of the results proved within the previous section.

Theorem 5.1 Suppose that (CQ_1) holds. Then the following assertions are equivalent:

(i) $x \in X,\ h(x) \leq_K 0 \Rightarrow f(x) \geq 0$;

(ii) there exist $p \in \mathbb{R}^n$ and $\beta \in K^*$ such that

$$f^*(p) + \big(\beta^T h\big)^*_X(-p) \leq 0.$$

Corollary 5.2 Assume that the hypothesis of Theorem 5.1 is fulfilled. Then either the inequality system

$$(I) \quad x \in X, h(x) \leq_K 0, f(x) < 0$$

has a solution or the system

$$(II) \; f^*(p) + \left(\beta^T h\right)_X^*(-p) \leq 0, \\ p \in \mathbb{R}^n, \beta \in K^*$$

has a solution, but never both.

Theorem 5.3 The statement (ii) in Theorem 5.1 is equivalent to

$$(0,0) \in \operatorname{epi}(f^*) + \bigcup_{\beta \in K^*} \operatorname{epi}\left((\beta^T h)_X^*\right). \tag{11}$$

Proof. By Theorem 4.3 we know that the statement (ii) in Theorem 5.1 is equivalent to

$$(0,0,0) \in \{0\} \times \mathcal{T}\left(\operatorname{epi}((\delta_{-K})^*)\right) + \operatorname{epi}(f^*) \times \{0\} + \bigcup_{\beta \in K^*} \left(\operatorname{epi}\left((\beta^T h)_X^*\right) \times \{-\beta\}\right).$$

Since

$$\operatorname{epi}\left((\delta_{-K})^*\right) = K^* \times [0, +\infty),$$

it is easy to see that the last relation can be equivalently written as

$$(0,0,0) \in \bigcup_{\beta \in K^*} \left(\{0\} \times [0,+\infty) \times K^* + \operatorname{epi}(f^*) \times \{0\} + \operatorname{epi}\left((\beta^T h)_X^*\right) \times \{-\beta\}\right).$$

This means that there exists $\beta \in K^*$ such that

$$(0,0) \in \{0\} \times [0,+\infty) + \operatorname{epi}(f^*) + \operatorname{epi}\left((\beta^T h)_X^*\right). \tag{12}$$

Using only the definition of the epigraph of a function it is easy to prove that

$$\{0\} \times [0,+\infty) + \operatorname{epi}(f^*) = \operatorname{epi}(f^*).$$

Therefore, by (12),

$$(0,0) \in \operatorname{epi}(f^*) + \bigcup_{\beta \in K^*} \operatorname{epi}\left((\beta^T h)_X^*\right),$$

and the proof is complete. $\square$

Let us consider now $h : \mathbb{R}^n \to \mathbb{R}^k$ and $K = \mathbb{R}^k_+$. The constraint qualification (CQ_1) becomes in this case

$$(CQ_1') \quad \exists x' \in \operatorname{ri}(X) \cap \operatorname{ri}\left(\operatorname{dom}(f)\right) : h(x') \in -\operatorname{ri}\left(\mathbb{R}^k_+\right),$$

which is actually the Slater constraint qualification

$$(CQ_1') \quad \exists x' \in \mathrm{ri}(X) \cap \mathrm{ri}\left(\mathrm{dom}(f)\right) : h(x') < 0.$$

As $\mathrm{ri}(X) \neq \emptyset$, the following equalities can be easily proved (cf. [3], [6])

$$\bigcup_{\beta \in K^*} \mathrm{epi}\left((\beta^T h)_X^*\right) = \bigcup_{\beta \geqq 0} \mathrm{epi}\left((\beta^T h)_X^*\right) = \mathrm{coneco}\left(\bigcup_{i=1}^{k} \mathrm{epi}(h_i^*)\right) + \mathrm{epi}(\sigma_X).$$

Then the following results are easy consequences of Theorem 5.1 and Theorem 5.3.

Theorem 5.4 Suppose that (CQ_1') holds. Then the following assertions are equivalent:

(i) $x \in X$, $h(x) \leqq 0 \Rightarrow f(x) \geq 0$;
(ii) there exist $p \in \mathbb{R}^n$ and $\beta \geqq 0$ such that

$$f^*(p) + \left(\beta^T h\right)_X^*(-p) \leq 0.$$

Theorem 5.5 The statement *(ii)* in Theorem 5.4 is equivalent with

$$(0,0) \in \mathrm{epi}(f^*) + \mathrm{coneco}\left(\bigcup_{i=1}^{k} \mathrm{epi}(h_i^*)\right) + \mathrm{epi}(\sigma_X).$$

As a last remark, let us mention that the last two theorems were obtained by Boţ and Wanka in [6], as a generalization of some results due to Jeyakumar ([9]).

6 Conclusions

Within the current paper we deal with conjugate duality and Farkas-type results in composed convex programming. The approach we use is based on conjugate duality for an optimization problem consisting in minimizing the sum between a convex function and the precomposition of an K-increasing and convex function with a K-convex vector function, where K is a closed convex cone. The result we present generalizes some Farkas-type results presented by Boţ and Wanka in [6] and by Jeyakumar in [9]. Moreover, the existing connections between the Farkas-type results and the theorems of the alternative and, respectively, the theory of duality are emphasized once more.

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