Brézis - Haraux - type approximation of the range of a monotone operator composed with a linear mapping

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Abstract. We give a Brézis - Haraux - type approximation of the range of the monotone operator $T_A = A^* \circ T \circ A$ when A is a linear continuous mapping between two Banach spaces and T is a maximal monotone operator. Then we specialize the result for a Brézis - Haraux - type approximation of the range of the subdifferential of the precomposition to A of a proper convex lower semicontinuous function defined on a Banach space, which is proven to hold under a weak sufficient condition. This extends and corrects some older results due to Riahi and Chbani that consist in the approximation of the range of the sum of the subdifferentials of two proper convex lower semicontinuous functions.

1 Introduction

Given two monotone operators, the sum of their ranges is usually larger than the range of their sum, but there are some situations where these sets are almost equal, i.e. their interiors and closures coincide. Brézis and Haraux ([7], [8]) pioneered the research on this subject giving some conditions that assured the mentioned result in Hilbert spaces. Since then the problem of finding conditions under which the sum of the ranges of two monotone operators is almost equal to the range of their sum is known as the *Brézis - Haraux approximation* problem and the original result has been extended in several directions. Reich ([19]), Chu ([12], [13]) and Simons ([23]) treated the problem in reflexive Banach spaces and Chbani and Riahi ([11]) and Riahi ([20]) in Banach spaces, while Pennanen ([17]), working in reflexive Banach spaces, extended the result from sums of monotone operators to monotone composite mappings of the form $A^* \circ T \circ A$ where A is a linear continuous mapping and T is a monotone operator.

The Brézis - Haraux approximation and its extensions are interesting not only for the results themselves, but also for their many applications. We mention here some of them, namely in variational inequality problems ([1]), Hammerstein equations and Neumann problem ([7], [8]), generalized equations ([16]), Kruzkov's solutions of the Burger - Carleman's system ([10]), projection algorithms ([2]), Bregman algorithms ([3]), Fenchel - Rockafellar - Moreau duality model ([16], [17]), optimization problems, Hammerstein differential inclusions and complementarity problems ([11]), and the list is far from being complete.

Within this paper we give a Brézis - Haraux - type approximation statement for $A^* \circ T \circ A$ in Banach spaces. Then we specialize the result to approximate the

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range of $\partial(f \circ A)$, where f is a proper convex lower semicontinuous function defined on the image space of A with extended real values, generalizing and correcting the result given in [11] and [20] for the sum of the subdifferentials of two proper convex lower semicontinuous functions which arises as special case. Moreover, the regularity condition we impose is weaker than the one considered in the mentioned papers in order to obtain the result. Finally we give two applications, one in optimization and the other to a complementarity problem.

The paper is structured as follows. The next section contains necessary preliminaries, notions and results used later, then we deal with the Brézis - Haraux - type approximation for $A^* \circ T \circ A$. Section 4 deals with the mentioned Brézis - Haraux - type approximations for $\partial(f \circ A)$ and its special case concerning the range of the sum of the subdifferentials of two proper convex lower semicontinuous functions, and it is followed by two applications. An ample list of references closes the paper.

2 Preliminaries

In order to make the paper self - contained we introduce here the context we work within and we recall the necessary notions and results. Let X and Y be two locally convex spaces, unless otherwise specified, and their continuous dual spaces X^* and Y^*, endowed with the weak* topologies $w(X^*, X)$ and $w(Y^*, Y)$, respectively. By $\langle x^*, x\rangle$ we denote the value of the linear continuous functional $x^* \in X^*$ at $x \in X$. Given a subset M of X, we denote by $\mathrm{int}(M)$ and $\mathrm{cl}(M)$ its *interior*, respectively its *closure* in the corresponding topology. We call it *closed regarding the subspace* $Z \subseteq X$ if $M \cap Z = \mathrm{cl}(M) \cap Z$ and we have its *indicator* function $\delta_M : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, defined by

$$\delta_M(x) = \begin{cases} 0, & \text{if } x \in M, \\ +\infty, & \text{otherwise.} \end{cases}$$

For a function $f : X \to \overline{\mathbb{R}}$, we have

- the *domain*: $\mathrm{dom}(f) = \{x \in X : f(x) < +\infty\}$,
- the *epigraph*: $\mathrm{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$,
- the *conjugate*: $f^* : X^* \to \overline{\mathbb{R}}$ given by $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\}$,
- the *subdifferential* of f at $x \in X$ where $f(x) \in \mathbb{R}$: $\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle \ \forall y \in X\}$,
- f is *proper*: $f(x) > -\infty \ \forall x \in X$ and $\mathrm{dom}(f) \neq \emptyset$.

When $f, g : X \to \overline{\mathbb{R}}$ are proper functions, their *infimal convolution* is defined by

$$f \Box g : X \to \overline{\mathbb{R}}, \ f \Box g(a) = \inf\{f(x) + g(a - x) : x \in X\}.$$

For $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$, we define the *product function*

$$(f \times g) : X \times Y \to \overline{\mathbb{R}} \times \overline{\mathbb{R}}, \ (f \times g)(x, y) = (f(x), g(y)) \ \forall (x, y) \in X \times Y.$$

Given a linear continuous mapping $A : X \to Y$, its *adjoint* is

$$A^* : Y^* \to X^*, \ \langle A^* y^*, x\rangle = \langle y^*, Ax\rangle \ \forall (x, y^*) \in X \times Y^*.$$

For a proper function $f : X \to \overline{\mathbb{R}}$ we recall also the definition of the *marginal* function of f through A as being

$$Af : Y \to \overline{\mathbb{R}}, \ Af(y) = \inf\big\{f(x) : x \in X, Ax = y\big\} \ \forall y \in Y.$$

Consider also the *identity* function on X defined by

$$\mathrm{id}_X : X \to X, \ \mathrm{id}_X(x) = x \ \forall x \in X.$$

Let us mention moreover that we write min (max) instead of inf (sup) when the infimum (supremum) is attained.

Proposition 1. ([6]) *Let $A : X \to Y$ be a linear continuous mapping and $f : Y \to \overline{\mathbb{R}}$ a proper, convex and lower semicontinuous function such that $f \circ A$ is proper. Then*

(i) $A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(f^*))$ *is closed in the product topology of* $(X^*, w(X^*, X)) \times \mathbb{R}$ *if and only if for any* $x^* \in X^*$ *one has*

$$(f \circ A)^*(x^*) = \min\{f^*(y^*) : A^*y^* = x^*\}.$$

(ii) If $A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(f^*))$ *is closed in the product topology of* $(X^*, w(X^*, X)) \times \mathbb{R}$*, then for any* $x \in \mathrm{dom}(f \circ A)$ *one has* $\partial(f \circ A)(x) = A^*\partial f(Ax)$*.*

The second part of this section in devoted to monotone operators and some of their properties. From now on we consider, within the whole paper, X and Y Banach spaces. We denote by $\|\cdot\|$ the norm on X, while the one on X^* is $\|\cdot\|_*$.

Definition 1. ([22]) A mapping (generally multivalued) $T : X \to 2^{X^*}$ is called *monotone operator* provided that for any $x, y \in X$ one has

$$\langle y^* - x^*, y - x\rangle \geq 0 \text{ whenever } x^* \in T(x) \text{ and } y^* \in T(y).$$

Definition 2. ([22]) For any monotone operator $T : X \to 2^{X^*}$ we have

· its *effective domain* $D(T) = \{x \in X : T(x) \neq \emptyset\}$,
· its *range* $R(T) = \cup\{T(x) : x \in X\}$,
· its *graph* $G(T) = \{(x, x^*) : x \in X, x^* \in T(x)\}$.

Definition 3. ([22]) A monotone operator $T : X \to 2^{X^*}$ is called *maximal* when its graph is not properly included in the graph of any other monotone operator $T' : X \to 2^{X^*}$.

Let τ_1 be the weakest topology on X^{**} which renders continuous the following real functions

$$X^{**} \to \mathbb{R} : x^{**} \mapsto \langle x^{**}, x^*\rangle \ \forall x^* \in X^*,$$
$$X^{**} \to \mathbb{R} : x^{**} \mapsto \|x^{**}\|.$$

The topology τ in $X^{**} \times X^*$ is the product topology of τ_1 and the strong (norm) topology of X^* (cf. [15]).

Definition 4. ([15]) A monotone operator $T : X \to 2^{X^*}$ is called of *dense type* provided that its *closure* operator $\overline{T} : X^{**} \to 2^{X^*}$,

$$G(\overline{T}) = \left\{(x^{**}, x^*) \in X^{**} \times X^* : \exists (x_i, x_i^*)_i \in G(T) \text{ with } (\hat{x}_i, x_i^*) \xrightarrow{\tau} (x^{**}, x^*)\right\}$$

is maximal monotone, where $\hat{y}$ denotes the canonical image of y in X^{**}.

Different to Riahi ([20]) and Chbani and Riahi ([11]), where these operators are called *densely maximal monotone*, respectively *densely monotone*, we decided to name them as Gossez ([15]) did when he introduced them. By Lemme 2.1 in [15], whenever the monotone operator $T : X \to 2^{X^*}$ is of dense type one has $(x^{**}, x^*) \in G(\overline{T})$ if and only if $\langle x^{**} - \hat{y}, x^* - y^*\rangle \geq 0 \ \forall (y, y^*) \in G(T)$.

The monotone operators belonging to the following class are also known as *star* monotone operators or operators of the *type* (BH), being first introduced in [8].

Definition 5. ([13], [17], [20]) A monotone operator $T : X \to 2^{X^*}$ is called 3^* - *monotone* if for all $x^* \in R(T)$ and $x \in D(T)$ there is some $\beta(x^*, x) \in \mathbb{R}$ such that $\inf_{(y,y^*)\in G(T)} \langle x^* - y^*, x - y\rangle \geq \beta(x^*, x)$.

The last collection of monotone operators we introduce consists of so - called *negative - infimum* monotone operators.

Definition 6. ([23], [24]) A monotone operator $T : X \to 2^{X^*}$ is called of *type* (NI) if for all $(x^{**}, x^*) \in X^{**} \times X^*$ one has $\inf_{(y,y^*)\in G(T)} \langle \hat{y} - x^{**}, y^* - x^*\rangle \leq 0$.

Remark 1. The subdifferential of a proper convex lower semicontinuous function on X is a typical example for all these classes of monotone operators. We refer to [15], [17], [18], [20], [21], [23], [24] and [25] for proofs and more on these subjects.

There are some other types of monotone operators, like cyclic monotone, but as they are not relevant for the results within this paper we do not mention them here. Between these classes of monotone operators there are various relations, let us recall the ones necessary for our purposes.

Proposition 2. ([15]) *In reflexive Banach spaces every maximal monotone operator is of dense type and coincides with its closure operator.*

We close the section by recalling an important result which proved to be useful in the present work.

Lemma 1. ([20]) *Given the dense type operator $T : X \to 2^{X^*}$ and the non - empty subset $E \subseteq X^*$ such that for any $x^* \in E$ there is some $x \in X$ fulfilling $\inf_{(y,y^*)\in G(T)} \langle x^* - y^*, x - y\rangle > -\infty$, one has $E \subseteq \operatorname{cl}(R(T))$ and $\operatorname{int}(E) \subseteq R(\overline{T})$.*

3 Brézis - Haraux - type approximation of the range of a monotone operator composed with a linear mapping

We give in this section the main results concerning the so - called Brézis - Haraux - type approximation (cf. [8], [23]) of the range of a composed operator T_A, defined below, respectively of the subdifferential of the precomposition of a linear continuous mapping with a proper convex lower semicontinuous function. Some results related to ours were obtained by Pennanen in [17], but in reflexive spaces, while we work in general Banach spaces.

Consider the monotone operator $T : Y \to 2^{Y^*}$ and the linear continuous mapping $A : X \to Y$. We introduce the composed operator $T_A := A^* \circ T \circ A : X \to 2^{X^*}$. It is known that T_A is a monotone operator and under certain conditions it is maximal monotone (cf. [4], for instance). We show first that it is 3^* monotone when T is 3^* monotone, too.

Proposition 3. *If $T : Y \to 2^{Y^*}$ is 3^* - monotone and $A : X \to Y$ is a linear continuous mapping, then T_A is 3^* - monotone, too.*

Proof. If $D(T_A) = \emptyset$, then the conclusion arises trivially. Elsewise take $x^* \in R(T_A)$, i.e. there is some $z \in X$ such that $x^* \in A^* \circ T \circ A(z)$. Thus there exists a $z^* \in T \circ A(z)$ satisfying $x^* = A^* z^*$. Clearly, $z^* \in R(T)$. Consider also an $x \in D(T_A)$

and denote $u = Ax \in D(T)$. When $y^* \in T_A(y)$ there is some $t^* \in T \circ A(y)$ such that $y^* = A^*t^*$. We have

$$\begin{aligned}\inf_{(y,y^*)\in G(T_A)} \langle x^* - y^*, x - y\rangle &= \inf_{(y,t^*)\in G(T\circ A)} \langle A^*z^* - A^*t^*, x - y\rangle \\ &= \inf_{(y,t^*)\in G(T\circ A)} \langle z^* - t^*, A(x - y)\rangle \\ &\geq \inf_{(v,t^*)\in G(T)} \langle z^* - t^*, u - v\rangle \geq \beta(z^*, u) \in \mathbb{R},\end{aligned}$$

as T is 3^* - monotone. Therefore, by definition, T_A is 3^* - monotone, too. □

Next we give an auxiliary result needed in order to prove the main statement of the section which comes after it.

Lemma 2. *If $T : Y \to 2^{Y^*}$ is 3^* - monotone and $A : X \to Y$ is a linear continuous mapping such that T_A is of dense type, then*

(i) $A^*(R(T)) \subseteq \mathrm{cl}(R(T_A))$, *and*
(ii) $\mathrm{int}(A^*(R(T))) \subseteq R(\overline{T_A})$.

Proof. The operator T_A being of dense type implies that $D(T_A) \neq \emptyset$, thus $D(T) \neq \emptyset$.

As T is 3^* - monotone, we have for any $s \in D(T)$ and any $s^* \in R(T)$ there is some $\beta(s^*, s) \in \mathbb{R}$ such that $\beta(s^*, s) \leq \inf_{(y,y^*)\in G(T)}\langle s^* - y^*, s - y\rangle$.

Take some $x^* \in A^*(R(T))$, thus there is an $z^* \in R(T)$ such that $x^* = A^*z^*$. As in the proof of Proposition 3, for some $x \in D(T_A)$ there holds

$$\inf_{(y,y^*)\in G(T_A)} \langle x^* - y^*, x - y\rangle > -\infty.$$

Now we can apply Lemma 1 for $E = A^*(R(T))$ and T_A and we obtain exactly (i) and (ii). □

Theorem 1. *If $T : Y \to 2^{Y^*}$ is 3^* - monotone and $A : X \to Y$ is a linear continuous mapping such that T_A is of dense type, then*

(i) $\mathrm{cl}(A^*(R(T))) = \mathrm{cl}(R(T_A))$, *and*
(ii) $\mathrm{int}(R(T_A)) \subseteq \mathrm{int}(A^*(R(T))) \subseteq \mathrm{int}(R(\overline{T_A}))$.

Proof. The operator T_A being of dense type implies that $D(T_A) \neq \emptyset$. Take some $x^* \in R(T_A)$. Then there are some $x \in X$ and $y^* \in T \circ A(x) \subseteq R(T)$ such that $x^* = A^*y^*$. Thus $x^* \in A^*(R(T))$, so $R(T_A) \subseteq A^*(R(T))$ and the same inclusion stands also between the closures, respectively the interiors, of these sets, i.e.

$$\mathrm{cl}(R(T_A)) \subseteq \mathrm{cl}(A^*(R(T))) \ \text{ and } \ \mathrm{int}(R(T_A)) \subseteq \mathrm{int}(A^*(R(T))). \tag{1}$$

On the other hand, by Lemma 2(i) we get immediately

$$\mathrm{cl}(A^*(R(T))) \subseteq \mathrm{cl}(R(T_A)) \ \text{ and } \ \mathrm{int}(A^*(R(T))) \subseteq \mathrm{int}(R(\overline{T_A})). \tag{2}$$

Relations (i) and (ii) follow immediately from (1) and (2). □

Remark 2. The previous statement generalizes Theorem 1 in [20], which can be obtained for $Y = X \times X$, $Ax = (x, x)$ and $T(y, z) = (T_1(y), T_2(z))$. The next assertion extends Corollary 1 in [20] which arises for the same choice of Y, A and T.

Corollary 1. *If X is a reflexive Banach space, $T : Y \to 2^{Y^*}$ is a 3^* - monotone operator and $A : X \to Y$ is a linear continuous mapping such that T_A is maximal monotone, then one has*

$$\mathrm{cl}(A^*(R(T))) = \mathrm{cl}(R(T_A)) \quad \textit{and} \quad \mathrm{int}(R(T_A)) = \mathrm{int}(A^*(R(T))).$$

Proof. As X is reflexive, Proposition 2 yields that T_A is maximal monotone of dense type and $\overline{T_A}$ and T_A coincide. Theorem 1 delivers the conclusion. □

4 The approximation of the range of the subdifferential of a function composed with a linear mapping

We generalize now Corollary 2 in [20] and Corollary 3.2 in [11], providing a Brézis - Haraux - type approximation of the range of the subdifferential of the precomposition of a proper convex lower semicontinuous function with a linear continuous mapping. Moreover we correct the mentioned results which are improved further by considering a weaker constraint qualification under which one can give the Brézis - Haraux - type approximation of the range of the sum of the subdifferentials of two proper convex lower semicontinuous functions. First we give the constraint qualification that guarantees our more general result,

(CQ) $A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(f^*))$ is closed in the product topology of $(X^*, w(X^*, X)) \times \mathbb{R}$.

Theorem 2. *Let the proper convex lower semicontinuous function $f : Y \to \overline{\mathbb{R}}$ and the linear continuous mapping $A : X \to Y$ such that $f \circ A$ is proper, and assume (CQ) valid. Then one has*

(i) $\mathrm{cl}(A^*(R(\partial f))) = \mathrm{cl}(R(\partial(f \circ A)))$, *and*
(ii) $\mathrm{int}(R(\partial(f \circ A))) \subseteq \mathrm{int}(A^*(R(\partial f))) \subseteq \mathrm{int}(D(\partial(A^* f^*)))$.

Proof. As $f \circ A$ is proper, convex and lower semicontinuous, by Théoréme 3.1 in [15] we know that $\partial(f \circ A)$ is an operator of dense type, while according to Theorem B in [21] (see also [17], [20]) ∂f is 3^* - monotone.

By Proposition 1(ii) we know that (CQ) implies $A^* \circ \partial f \circ A = \partial(f \circ A)$. Therefore $A^* \circ \partial f \circ A$ is an operator of dense type, too.
Applying Theorem 1 for $T = \partial f$ we get

$$\mathrm{cl}(A^*(R(\partial f))) = \mathrm{cl}(R(A^* \circ \partial f \circ A)) = \mathrm{cl}(R(\partial(f \circ A)))$$

and

$$\mathrm{int}(R(A^* \circ \partial f \circ A)) \subseteq \mathrm{int}(A^*(R(\partial f))) \subseteq \mathrm{int}(R(\overline{A^* \circ \partial f \circ A})).$$

The relation above that involves closures yields (i), while the other becomes

$$\mathrm{int}(R(\partial(f \circ A))) \subseteq \mathrm{int}(A^*(R(\partial f))) \subseteq \mathrm{int}(R(\overline{\partial(f \circ A)})). \tag{3}$$

As from Proposition 1(i) one may deduce that under (CQ) $A^* f^* = (f \circ A)^*$, by Théoréme 3.1 in [15] we get $R(\overline{\partial(f \circ A)}) = D(\partial(f \circ A)^*) = D(\partial(A^* f^*))$. Putting this into (3) we get (ii). □

When one takes $Y = X \times X$, $Ax = (x, x)$ and $f(x, y) = g(x) + h(y)$, for $x, y \in X$, where g and h are functions defined on X with extended - real values, the constraint qualification (CQ) becomes (cf. [6])

(CQ^s) $\operatorname{epi}(g^*) + \operatorname{epi}(h^*)$ is closed in the product topology of $(X^*, w(X^*, X)) \times \mathbb{R}$

and one obtains the following statement.

Corollary 2. (see also [5]) *Let g and h be two proper convex lower semicontinuous functions on the Banach space X with extended real values such that $\operatorname{dom}(g) \cap \operatorname{dom}(h) \neq \emptyset$. Assume (CQ^s) satisfied. Then one has*

(i) $\operatorname{cl}(R(\partial g) + R(\partial h)) = \operatorname{cl}(R(\partial(g + h)))$, *and*

(ii) $\operatorname{int}(R(\partial g + \partial h)) \subseteq \operatorname{int}(R(\partial g) + R(\partial h)) \subseteq \operatorname{int}(D(\partial(g^*\Box h^*))) = \operatorname{int}(D(\partial((g + h)^*)))$.

Proof. We apply Theorem 2 and Proposition 1 for $Ax = (x, x)$ and $f(y, z) = g(y) + h(z)$ for any $(y, z) \in Y = X \times X$. One can easily verify that $(f \circ A)(x) = g(x)+h(x)$, $A^*(y^*, z^*) = y^*+z^*$ $\forall (y^*, z^*) \in X^* \times X^*$ and $A^*f^* = g^*\Box h^*$. Moreover, $A^*(R(\partial f)) = A^*(R(\partial g)\times R(\partial h)) = R(\partial g)+R(\partial h)$ and $A^*\times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(f^*)) = \operatorname{epi}(g^*)+\operatorname{epi}(h^*)$.

By Proposition 1(ii) we have that (CQ^s) yields $\partial(g + h) = \partial g + \partial h$.

Using the remarks above, from Theorem 2 we get $\operatorname{cl}(R(\partial(g + h))) = \operatorname{cl}(R(\partial g) + R(\partial h))$ and

$$\operatorname{int}(R(\partial(g + h))) \subseteq \operatorname{int}(R(\partial g) + R(\partial h)) \subseteq \operatorname{int} D(\partial(g^*\Box h^*)) = \operatorname{int}(D(\partial(g + h)^*)),$$

the last equality arising by Proposition 1(ii). $\square$

We remark that this proof is different from the one in [5].

Similar results have been obtained by Riahi in Corollary 2 in [20] and by Chbani and Riahi in Corollary 3.2 in [11], under the constraint qualification

$$(CQ_R) \qquad \bigcup_{t>0} t(\operatorname{dom}(g) - \operatorname{dom}(h)) \text{ is a closed linear subspace of } X.$$

In [20] (CQ_R) is said to imply

$$\operatorname{cl}(R(\partial g) + R(\partial h)) = \operatorname{cl}(R(\partial(g + h))) \text{ and } \operatorname{int}(R(\partial g) + R(\partial h)) = \operatorname{int}(D(\partial(g^*\Box h^*))),$$

while according to [11] it yields

$$\operatorname{cl}(R(\partial g) + R(\partial h)) = \operatorname{cl}(R(\partial(g + h))) \text{ and } \operatorname{int}(R(\partial g) + R(\partial h)) = \operatorname{int}(D(\partial(g + h)^*)).$$

We prove that the latter is not always true when (CQ_R) stands. For a proper, convex and lower semicontinuous function $g : X \to \overline{\mathbb{R}}$ (by taking $h \equiv 0$) Riahi's relation would become $\operatorname{int}(R(\partial g)) = \operatorname{int}(D(\partial g^*))$, which is equivalent, by Théoréme 3.1 in [15], to

$$\operatorname{int}(R(\partial g)) = \operatorname{int}(R(\overline{\partial g})). \tag{4}$$

From Théoréme 3.1 in [15] we know that ∂g is a monotone operator of dense type and, from [21], that it is maximal monotone, too. By [24] we know that ∂g is also of type (NI).

By Theorem 20 in ([24]) we get that $\operatorname{int}(R(\overline{\partial g}))$ is convex, so (4) yields that $\operatorname{int}(R(\partial g))$ is convex, too.

Unfortunately this is not always true, as Example 2.21 in [18], originally given by Fitzpatrick, shows. Take $X = c_0$, the space of the real sequences converging to 0, which is a non - reflexive Banach space with the usual norm $\|x\| = \sup_{n\geq 1} |x_n|$ $\forall x = (x_n)_{n\geq 1} \in c_0$, and $g(x) = \|x\|+\|x-e_1\|$, for all $x \in c_0$, where $e_1 = (1, 0, 0, ...) \in c_0$. It is clear that g is proper, convex and continuous on c_0, since $\|\cdot\|$ has these properties. Moreover for any $x \in c_0$ one has $\partial g(x) = \partial\|\cdot\|(x) + \partial\|\cdot -e_1\|(x)$.

The dual space of c_0 is l^1, which consists of all the sequences $y = (y_n)_{n\geq 1}$ such that $\|y\|_* = \sum_{n=1}^{+\infty} |y_n| < +\infty$. Denote by F the set of sequences in l^1 having finitely many non - zero entries and by B^* the closed unit ball in l^1.

It is known that $\|\cdot\|^*(y) = 0$ if $\|y\|_* \leq 1$ and $\|\cdot\|^*(y) = +\infty$ otherwise, which leads to $\partial\|\cdot\|(x) = B^*$ if $x = 0$, $\partial\|\cdot\|(e_1) = \{e_1^*\}$, $\partial\|\cdot\|(-e_1) = \{-e_1^*\}$ and $\partial\|\cdot\|(x) = \{y \in l^1 : \|y\|_* \leq 1,\ \langle y, x\rangle = \|x\|\} \subseteq F$, otherwise, where $e_1^* = (1, 0, 0, ...) \in l^1$. Moreover we have $\partial\|\cdot - e_1\|(x) = \partial\|\cdot\|(x - e_1)$ for any $x \in c_0$. Further one gets $\partial g(0) = -e_1^* + B^*$ and $\partial g(e_1) = e_1^* + B^*$. Otherwise, i.e. if $x \in c_0\backslash\{0, e_1\}$, $\partial g(x) \subseteq F$. Therefore

$$R(\partial g) \subseteq (-e_1^* + B^*) \cup (e_1^* + B^*) \cup F. \tag{5}$$

Since $\text{int}(R(\partial g))$ includes $\text{int}(B^*) \pm e_1^*$, assuming it convex yields $0 = 1/2(e_1^* - e_1^*) \in \text{int}(R(\partial g))$. Hence there is a neighborhood of 0, say U, completely included in $R(\partial g)$. Take some $\lambda > 0$ sufficiently small such that

$$\nu(\lambda) = \left(0, \frac{\lambda}{2^2}, \frac{\lambda}{2^3}, \frac{\lambda}{2^4}, ...\right) \in U.$$

Thus $\nu(\lambda) \in R(\partial g)$. One can check that $\|\nu(\lambda) \pm e_1^*\|_* = 1 + \frac{\lambda}{2} > 1$, so, taking into consideration (5), $\nu(\lambda)$ must be in F. It is clear that this does not happen, thus we have obtained a contradiction. Therefore $\text{int}(R(\partial g))$ is not convex, unlike $\text{int}(R(\overline{\partial g}))$. Thus (4) is false and the same happens to the allegations concerning the interior of the sum of the ranges of two subdifferentials in [11] and [20].

Remark 3. As proven in Proposition 3.1 in [9] (see also [6]), (CQ_R) implies (CQ^s), but the converse is not true, as shown by Example 3.1 in [9]. Therefore our Corollary 2 extends, by weakening the constraint qualification, and corrects Corollary 3.2 in [11] and Corollary 2 in [20].

5 Applications

We give in the following two applications of the results we have presented in the previous section. Both of them generalize some earlier statements that are available in [11] under stronger requirements.

5.1 Existence of a solution to an optimization problem

We work within the framework of Corollary 2, i.e. let g and h be two proper convex lower semicontinuous functions on the Banach space X with extended real values such that $\text{dom}(g) \cap \text{dom}(h) \neq \emptyset$.

Theorem 3. *Assume (CQ^s) satisfied and moreover that $0 \in \text{int}(R(\partial g) + R(\partial h))$. Then there is a neighborhood V of 0 in X^* such that $\forall x^* \in V$ there is an $\bar{x} \in \text{dom}(g) \cap \text{dom}(h)$ where*

$$g(\bar{x}) + h(\bar{x}) - \langle x^*, \bar{x}\rangle = \min_{x\in X}[g(x) + h(x) - \langle x^*, x\rangle].$$

Proof. By Corollary 2 we have $\text{int}(R(\partial g) + R(\partial h)) \subseteq \text{int}(D(\partial((g + h)^*)))$, thus $0 \in \text{int}(D(\partial((g + h)^*)))$, i.e. there is a neighborhood V of 0 in X^* such that $V \subseteq D(\partial((g + h)^*))$. Fix some $x^* \in V$. Immediately one gets that there is some $\bar{x} \in \text{dom}(g) \cap \text{dom}(h)$ such that $(g + h)^*(x^*) + ((g + h)^*)^*(\bar{x}) = \langle x^*, \bar{x}\rangle$. As $g + h$ is a proper convex lower semicontinuous function we have $(g+h)^{**} = ((g+h)^*)^* = g+h$, thus the equality above becomes

$$g(\bar{x}) + h(\bar{x}) - \langle x^*, \bar{x}\rangle = -(g + h)^*(x^*) = -\sup_{x\in X}\{\langle x^*, x\rangle - g(x) - h(x)\}.$$

This means actually that the conclusion stands. Because of (CQ^s) we know (cf. [6]) that

$$\inf_{x\in X}[g(x)+h(x)-\langle x^*,x\rangle]=\max_{p\in X^*}\{-g^*(p)-h^*(x^*-p)\},$$

so one may notice that under the assumptions of the problem we obtain something that may be called locally stable total Fenchel duality, i.e. the situation where both problems, the primal on the left-hand side and the dual on the right-hand side, have optimal solutions and their values coincide for small enough linear perturbations of the objective function of the primal problem. Let us notice moreover that as $0\in V$, for $x^*=0$ we obtain also the classical Fenchel strong duality statement, but where moreover the primal problem has a solution, too. □

5.2 Existence of a solution to a complementarity problem

Further consider X a reflexive Banach space, let $C\subseteq X$ be a closed convex cone and $S:X\to 2^{X^*}$ a monotone operator. In order to formulate the statement we have to introduce some new notions and to recall a recent result of ours.

To a monotone operator $T:X\to 2^{X^*}$ Fitzpatrick ([14], see also [4]) attached the function

$$\varphi_T:X\times X^*\to\overline{\mathbb{R}},\ \varphi_T(x,x^*)=\sup\big\{\langle y^*,x\rangle+\langle x^*,y\rangle-\langle y^*,y\rangle:(y,y^*)\in G(T)\big\}.$$

For any monotone operator T it is quite clear that φ_T is a convex lower semicontinuous function as an affine supremum. Denote also $\varDelta_X=\{(x,x):x\in X\}$.

Theorem 4. ([5]) *Given two maximal monotone operators T_1 and T_2 on X. If the constraint qualification*

$(\widetilde{CQ})$ $\{(x^*+y^*,x,y,r):\varphi^*_{T_1}(x^*,x)+\varphi^*_{T_2}(y^*,y)\le r\}$ *is closed regarding the subspace $X^*\times\varDelta_X\times\mathbb{R}$,*

is fulfilled then T_1+T_2 is a maximal monotone operator.

Consider the complementarity problem

$$(CP)\qquad \begin{cases} x\in C,\ x^*\in C^*,\\ \langle x^*,x\rangle=0,\\ x^*\in S(x).\end{cases}$$

and the constraint qualification

$(\overline{CQ})$ $\{(x^*+y^*,x,y,r)$: $(x^*,x,r)\in\operatorname{epi}(\varphi^*_S),\ y\in C,\ y^*\in -C^*\}$ is closed regarding the subspace $X^*\times\varDelta_X\times\mathbb{R}$.

Theorem 5. *Suppose that S is simultaneously maximal and 3^* monotone, assume $(\overline{CQ})$ fulfilled and moreover that $0\in\operatorname{int}(R(S)-C^*)$. Then the complementarity problem (CP) admits a solution.*

Proof. The conjugate function to δ_C and its subdifferential are

$$\delta^*_C(y^*)=\begin{cases}0, & \text{if } y^*\in -C^*,\\ +\infty, & \text{otherwise.}\end{cases}\quad\text{and } \partial\delta_C(x)=\{y^*\in -C^*:\langle y^*,x\rangle=0\}\ \forall x\in C.$$

It is easy to notice that $R(\partial\delta_C)\subseteq -C^*$ and $\partial\delta_C(0)=-C^*$, thus $R(\partial\delta_C)=-C^*$.

It is also straightforward to see that finding a solution to (CP) is equivalent to proving the existence of some $x \in C$ such that $0 \in S(x) + \partial\delta_C(x) = (S + \partial\delta_C)(x)$.

In order to apply Corollary 1 we need the maximal monotonicity of $S+\partial\delta_C$. As suggested by Theorem 4 we calculate the Fitzpatrick function attached to $\partial\delta_C$ and its conjugate. We have for some pair $(x, x^*) \in X \times X^*$

$$\begin{aligned}
\varphi_{\partial\delta_C}(x, x^*) &= \sup_{(y,y^*)\in G(\partial\delta_C)} \{\langle y^*, x\rangle + \langle x^*, y\rangle - \langle y^*, y\rangle\} \\
&= \sup_{\substack{y\in C, y^*\in -C^*,\\ \langle y^*, y\rangle = 0}} \{\langle y^*, x\rangle + \langle x^*, y\rangle\} \\
&= \begin{cases} 0, & \text{if } x \in C, x^* \in -C^*, \\ +\infty, & \text{otherwise.} \end{cases}
\end{aligned}$$

Its conjugate is, for $(z^*, z) \in X^* \times X$,

$$\varphi^*_{\partial\delta_C}(z^*, z) = \sup_{\substack{x\in C,\\ x^*\in -C^*}} \{\langle z^*, x\rangle + \langle x^*, z\rangle\} = \begin{cases} 0, & \text{if } z \in C,\ z^* \in -C^*, \\ +\infty, & \text{otherwise.} \end{cases}$$

It is not difficult to observe now that for $T_1 = S$ and $T_2 = \partial\delta_C$ the constraint qualification $(\widetilde{CQ})$ turns into $(\overline{CQ})$. This leads, by Theorem 4, to the maximal monotonicity of $S + \partial\delta_C$, so by Corollary 1, for the same choice of Y, A and S as in Remark 2, one gets

$$\operatorname{int}(R(S) - C^*) = \operatorname{int}(R(S) + R(\partial\delta_C)) = \operatorname{int}(R(S + \partial\delta_C)),$$

as in this case $T_A = S + \partial\delta_C$ and $A^*R(T) = R(S) + R(\partial\delta_C)$.

From the hypothesis we get $0 \in \operatorname{int}(R(S+\partial\delta_C))$, thus $0 \in R(S+\partial\delta_C)$, i.e. there is some $x \in C$ such that $0 \in S(x) + \partial\delta_C(x) = (S + \partial\delta_C)(x)$. As remarked above, this is equivalent to the fact that (CP) admits a solution. □

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