Almost Convex Functions: Conjugacy and Duality

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Summary. We prove that the formulae of the conjugates of the precomposition with a linear operator, of the sum of finitely many functions and of the sum between a function and the precomposition of another one with a linear operator hold even when the convexity assumptions are replaced by almost convexity or nearly convexity. We also show that the duality statements due to Fenchel hold when the functions involved are taken only almost convex, respectively nearly convex.

Key words: Fenchel duality, conjugate functions, almost convex functions, nearly convex functions
AMS subject classification (2000). 26A51, 42A50, 49N15

1 Introduction

Convexity is an important tool in many fields of Mathematics having applications in different areas, including optimization. Various generalizations of the convexity were given in the literature, so a natural consequence was to verify their applicability in optimization. We mention here the papers [3], [4], [6], [8], [9], [10], [12], [13] and [15], where properties of the convex functions and statements in convex analysis and optimization were extended by using functions and sets that are not convex but nearly convex, closely convex, convexlike, evenly convex, quasiconvex or weakly convex. Comparisons between some classes of generalized convexities were also performed, let us remind here just [6] and [9] among many others.

Within this article we work with three types of generalized convexity. Our main results concern almost convex functions, which are defined as they were introduced by Frenk and Kassay in [9]. We need to mention this because there are in the literature some other types of functions called almost convex, too. We wrote our paper motivated by the lack of known results concerning almost convex functions (cf. [9]), but also in order to introduce new and to rediscover some of our older ([3]) statements for nearly convex functions. Introduced by Aleman ([1]) as p-convex functions, the latter ones were quite intensively studied recently under the name of nearly convex functions in papers like [3], [4], [6], [10], [12], [15] and [17], while for studies on nearly convex sets we refer to [7] and [14]. Closely convexity (cf. [2], [17]) is used to illustrate some properties of the already mentioned types of functions. We have also shown that there are differences between the classes of almost convex functions and nearly convex functions, both of them being moreover larger than the one of the convex functions.

Our paper is dedicated to the extension of some results from Convex Analysis in the sense that we prove that they hold not only when the functions involved are convex, but also when they are only almost convex, respectively nearly convex. The statements we generalize concern conjugacy and duality, as follows. We prove that the formulae of some conjugates, namely of the precomposition with a linear operator, of the sum of finitely many functions and of the sum between a function and the precomposition of another one with a linear operator hold even when the convexity assumptions are replaced by almost or nearly convexity. After these, we show that the well-known duality statements due to Fenchel hold when the functions involved are taken only almost convex, respectively nearly convex. The paper is divided into five sections. After the introduction and the necessary preliminaries we give some properties of the almost convex functions, then we deal with conjugacy and Fenchel duality for this kind of functions. Some short but comprehensive conclusions and the list of references close the paper.

2 Preliminaries

This section is dedicated to the exposition of some notions and results used within our paper. Not all the results we present here are so widely-known, thus we consider necessary to recall them.

As usual, $\mathbb{R}^n$ denotes the *n-dimensional real space*, for $n \in \mathbb{N}$, and $\mathbb{Q}$ is the set of all *rational* real numbers. Throughout this paper all the vectors are considered as column vectors belonging to $\mathbb{R}^n$, unless otherwise specified. An upper index ^{T} transposes a column vector to a row one and viceversa. The *inner product* of two vectors $x = \big(x_1, ..., x_n\big)^T$ and $y = \big(y_1, ..., y_n\big)^T$ in the n-dimensional real space is denoted by $x^T y = \sum_{i=1}^n x_i y_i$. The *closure* of a certain set is distinguished from the set itself by the preceding particle cl, while the leading ri denotes the *relative interior* of the set. If $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then by $A^* : \mathbb{R}^m \to \mathbb{R}^n$ we denote its *adjoint* defined by $(Ax)^T y = x^T(A^*y)\ \forall x \in \mathbb{R}^n\ \forall y \in \mathbb{R}^m$. For some set $X \subseteq \mathbb{R}^n$ we have the *indicator* function $\delta_X : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\delta_X(x) = \begin{cases} 0, & \text{if } x \in X, \\ +\infty, & \text{if } x \notin X. \end{cases}$$

Definition 1. *For a function $f : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ we consider the following notions*

(*i*) epigraph*:* $\text{epi}(f) = \big\{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \le r\big\}$,
(*ii*) (effective) domain*:* $\text{dom}(f) = \big\{x \in \mathbb{R}^n : f(x) < +\infty\big\}$,
(*iii*) f *is called* proper *if* $\text{dom}(f) \neq \emptyset$ *and* $f(x) > -\infty\ \forall x \in \mathbb{R}^n$,
(*iv*) $\bar{f}$ *is called the* lower-semicontinuous hull of f *if* $\text{epi}(\bar{f}) = \text{cl}(\text{epi}(f))$.
(*v*) subdifferential of f at x *(where $f(x) \in \mathbb{R}$):*

$$\partial f(x) = \{p \in \mathbb{R}^n : f(y) - f(x) \ge p^T(y - x)\ \forall y \in \mathbb{R}^n\}.$$

Remark 1. For any function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ we have $\text{dom}(f) \subseteq \text{dom}(\bar{f}) \subseteq \text{cl}(\text{dom}(f))$, which implies $\text{cl}(\text{dom}(f)) = \text{cl}(\text{dom}(\bar{f}))$.

Definition 2. *A set $X \subseteq \mathbb{R}^n$ is called* nearly convex *if there is a constant $\alpha \in]0, 1[$ such that for any x and y belonging to X one has $\alpha x + (1 - \alpha)y \in X$.*

An example of a nearly convex set which is not convex is $\mathbb{Q}$. Important properties of the nearly convex sets follow.

Lemma 1. *([1]) For every nearly convex set* $X \subseteq \mathbb{R}^n$ *the following properties are valid*

(i) $\mathrm{ri}(X)$ *is convex (may be empty),*
(ii) $\mathrm{cl}(X)$ *is convex,*
(iii) *for every* $x \in \mathrm{cl}(X)$ *and* $y \in \mathrm{ri}(X)$ *we have* $tx + (1-t)y \in \mathrm{ri}(X)$ *for each* $0 \le t < 1$.

Definition 3. *([6], [9]) A function* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *is called*

(i) almost convex *if* $\bar{f}$ *is convex and* $\mathrm{ri}(\mathrm{epi}(\bar{f})) \subseteq \mathrm{epi}(f)$,
(ii) nearly convex *if* $\mathrm{epi}(f)$ *is nearly convex,*
(iii) closely convex *if* $\mathrm{epi}(\bar{f})$ *is convex (i.e.* $\bar{f}$ *is convex).*

Connections between these kinds of functions arise from the following observations, while to show that there are differences between them we give Example 1 within the next section.

Remark 2. Any almost convex function is also closely convex.

Remark 3. Any nearly convex function has a nearly convex effective domain. Moreover, as its epigraph is nearly convex, the function is also closely convex, according to Lemma 1(ii).

Although cited from the literature, the following auxiliary results are not so widely known, thus we have included them here.

Lemma 2. *([4], [9]) For a convex set* $C \subseteq \mathbb{R}^n$ *and any non-empty set* $X \subseteq \mathbb{R}^n$ *satisfying* $X \subseteq C$ *we have* $\mathrm{ri}(C) \subseteq X$ *if and only if* $\mathrm{ri}(C) = \mathrm{ri}(X)$.

Lemma 3. *([4]) Let* $X \subseteq \mathbb{R}^n$ *be a non-empty nearly convex set. Then* $\mathrm{ri}(X) \neq \emptyset$ *if and only if* $\mathrm{ri}(\mathrm{cl}(X)) \subseteq X$.

Lemma 4. *([4]) For a non-empty nearly convex set* $X \subseteq \mathbb{R}^n$, $\mathrm{ri}(X) \neq \emptyset$ *if and only if* $\mathrm{ri}(X) = \mathrm{ri}(\mathrm{cl}(X))$.

Using the last remark and Lemma 3 we deduce the following statement.

Proposition 1. *If* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *is a nearly convex function satisfying* $\mathrm{ri}(\mathrm{epi}(f)) \neq \emptyset$, *then it is almost convex.*

Remark 4. Each convex function is both nearly convex and almost convex.

The first observation is obvious, while the second can be easily proven. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a convex function. If $f(x) = +\infty$ everywhere then $\mathrm{epi}(f) = \emptyset$, which is closed, so $\bar{f} = f$ and it follows f almost convex. Otherwise, $\mathrm{epi}(f)$ is non-empty and, being convex because of f's convexity, it has a non-empty relative interior (cf. Theorem 6.2 in [16]) so, by Proposition 1, is almost convex.

3 Properties of the almost convex functions

Within this part of our paper we present some properties of the almost convex functions and some examples that underline the differences between this class of functions and the nearly convex functions.

Theorem 1. *([9]) Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *having non-empty domain. The function* f *is almost convex if and only if* $\bar{f}$ *is convex and* $\bar{f}(x) = f(x)\ \forall x \in \mathrm{ri}(\mathrm{dom}(\bar{f}))$.

Proof. "$\Rightarrow$" When f is almost convex, $\bar{f}$ is convex. As $\mathrm{dom}(f) \neq \emptyset$, we have $\mathrm{dom}(\bar{f}) \neq \emptyset$. It is known (cf. [16]) that

$$\mathrm{ri}(\mathrm{epi}(\bar{f})) = \big\{(x, r) : \bar{f}(x) < r, x \in \mathrm{ri}(\mathrm{dom}(\bar{f}))\big\} \tag{1}$$

so, as the definition of the almost convexity includes $\mathrm{ri}(\mathrm{epi}(\bar{f})) \subseteq \mathrm{epi}(f)$, it follows that for any $x \in \mathrm{ri}(\mathrm{dom}\bar{f}))$ and $\varepsilon > 0$ one has $(x, \bar{f}(x) + \varepsilon) \in \mathrm{epi}(f)$. Thus $\bar{f}(x) \geq f(x)$ $\forall x \in \mathrm{ri}(\mathrm{dom}(\bar{f}))$ and the definition of $\bar{f}$ yields the coincidence of f and $\bar{f}$ over $\mathrm{ri}(\mathrm{dom}(\bar{f}))$.

"$\Leftarrow$" We have $\bar{f}$ convex and $\bar{f}(x) = f(x)$ $\forall x \in \mathrm{ri}(\mathrm{dom}(\bar{f}))$. Thus $\mathrm{ri}(\mathrm{dom}(\bar{f})) \subseteq \mathrm{dom}(f)$. By Lemma 2 and Remark 1 one gets $\mathrm{ri}(\mathrm{dom}(\bar{f})) \subseteq \mathrm{dom}(f)$ if and only if $\mathrm{ri}(\mathrm{dom}(\bar{f})) = \mathrm{ri}(\mathrm{dom}(f))$, therefore this last equality holds. Using this and (1) it follows $\mathrm{ri}(\mathrm{epi}(\bar{f})) = \big\{(x, r) : f(x) < r, x \in \mathrm{ri}(\mathrm{dom}(f))\big\}$, so $\mathrm{ri}(\mathrm{epi}(\bar{f})) \subseteq \mathrm{epi}(f)$. This and the hypothesis $\bar{f}$ convex yield that f is almost convex. □

Remark 5. From the previous proof we obtain also that if f is almost convex and has a non-empty domain then $\mathrm{ri}(\mathrm{dom}(f)) = \mathrm{ri}(\mathrm{dom}(\bar{f})) \neq \emptyset$. We have also $\mathrm{ri}(\mathrm{epi}(\bar{f})) \subseteq \mathrm{epi}(f)$, from which, by the definition of $\bar{f}$, follows

$$\mathrm{ri}(\mathrm{cl}(\mathrm{epi}(f))) \subseteq \mathrm{epi}(f) \subseteq \mathrm{cl}(\mathrm{epi}(f)).$$

Applying Lemma 2 we get $\mathrm{ri}(\mathrm{epi}(f)) = \mathrm{ri}(\mathrm{cl}(\mathrm{epi}(f))) = \mathrm{ri}(\mathrm{epi}(\bar{f}))$.

In order to avoid confusions between the nearly convex functions and the almost convex functions we give below some examples showing that there is no inclusion between these two classes of functions. Their intersection is not empty, as Remark 4 states that the convex functions are concomitantly almost convex and nearly convex.

Example 1. (i) Let $f : \mathbb{R} \to \mathbb{R}$ be any discontinuous solution of Cauchy's functional equation $f(x+y) = f(x) + f(y)$ $\forall x, y \in \mathbb{R}$. For each of these functions, whose existence is guaranteed in [11], one has

$$f\Big(\frac{x+y}{2}\Big) = \frac{f(x) + f(y)}{2} \ \forall x, y \in \mathbb{R},$$

i.e. these functions are nearly convex. None of these functions is convex because of the absence of continuity. We have that $\mathrm{dom}(f) = \mathbb{R} = \mathrm{ri}(\mathrm{dom}(f))$. Suppose f is almost convex. Then Theorem 1 yields $\bar{f}$ convex and $f(x) = \bar{f}(x)$ $\forall x \in \mathbb{R}$. Thus f is convex, but this is false. Therefore f is nearly convex, but not almost convex.

(ii) Consider the set $X = \big([0,2]\times[0,2]\big)\setminus\big(\{0\}\times]0,1[\big)$ and let $g : \mathbb{R}^2 \to \overline{\mathbb{R}}$, $g = \delta_X$. We have $\mathrm{epi}(g) = X\times[0,+\infty)$, so $\mathrm{epi}(\bar{g}) = \mathrm{cl}(\mathrm{epi}(g)) = [0,2]\times[0,2]\times[0,+\infty)$. As this is a convex set, $\bar{g}$ is a convex function. We also have $\mathrm{ri}(\mathrm{epi}(\bar{g})) =]0,2[\times]0,2[\times]0,+\infty)$, which is clearly contained inside $\mathrm{epi}(g)$. Thus g is almost convex. On the other hand, $\mathrm{dom}(g) = X$ and X is not a nearly convex set, because for any $\alpha \in]0,1[$ we have $\alpha(0,1) + (1-\alpha)(0,0) = (0,\alpha) \notin X$. By Remark 3 it follows that the almost convex function g is not nearly convex.

Using Remark 4 and the facts above we see that there are almost convex and nearly functions which are not convex, i.e. both these classes are larger than the one of convex functions.

The following assertion states an interesting and important property of the almost convex functions that is not applicable for nearly convex functions.

Theorem 2. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ be proper almost convex functions. Then the function $F : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ defined by $F(x,y) = f(x) + g(y)$ is almost convex, too.*

Proof. Consider the linear operator $L : (\mathbb{R}^n \times \mathbb{R}) \times (\mathbb{R}^m \times \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ defined as $L(x, r, y, s) = (x, y, r + s)$. Let us first show that $L(\text{epi}(f) \times \text{epi}(g)) = \text{epi}(F)$.

Taking the pairs $(x, r) \in \text{epi}(f)$ and $(y, s) \in \text{epi}(g)$ we have $f(x) \leq r$ and $g(y) \leq s$, so $F(x, y) = f(x) + g(y) \leq r + s$, i.e. $(x, y, r + s) \in \text{epi}(F)$. Thus $L(\text{epi}(f) \times \text{epi}(g)) \subseteq \text{epi}(F)$.

On the other hand, for $(x, y, t) \in \text{epi}(F)$ one has $F(x, y) = f(x) + g(y) \leq t$, so $f(x)$ and $g(y)$ are finite. It follows $(x, f(x), y, t - f(x)) \in \text{epi}(f) \times \text{epi}(g)$, i.e. $(x, y, t) \in L(\text{epi}(f) \times \text{epi}(g))$ meaning $\text{epi}(F) \subseteq L(\text{epi}(f) \times \text{epi}(g))$.

Therefore $L(\text{epi}(f) \times \text{epi}(g)) = \text{epi}(F)$. We prove that $\text{cl}(\text{epi}(F))$ is convex, which means $\bar{F}$ convex.

Let (x, y, r) and (u, v, s) in $\text{cl}(\text{epi}(F))$. There are two sequences, $(x_k, y_k, r_k)_{k \geq 1}$ and $(u_k, v_k, s_k)_{k \geq 1}$ in $\text{epi}(F)$, the first converging towards (x, y, r) and the second to (u, v, s). Then we also have the sequences of reals $(r_k^1)_{k \geq 1}$, $(r_k^2)_{k \geq 1}$, $(s_k^1)_{k \geq 1}$ and $(s_k^2)_{k \geq 1}$ fulfilling for each $k \geq 1$ the following $r_k^1 + r_k^2 = r_k$, $s_k^1 + s_k^2 = s_k$, $(x_k, r_k^1) \in \text{epi}(f)$, $(y_k, r_k^2) \in \text{epi}(g)$, $(u_k, s_k^1) \in \text{epi}(f)$ and $(v_k, s_k^2) \in \text{epi}(g)$. Let $\lambda \in [0, 1]$. We have, due to the convexity of the lower-semicontinuous hulls of f and g, $(\lambda x_k + (1-\lambda)u_k, \lambda r_k^1 + (1-\lambda)s_k^1) \in \text{cl}(\text{epi}(f)) = \text{epi}(\bar{f})$ and $(\lambda y_k + (1-\lambda)v_k, \lambda r_k^2 + (1-\lambda)s_k^2) \in \text{cl}(\text{epi}(g)) = \text{epi}(\bar{g})$. Further, $(\lambda x_k + (1-\lambda)u_k, \lambda y_k + (1-\lambda)v_k, \lambda r_k + (1-\lambda)s_k) \in L(\text{cl}(\text{epi}(f)) \times \text{cl}(\text{epi}(g))) = L(\text{cl}(\text{epi}(f) \times \text{epi}(g))) \subseteq \text{cl}(L(\text{epi}(f) \times \text{epi}(g)))$ for all $k \geq 1$. Letting k converge towards $+\infty$ we get $(\lambda x + (1-\lambda)u, \lambda y + (1-\lambda)v, \lambda r + (1-\lambda)s) \in \text{cl}(L(\text{epi}(f) \times \text{epi}(g))) = \text{cl}(\text{epi}(F))$. As this happens for any $\lambda \in [0, 1]$ it follows $\text{cl}(\text{epi}(F))$ convex, so $\text{epi}(\bar{F})$ is convex, i.e. $\bar{F}$ is a convex function.

Therefore, in order to obtain that F is almost convex we have to prove only that $\text{ri}(\text{cl}(\text{epi}(F))) \subseteq \text{epi}(F)$. Using some basic properties of the closures and relative interiors and also that f and g are almost convex we have $\text{ri}(\text{cl}(\text{epi}(f) \times \text{epi}(g))) = \text{ri}(\text{cl}(\text{epi}(f)) \times \text{cl}(\text{epi}(g))) = \text{ri}(\text{cl}(\text{epi}(f))) \times \text{ri}(\text{cl}(\text{epi}(g))) \subseteq \text{epi}(f) \times \text{epi}(g)$. Applying the linear operator L to both sides we get $L(\text{ri}(\text{cl}(\text{epi}(f) \times \text{epi}(g)))) \subseteq L(\text{epi}(f) \times \text{epi}(g)) = \text{epi}(F)$. One has $\text{cl}(\text{epi}(f) \times \text{epi}(g)) = \text{cl}(\text{epi}(f)) \times \text{cl}(\text{epi}(g)) = \text{epi}(\bar{f}) \times \text{epi}(\bar{g})$, which is a convex set, so also $L(\text{cl}(\text{epi}(f) \times \text{epi}(g)))$ is convex. As for any linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$ and any convex set $X \subseteq \mathbb{R}^n$ one has $A(\text{ri}(X)) = \text{ri}(A(X))$ (see for instance Theorem 6.6 in [16]), it follows

$$\text{ri}(L(\text{cl}(\text{epi}(f) \times \text{epi}(g)))) = L(\text{ri}(\text{cl}(\text{epi}(f) \times \text{epi}(g)))) \subseteq \text{epi}(F). \tag{2}$$

On the other hand, $\text{epi}(F) = L(\text{epi}(f) \times \text{epi}(g)) \subseteq L(\text{cl}(\text{epi}(f) \times \text{epi}(g))) \subseteq \text{cl}(L(\text{epi}(f) \times \text{epi}(g)))$, so $\text{cl}(L(\text{epi}(f) \times \text{epi}(g))) = \text{cl}(L(\text{cl}(\text{epi}(f) \times \text{epi}(g))))$ and further

$$\text{ri}(\text{cl}(L(\text{epi}(f) \times \text{epi}(g)))) = \text{ri}(\text{cl}(L(\text{cl}(\text{epi}(f) \times \text{epi}(g))))).$$

As for any convex set $X \subseteq \mathbb{R}^n$ $\text{ri}(\text{cl}(X)) = \text{ri}(X)$ (see Theorem 6.3 in [16]), we have $\text{ri}(\text{cl}(L(\text{cl}(\text{epi}(f) \times \text{epi}(g))))) = \text{ri}(L(\text{cl}(\text{epi}\ (f) \times \text{epi}(g))))$, which implies $\text{ri}(\text{cl}(L(\text{epi}(f) \times \text{epi}(g)))) = \text{ri}(L(\text{cl}(\text{epi}(f) \times \text{epi}(g))))$. Using (2) it follows that $\text{ri}(\text{epi}(\bar{F})) = \text{ri}(\text{cl}(\text{epi}(F))) = \text{ri}(L(\text{cl}(\text{epi}(f) \times \text{epi}(g)))) \subseteq \text{epi}(F)$. Because $\bar{F}$ is a convex function it follows by definition that F is almost convex. □

Corollary 1. *Using the previous statement it can be shown that if* $f_i : \mathbb{R}^{n_i} \to \overline{\mathbb{R}}$, $i = 1, ..., k$, *are proper almost convex functions, then* $F : R^{n_1} \times ... \times \mathbb{R}^{n_k} \to \overline{\mathbb{R}}$, $F(x^1, ..., x^k) = \sum_{i=1}^k f_i(x^i)$ *is almost convex, too.*

Next we give an example that shows that the property just proven to hold for almost convex functions does not apply for nearly convex functions.

Example 2. Consider the sets

$$X_1 = \bigcup_{n \geq 1} \left\{ \tfrac{k}{2^n} : 0 \leq k \leq 2^n \right\} \qquad \text{and} \qquad X_2 = \bigcup_{n \geq 1} \left\{ \tfrac{k}{3^n} : 0 \leq k \leq 3^n \right\}.$$

They are both nearly convex, X_1 for $\alpha = 1/2$ and X_2 for $\alpha = 1/3$, for instance. It is easy to notice that δ_{X_1} and δ_{X_2} are nearly convex functions. Taking $F : \mathbb{R}^2 \to \overline{\mathbb{R}}$, $F(x_1, x_2) = \delta_{X_1}(x_1) + \delta_{X_2}(x_2)$, we have $\mathrm{dom}(F) = X_1 \times X_2$, which is not nearly convex, thus F is not a nearly convex function. To show this, we have $(0,0), (1,1) \in \mathrm{dom}(F)$ and assuming $\mathrm{dom}(F)$ nearly convex with the constant $\bar{\alpha} \in]0,1[$, one gets $(\bar{\alpha}, \bar{\alpha}) \in \mathrm{dom}(F)$. This yields $\bar{\alpha} \in X_1 \cap X_2$ and, so, $\bar{\alpha} \in \{0,1\}$, which is false. Therefore F is not nearly convex.

4 Conjugacy and Fenchel duality for almost convex functions

This section is dedicated to the generalization of some well-known results concerning the conjugate of convex functions. We prove that they keep their validity when the functions involved are taken almost convex, too. Moreover, these results are proven to stand also when the functions are nearly convex and their epigraphs have non-empty relative interiors.

First we deal with the conjugate of the precomposition with a linear operator (see, for instance, Theorem 16.3 in [16]).

Theorem 3. *Let $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ be an almost convex function and $A : \mathbb{R}^n \to \mathbb{R}^m$ a linear operator such that there is some $x' \in \mathbb{R}^n$ satisfying $Ax' \in \mathrm{ri}(\mathrm{dom}(f))$. Then for any $p \in \mathbb{R}^m$ one has*

$$(f \circ A)^*(p) = \inf\big\{f^*(q) : A^*q = p\big\},$$

and the infimum is attained.

Proof. We first prove that $(f \circ A)^*(p) = (\bar{f} \circ A)^*(p)$ $\forall p \in \mathbb{R}^n$. By Remark 5 we get $Ax' \in \mathrm{ri}(\mathrm{dom}(\bar{f}))$. Assume first that $\bar{f}$ is not proper. Corollary 7.2.1 in [16] yields $\bar{f}(y) = -\infty$ $\forall y \in \mathrm{dom}(\bar{f})$. As $\mathrm{ri}(\mathrm{dom}(\bar{f})) = \mathrm{ri}(\mathrm{dom}(f))$ and $\bar{f}(y) = f(y)$ $\forall y \in \mathrm{ri}(\mathrm{dom}(\bar{f}))$, one has $\bar{f}(Ax') = f(Ax') = -\infty$. It follows easily $(\bar{f} \circ A)^*(p) = (f \circ A)^*(p) = +\infty$ $\forall p \in \mathbb{R}^n$.

Now take $\bar{f}$ proper. By definition one has $(\bar{f} \circ A)(x) \leq (f \circ A)(x)$ $\forall x \in \mathbb{R}^n$ and, by simple calculations, one gets $(\bar{f} \circ A)^*(p) \geq (f \circ A)^*(p)$ for any $p \in \mathbb{R}^n$. Take some $p \in \mathbb{R}^n$ and denote $\beta := (\bar{f} \circ A)^*(p) \in] - \infty, +\infty]$. Assume $\beta \in \mathbb{R}$. We have $\beta = \sup_{x \in \mathbb{R}^n} \big\{p^T x - \bar{f} \circ A(x)\big\}$. Let $\varepsilon > 0$. Then there is an $\bar{x} \in \mathbb{R}^n$ such that $p^T \bar{x} - \bar{f} \circ A(\bar{x}) \geq \beta - \varepsilon$, so $A\bar{x} \in \mathrm{dom}(\bar{f})$. As $Ax' \in \mathrm{ri}(\mathrm{dom}(\bar{f}))$, we get, because of the linearity of A and of the convexity of $\mathrm{dom}(\bar{f})$, by Theorem 6.1 in [16] that for any $\lambda \in]0,1]$ it holds $A((1-\lambda)\bar{x} + \lambda x') = (1-\lambda)A\bar{x} + \lambda Ax' \in \mathrm{ri}(\mathrm{dom}(\bar{f}))$. Applying Theorem 1 and using the convexity of $\bar{f}$ we have

$$p^T((1-\lambda)\bar{x} + \lambda x') - f(A((1-\lambda)\bar{x} + \lambda x')) = p^T((1-\lambda)\bar{x} + \lambda x')$$
$$-\bar{f}(A((1-\lambda)\bar{x} + \lambda x')) \geq p^T((1-\lambda)\bar{x} + \lambda x') - (1-\lambda)\bar{f} \circ A(\bar{x})$$
$$-\lambda \bar{f} \circ A(x') = p^T \bar{x} - \bar{f} \circ A(\bar{x}) + \lambda\big[p^T(x' - \bar{x}) - (\bar{f} \circ A(x') - \bar{f} \circ A(\bar{x}))\big].$$

As Ax' and $A\bar{x}$ belong to the domain of the proper function $\bar{f}$, there is a $\bar{\lambda} \in]0,1]$ such that $\bar{\lambda}\big[p^T(x' - \bar{x}) - (\bar{f} \circ A(x') - \bar{f} \circ A(\bar{x}))\big] > -\varepsilon$.

The calculations above lead to

$$(f \circ A)^*(p) \geq p^T((1-\bar{\lambda})\bar{x} + \bar{\lambda}x') - (\bar{f} \circ A)((1-\bar{\lambda})\bar{x} + \bar{\lambda}x') \geq \beta - 2\varepsilon.$$

As ε is an arbitrarily chosen positive number, let it converge towards 0. We get $(f \circ A)^*(p) \geq \beta = (\bar{f} \circ A)^*(p)$. Because the opposite inequality is always true, we get $(f \circ A)^*(p) = (\bar{f} \circ A)^*(p)$.

Consider now the last possible situation, $\beta = +\infty$. Then for any $k \geq 1$ there is an $x_k \in \mathbb{R}^n$ such that $p^T x_k - \bar{f}(Ax_k) \geq k+1$. Thus $Ax_k \in \operatorname{dom}(\bar{f})$ and by Theorem 6.1 in [16] we have, for any $\lambda \in]0,1]$,

$$p^T((1-\lambda)x_k + \lambda x') - f \circ A((1-\lambda)x_k + \lambda x') = p^T((1-\lambda)x_k + \lambda x')$$
$$-\bar{f} \circ A((1-\lambda)x_k + \lambda x') \geq p^T((1-\lambda)x_k + \lambda x') - (1-\lambda)\bar{f} \circ A(x_k)$$
$$-\lambda \bar{f} \circ A(x') = p^T x_k - \bar{f} \circ A(x_k) + \lambda\big[p^T(x' - x_k) - (\bar{f} \circ A(x') - \bar{f} \circ A(x_k))\big].$$

Like before, there is some $\bar{\lambda} \in]0,1[$ such that

$$\bar{\lambda}\big[p^T(x' - x_k) - (\bar{f} \circ A(x') - \bar{f} \circ A(x_k))\big] \geq -1.$$

Denoting $z_k := (1-\bar{\lambda})x_k + \bar{\lambda}x'$ we have $z_k \in \mathbb{R}^n$ and $p^T z_k - f \circ A(z_k) \geq k+1-1 = k$. As $k \geq 1$ is arbitrarily chosen, one gets

$$(f \circ A)^*(p) = \sup_{x \in \mathbb{R}^n} \big\{p^T x - f \circ A(x)\big\} = +\infty,$$

so $(f \circ A)^*(p) = +\infty = (\bar{f} \circ A)^*(p)$. Therefore, as $p \in \mathbb{R}^n$ has been arbitrary chosen, we get

$$(f \circ A)^*(p) = (\bar{f} \circ A)^*(p) \ \forall p \in \mathbb{R}^n. \tag{3}$$

By Theorem 16.3 in [16] we have, as $\bar{f}$ is convex and $Ax' \in \operatorname{ri}(\operatorname{dom}(\bar{f})) = \operatorname{ri}(\operatorname{dom}(f))$,

$$(\bar{f} \circ A)^*(p) = \inf\big\{(\bar{f})^*(q) : A^* q = p\big\},$$

with the infimum attained at some $\bar{q}$. But $f^* = (\bar{f})^*$ (cf. [16]), so the relation above gives

$$(\bar{f} \circ A)^*(p) = \inf\big\{f^*(q) : A^* q = p\big\}.$$

Finally, by (3), this turns into

$$(f \circ A)^*(p) = \inf\big\{f^*(q) : A^* q = p\big\},$$

and the infimum is attained at $\bar{q}$. □

The following statement follows from Theorem 3 immediately by Proposition 1.

Corollary 2. *If $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ is a nearly convex function satisfying $\operatorname{ri}(\operatorname{epi}(f)) \neq \emptyset$ and $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator such that there is some $x' \in \mathbb{R}^n$ fulfilling $Ax' \in \operatorname{ri}(\operatorname{dom}(f))$, then for any $p \in \mathbb{R}^m$ one has*

$$(f \circ A)^*(p) = \inf\big\{f^*(q) : A^* q = p\big\},$$

and the infimum is attained.

Now we give a statement concerning the conjugate of the sum of finitely many proper functions, which is actually the infimal convolution of their conjugates also when the functions are almost convex functions, provided that the relative interiors of their domains have a point in common.

Theorem 4. *Let $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, ..., k$, be proper and almost convex functions whose domains satisfy $\cap_{i=1}^k \operatorname{ri}(\operatorname{dom}(f_i)) \neq \emptyset$. Then for any $p \in \mathbb{R}^n$ we have*

$$(f_1 + ... + f_k)^*(p) = \inf\left\{\sum_{i=1}^k f_i^*(p^i) : \sum_{i=1}^k p^i = p\right\}, \tag{4}$$

with the infimum attained.

Proof. Let $F : \mathbb{R}^n \times ... \times \mathbb{R}^n \to \overline{\mathbb{R}}$, $F(x^1, ..., x^k) = \sum_{i=1}^k f_i(x^i)$. By Corollary 1 we know that F is almost convex. We have $\mathrm{dom}(F) = \mathrm{dom}(f_1) \times ... \times \mathrm{dom}(f_k)$, so $\mathrm{ri}(\mathrm{dom}(F)) = \mathrm{ri}(\mathrm{dom}(f_1)) \times ... \times \mathrm{ri}(\mathrm{dom}(f_k))$. Consider also the linear operator $A : \mathbb{R}^n \to \mathbb{R}^n \times ... \times \mathbb{R}^n$, $Ax = (\underbrace{x, ..., x}_{k})$. The existence of the element $x' \in \cap_{i=1}^k \mathrm{ri}(\mathrm{dom}(f_i))$ gives $(x', ..., x') \in \mathrm{ri}(\mathrm{dom}(F))$, so $Ax' \in \mathrm{ri}(\mathrm{dom}(F))$. By Theorem 3 we have for any $p \in \mathbb{R}^n$

$$(F \circ A)^*(p) = \inf \{F^*(q) : A^*q = p\}, \tag{5}$$

with the infimum attained at some $\bar{q} \in \mathbb{R}^n \times ...\mathbb{R}^n$. For the conjugates above we have for any $p \in \mathbb{R}^n$

$$(F \circ A)^*(p) = \sup_{x \in \mathbb{R}^n} \left\{ p^T x - \sum_{i=1}^k f_i(x) \right\} = \left(\sum_{i=1}^k f_i \right)^* (p)$$

and for every $q = (p^1, ..., p^k) \in \mathbb{R}^n \times ... \times \mathbb{R}^n$,

$$F^*(q) = \sup_{\substack{x^i \in \mathbb{R}^n,\\ i=1,...,k}} \left\{ \sum_{i=1}^k (p^i)^T x^i - \sum_{i=1}^k f_i(x^i) \right\} = \sum_{i=1}^k f_i^*(p^i),$$

so, as $A^*q = \sum_{i=1}^k p^i$, (5) delivers (4). □

In [16] the formula (4) is given assuming the functions f_i, $i = 1, ..., k$, proper and convex and the intersection of the relative interiors of their domains non-empty. We have proven above that it holds even under the much weaker than convexity assumption of almost convexity imposed on these functions, when the other two conditions, i.e. their properness and the non-emptiness of the intersection of the relative interiors of their domains, stand. As the following assertion states, the formula is valid under the assumption regarding the domains also when the functions are proper and nearly convex, provided that the relative interiors of their epigraphs are non-empty.

Corollary 3. *If $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, ..., k$, are proper nearly convex functions whose epigraphs have non-empty relative interiors and with their domains satisfying $\cap_{i=1}^k \mathrm{ri}(\mathrm{dom}(f_i)) \neq \emptyset$, then for any $p \in \mathbb{R}^n$ one has*

$$(f_1 + ... + f_k)^*(p) = \inf \left\{ \sum_{i=1}^k f_i^*(p_i) : \sum_{i=1}^k p_i = p \right\},$$

with the infimum attained.

Next we show that another important conjugacy formula remains true when imposing almost convexity (or nearly convexity) instead of convexity for the functions in discussion.

Theorem 5. *Given two proper almost convex functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ and the linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$ for which is guaranteed the existence of some $x' \in \mathrm{ri}(\mathrm{dom}(f))$ satisfying $Ax' \in \mathrm{ri}(\mathrm{dom}(g))$, one has for all $p \in \mathbb{R}^n$*

$$(f + g \circ A)^*(p) = \inf \{f^*(p - A^*q) + g^*(q) : q \in \mathbb{R}^m\}, \tag{6}$$

with the infimum attained.

Proof. Consider the linear operator $B : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ defined by $Bz = (z, Az)$ and the function $F : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$, $F(x, y) = f(x) + g(y)$. By Theorem 2, F is an almost convex function and we have $\mathrm{dom}(F) = \mathrm{dom}(f) \times \mathrm{dom}(g)$. From the hypothesis one gets

$$Bx' = (x', Ax') \in \mathrm{ri}(\mathrm{dom}(f)) \times \mathrm{ri}(\mathrm{dom}(g)) = \mathrm{ri}(\mathrm{dom}(f) \times \mathrm{dom}(g)) = \mathrm{ri}(\mathrm{dom}(F)),$$

thus $Bx' \in \mathrm{ri}(\mathrm{dom}(F))$. Theorem 3 is applicable, leading to

$$(F \circ B)^*(p) = \inf \big\{ F^*(q_1, q_2) : B^*(q_1, q_2) = p, (q_1, q_2) \in \mathbb{R}^n \times \mathbb{R}^m \big\}$$

where the infimum is attained for any $p \in \mathbb{R}^n$. Since for each $p \in \mathbb{R}^n$

$$\begin{aligned}(F \circ B)^*(p) &= \sup_{x \in \mathbb{R}^n} \big\{ p^T x - F(B(x)) \big\} = \sup_{x \in \mathbb{R}^n} \big\{ p^T x - F(x, Ax) \big\} \\ &= \sup_{x \in \mathbb{R}^n} \big\{ p^T x - f(x) - g(Ax) \big\} = (f + g \circ A)^*(p),\end{aligned}$$

$F^*(q_1, q_2) = f^*(q_1) + g^*(q_2)\ \forall (q_1, q_2) \in \mathbb{R}^n \times \mathbb{R}^m$ and

$$B^*(q_1, q_2) = q_1 + A^* q_2\ \forall (q_1, q_2) \in \mathbb{R}^n \times \mathbb{R}^m,$$

the relation above becomes

$$\begin{aligned}(f + g \circ A)^*(p) &= \inf \big\{ f^*(q_1) + g^*(q_2) : q_1 + A^* q_2 = p \big\} \\ &= \inf \big\{ f^*(p - A^* q_2) + g^*(q_2) : q_2 \in \mathbb{R}^m \big\},\end{aligned}$$

where the infimum is attained for any $p \in \mathbb{R}^n$, i.e. (6) stands. □

Corollary 4. *Let the proper nearly convex functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ satisfying $\mathrm{ri}(\mathrm{epi}(f)) \neq \emptyset$ and $\mathrm{ri}(\mathrm{epi}(g)) \neq \emptyset$ and the linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$ such that there is some $x' \in \mathrm{ri}(\mathrm{dom}(f))$ fulfilling $Ax' \in \mathrm{ri}(\mathrm{dom}(g))$. Then (6) holds for any $p \in \mathbb{R}^n$ and the infimum is attained.*

Remark 6. Assuming the hypotheses of Theorem 5, respectively, Corollary 4 fulfilled, one has from (6) that the following so-called subdifferential sum formula holds (for the proof see, for example, [5])

$$\partial(f + g \circ A)(x) = \partial f(x) + A^* \partial g(Ax)\ \forall x \in \mathrm{dom}(f) \cap A^{-1}(\mathrm{dom}(g)).$$

After weakening the conditions under which some widely-used formulae concerning the conjugation of functions take place, we switch to duality where we prove important results which hold even when replacing the convexity with almost convexity or nearly convexity.

The following duality statements are immediate consequences of Theorem 5, respectively Corollary 4, by taking $p = 0$ in (6).

Theorem 6. *Given two proper almost convex functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ and the linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$ for which is guaranteed the existence of some $x' \in \mathrm{ri}(\mathrm{dom}(f))$ satisfying $Ax' \in \mathrm{ri}(\mathrm{dom}(g))$, one has*

$$\inf_{x \in \mathbb{R}^n} \big[f(x) + g(Ax) \big] = -(f + g \circ A)^*(0) = \sup_{q \in \mathbb{R}^m} \big\{ -f^*(A^* q) - g^*(-q) \big\}, \tag{7}$$

with the supremum in the right-hand side attained.

Remark 7. This statement generalizes Corollary 31.2.1 in [16] as we take the functions f and g almost convex instead of convex and, moreover, we remove the closedness assumption required in the mentioned book. It is easy to notice that when f and g are convex there is no need to consider them moreover closed in order to obtain the formula (7).

Remark 8. Theorem 6 states actually the so-called strong duality between the primal problem (P_A) $\inf_{x\in\mathbb{R}^n}\big[f(x)+g(Ax)\big]$ and its Fenchel dual (D_A) $\sup_{q\in\mathbb{R}^m}\big\{-f^*(A^*q)-g^*(-q)\big\}$.

Using Proposition 1 and Theorem 6 we rediscover the assertion in Theorem 4.1 in [3], which follows.

Corollary 5. *Let $f:\mathbb{R}^n\to\overline{\mathbb{R}}$ and $g:\mathbb{R}^m\to\overline{\mathbb{R}}$ two proper nearly convex functions whose epigraphs have non-empty relative interiors and consider the linear operator $A:\mathbb{R}^n\to\mathbb{R}^m$. If there is an $x'\in\operatorname{ri}(\operatorname{dom}(f))$ such that $Ax'\in\operatorname{ri}(\operatorname{dom}(g))$, then* (7) *holds and the dual problem (D_A) has a solution.*

In the end we give a generalization of the well-known Fenchel's duality theorem (Theorem 31.1 in [16]). It follows immediately from Theorem 6, for A the identity mapping, thus we skip the proof.

Theorem 7. *Let f and g be proper almost convex functions on $\mathbb{R}^n$ with values in $\overline{\mathbb{R}}$. If $\operatorname{ri}(\operatorname{dom}(f))\cap\operatorname{ri}(\operatorname{dom}(g))\neq\emptyset$, one has*

$$\inf_{x\in\mathbb{R}^n}\big[f(x)+g(x)\big]=\sup_{q\in\mathbb{R}^n}\big\{-f^*(q)-g^*(-q)\big\},$$

with the supremum attained.

When f and g are nearly convex functions we have, as in Theorem 3.1 in [3], the following statement.

Corollary 6. *Let f and g be proper nearly convex functions on $\mathbb{R}^n$ with values in $\overline{\mathbb{R}}$. If $\operatorname{ri}(\operatorname{epi}(f))\neq\emptyset$, $\operatorname{ri}(\operatorname{epi}(g))\neq\emptyset$ and $\operatorname{ri}(\operatorname{dom}(f))\cap\operatorname{ri}(\operatorname{dom}(g))\neq\emptyset$, one has*

$$\inf_{x\in\mathbb{R}^n}\big[f(x)+g(x)\big]=\sup_{q\in\mathbb{R}^n}\big\{-f^*(q)-g^*(-q)\big\},$$

with the supremum attained.

Remark 9. The last two assertions give actually the strong duality between the primal problem (P) $\inf_{x\in\mathbb{R}^n}\big[f(x)+g(x)\big]$ and its Fenchel dual (D) $\sup_{q\in\mathbb{R}^m}\big\{-f^*(q)-g^*(-q)\big\}$. In both cases we have weakened the initial assumptions required in [16] to guarantee strong duality between (P) and (D) by asking the functions f and g to be almost convex, respectively nearly convex, instead of convex.

Remark 10. Let us notice that the relative interior of the epigraph of a proper nearly convex function f with $\operatorname{ri}(\operatorname{dom}(f))\neq\emptyset$ may be empty (see for instance the function in Example 1(i)).

As proven in Example 1 there are almost convex functions which are not convex, so our Theorems 3–7 extend some results in [16]. An example given in [3] shows that also the Corollaries 2–6 generalize indeed the corresponding results from Rockafellar's book [16], as a nearly function whose epigraph has a non-empty interior is not necessarily convex.

5 Conclusions

After recalling the definitions of three generalizations of the convexity, we have shown that there are differences between the classes of almost convex functions and nearly convex functions, both of them being indeed larger than the one of the convex functions. Then we proved that the formulae of some conjugates, namely of the precomposition with a linear operator, of the sum of finitely many functions and of the sum between a function and the precomposition of another one with a linear operator hold even when the convexity assumptions are replaced by almost (or nearly) convexity. The last results we give show that the well-known duality statements due to Fenchel hold when the functions involved are taken only almost convex, respectively nearly convex.

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