On some general Farkas-type results and their applications

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Abstract. In this paper we present first some Farkas-type results for inequality systems with convex and with composed convex functions, respectively, expressed by means of the conjugate functions of the functions involved. It is also shown that Motzkin's theorem of the alternative is actually a special instance of the general result we give. Another application we present is concerning the dual characterization of the containment of a polyhedral set in the reverse of an open polyhedral set.

Key Words. Farkas-type results, composed convex functions, conjugate functions, conjugate duality

1 Introduction

To an optimization problem with a convex objective function and finitely many convex inequality constraints one can attach various dual problems, from which we mention here only the Lagrange and Fenchel dual problems. In [14], using an approach based on perturbations and conjugacy, Boţ and Wanka have constructed a new dual to a primal problem, called Fenchel-Lagrange dual. Like the name already suggests, the new dual is a "combination" of the classical Fenchel and Lagrange dual problems. Regarding

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the new dual problem, they proved that the optimal objective values of the Fenchel and Lagrange dual problems lie between the optimal objective value of the Fenchel-Lagrange dual problem and the optimal objective value of the primal problem. The previous inequalities ensure that in the case of strong duality between the primal problem and the Fenchel-Lagrange dual problem, strong duality holds between the primal problem and its Lagrange and, respectively, Fenchel dual problem, too. We would like to mention that the same results are true also in a more general setting when the convexity assumptions are replaced by generalized convexity ones (for more details the reader can consult [4]).

The Fenchel-Lagrange dual problem has proved to be a very useful dual, since it can be successfully used for more general problems, like the ones which involve DC and composed convex functions (see [7], [10], [11], [15]). More information regarding the Fenchel-Lagrange dual are to be found in [1] and [3–6] and in the references therein.

By means of the weak and strong duality between a convex optimization problem and its Fenchel-Lagrange dual, Boţ and Wanka have presented in [5] some Farkas-type results for inequality systems involving finitely many convex functions. Since the Fenchel-Lagrange dual can be successfully employed also for optimization problems which involve the composition of two convex functions as objective function, the results in [3] and [6] naturally extend the ones from [5] to inequality systems which involve also composed convex functions. Moreover, some results presented in [8] and [12] are rediscovered as special cases of the ones presented in [6].

As an application of the Farkas-type results they proved, Boţ and Wanka gave in [6] some dual characterizations of the containment of a given set in another one, along with the proof of how some classical theorems of the alternative can be rediscovered as a special instance of their results.

Before going further we would like to mention that some of the proofs of the results we present in the following are omitted. For more details the interested reader can consult the literature given at the end of the paper.

The paper is organized as follows. In Section 2 we present some definitions and results needed later within the paper. We give a dual for the ordinary optimization problem with convex inequality constraints and establish the weak and strong duality assertions in the third section. A Farkas-type result is proved using the weak and strong duality assertions already proved. Within the fourth section, using the duality acquired in Section 3 we give a dual for the optimization problem having a composed convex function as objective function, together with the weak and strong duality assertions. Moreover, a Farkas-type result for inequality systems with composed convex functions is presented. In the last section some special instances of the results presented in the previous sections are presented.

2 Notations and preliminaries

The notations we use throughout the paper and some preliminary results, as well as some well-known concepts, are presented in the following. We consider all vectors as column vectors. Any column vector can be transposed to a row vector by an upper index ^{T}. By $x^T y = \sum_{i=1}^n x_i y_i$ is denoted the usual inner product of two vectors $x = (x_1, ..., x_n)^T$ and $y = (y_1, ..., y_n)^T$ in the real space $\mathbb{R}^n$. By "$\leqq$" we denote the partial order introduced by the non-negative orthant $\mathbb{R}^n_+$, defined by

$$x \leqq y \Leftrightarrow x_i \leq y_i, \forall i = 1, ..., n.$$

For an arbitrary set $X \subseteq \mathbb{R}^n$, by $\operatorname{ri}(X)$ is denoted the *relative interior* of the set X. By $v(P)$ we denote the optimal objective value of an optimization problem (P).

For any set $X \subseteq \mathbb{R}^n$ we consider the *indicator function* of X

$$\delta_X : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}, \quad \delta_X(x) = \begin{cases} 0, & x \in X, \\ +\infty, & \text{otherwise}, \end{cases}$$

and the *support function* of X

$$\sigma_X : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}, \quad \sigma_X(u) = \sup_{x \in X} u^T x.$$

For $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, we denote by $\operatorname{dom}(f) = \big\{x \in \mathbb{R}^n : f(x) < +\infty\big\}$ its *effective domain*. The function f is called *proper* if its effective domain is a nonempty set and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$.

When X is a nonempty subset of $\mathbb{R}^n$ we define for the function f the *conjugate relative to the set* X by

$$f^*_X : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad f^*_X(p) = \sup_{x \in X} \big\{p^T x - f(x)\big\}.$$

One can notice that for $X = \mathbb{R}^n$ the conjugate relative to the set X is actually the *(Fenchel-Moreau) conjugate function* of f denoted by f^*. Even more, it can be easily proved that

$$f_X^* = (f + \delta_X)^* \text{ and } \delta_X^* = \sigma_X.$$

Definition 2.1 Let the function $f : \mathbb{R}^k \to \overline{\mathbb{R}}$ be given. The function is called $\mathbb{R}^k_+$*-increasing* if for all $x = (x_1, ..., x_k)^T$ and $y = (y_1, ..., y_k)^T$ in $\mathbb{R}^k$ such that $x_i \leq y_i$, $i = 1, ..., k$, it holds $f(x) \leq f(y)$.

The following statement closes this preliminary section.

Theorem 2.1 (cf. [13]) Let $f_1, ..., f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions. If the set $\bigcap_{i=1}^m \operatorname{ri}(\operatorname{dom}(f_i))$ is nonempty, then

$$\left(\sum_{i=1}^m f_i \right)^* (p) = \inf \left\{ \sum_{i=1}^m f_i^*(p_i) : p = \sum_{i=1}^m p_i \right\},$$

and for each $p \in \mathbb{R}^n$ the infimum is attained.

3 The Fenchel-Lagrange dual problem of a convex optimization problem

The optimization problem we treat within this section is

$$(P) \qquad \inf_{\substack{x \in X, \\ g(x) \leqq 0}} f(x),$$

where X is a nonempty convex set in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$, $g = \big(g_1, ..., g_m\big)^T$, are such that f is proper and convex, while $g_1, ..., g_m$ are convex. Moreover, we assume that

$$\operatorname{dom}(f) \cap X \cap g^{-1}\big(- \mathbb{R}^m_+\big) \neq \emptyset, \tag{1}$$

where $g^{-1}\big(- \mathbb{R}^m_+\big) = \{x \in \mathbb{R}^n : g(x) \leqq 0\}$. It is easy to see that the above condition implies that $v(P) < +\infty$.

To (P) one can attach different dual problems, like for example the classical Wolfe, Mond-Weir, Lagrange and Fenchel duals, but also the Fenchel-Lagrange dual problem. Before going further, we would like to mention that the last one has been obtained by Wanka and Boţ in [14] using an approach based on conjugacy and perturbation. As the name already suggests, it is a "combination" of the Fenchel and Lagrange dual problems. For more details regarding this kind of dual, the reader can consult [1–6].

The Fenchel-Lagrange dual problem of (P) is defined as

$$(D) \qquad \sup_{\substack{p\in\mathbb{R}^n,\\ q\geqq 0}} \Big\{ - f^*(p) - (q^T g)^*_X(-p) \Big\}.$$

The proof of the following results can be found in [14].

Theorem 3.1 Between the primal problem (P) and the dual problem (D) weak duality always hold, namely $v(P) \geq v(D)$.

Since the inequality in the previous theorem can be strict (see [14] for an example), in order to guarantee the equality of the optimal objective values of the optimization problems (P) and (D) and the existence of a solution for the dual we impose the following constraint qualification

$$(CQ) \qquad \exists x' \in \operatorname{ri}\big(\operatorname{dom}(f)\big) \cap \operatorname{ri}(X) \text{ s.t. } \begin{cases} g_i(x') \leq 0, & i \in L, \\ g_i(x') < 0. & i \in N, \end{cases}$$

where $L := \big\{i \in \{1, ..., m\} : g_i \text{ is an affine function}\big\}$ and $N := \{1, ..., m\} \setminus L$.

Theorem 3.2 (cf. [4, 14]) If the condition (CQ) is fulfilled, then strong duality holds between (P) and (D), i.e., $v(P) = v(D)$ and the dual problem (D) has an optimal solution.

As a last remark, we would like to mention that the proofs of the previous theorems are given in [4] under some more general assumptions than the ones imposed above, namely in the context in which the convexity assumptions are replaced by nearly convexity assumptions.

The next theorem states a first Farkas-type result.

Theorem 3.3 Suppose that (CQ) holds. Then the following assertions are equivalent:

(i) $x \in X,\ g(x) \leqq 0 \Rightarrow f(x) \geq 0$;

(ii) there exist $p \in \mathbb{R}^n$ and $q \geqq 0$ such that

$$f^*(p) + (q^T g)^*_X(-p) \leq 0. \tag{2}$$

Proof. "$(i) \Rightarrow (ii)$" The statement (i) implies $v(P) \geq 0$ and, since the assumptions of Theorem 3.2 are fulfilled, strong duality holds, i.e. $v(D) = v(P) \geq 0$ and the dual (D) has an optimal solution. Thus there exist $p \in \mathbb{R}^n$ and $q \geqq 0$ fulfilling (2).

"$(ii) \Rightarrow (i)$" As we can find some $p \in \mathbb{R}^n$ and $q \geqq 0$ fulfilling (2), it follows right away that

$$v(D) \geq -f^*(p) - (q^T g)^*_X(-p) \geq 0.$$

Weak duality between (P) and (D) always holds and thus we obtain $v(P) \geq 0$, i.e. (i) is true. □

The theorem above can be reformulated as a theorem of the alternative.

Theorem 3.4 Assume that the hypothesis of Theorem 3.3 is fulfilled. Then either the inequality system

$$(I) \quad x \in X, g(x) \leqq 0, f(x) < 0$$

has a solution or the system

$$(II) \quad \begin{array}{l} f^*(p) + (q^T g)^*_X(-p) \leq 0, \\ p \in \mathbb{R}^n, q \geqq 0 \end{array}$$

has a solution, but never both.

4 A Farkas-type result for inequality systems with composed convex functions

Let X be a nonempty convex set in $\mathbb{R}^n$. Consider the functions $f : \mathbb{R}^k \to \overline{\mathbb{R}}$, $F : \mathbb{R}^n \to \mathbb{R}^k$, $F = \big(F_1, ..., F_k\big)^T$ and $g : \mathbb{R}^n \to \mathbb{R}^m$, $g = \big(g_1, ..., g_m\big)^T$ such that f is proper, $\mathbb{R}^k_+$-increasing and convex, while $F_1, ..., F_k$ and $g_1, ..., g_m$ are convex. Moreover, assume that

$$F^{-1}(\mathrm{dom}(f)) \cap X \cap g^{-1}(-\mathbb{R}^m_+) \neq \emptyset, \tag{3}$$

where $F^{-1}\big(\mathrm{dom}(f)\big) = \{x \in \mathbb{R}^n : F(x) \in \mathrm{dom}(f)\}$. The optimization problem we treat within this section is

$$(P_c) \qquad \inf_{\substack{x\in X,\\ g(x)\leqq 0}} f\big(F(x)\big).$$

One can see that the function $f \circ F$ is actually a convex function and thus the problem (P_c) is nothing but a convex optimization problem with convex objective function and finitely many convex inequality constraints. To the problem (P_c) we attach the following optimization problem

$$(P_c') \qquad \inf_{\substack{x\in X,\, y\in\mathbb{R}^k,\\ g(x)\leqq 0,\\ F(x)-y\leqq 0}} f(y),$$

Since the equality $v(P_c) = v(P_c')$ holds (for a proof see [3]), any dual problem of (P_c') turns out to be automatically a dual problem of (P_c). That is why, in order to provide a dual problem to (P_c), we actually provide a dual problem to (P_c').

Thus we consider the functions

$$\widetilde{f} : \mathbb{R}^n \times \mathbb{R}^k \to \overline{\mathbb{R}}, \quad \widetilde{f}(x, y) = f(y)$$

and

$$\widetilde{g} : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m \times \mathbb{R}^k, \quad \widetilde{g}(x, y) = \big(g(x), F(x) - y\big)^T$$

and we equivalently rewrite the problem (P_c') as

$$(P_c') \qquad \inf_{\substack{(x,y)\in X\times\mathbb{R}^k,\\ \widetilde{g}(x,y)\leqq 0}} \widetilde{f}(x, y).$$

Since $\widetilde{f}$ and $\widetilde{g}$ are convex functions (this is a trivial consequence of the convexity of the functions f, F and g), the problem (P_c') is a convex optimization problem. To (P_c') we attach first the Lagrange dual problem with $(\alpha, \beta) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$, $\alpha = (\alpha_1, ..., \alpha_m)^T$ and $\beta = (\beta_1, ..., \beta_k)^T$ as dual variables

$$(D_c) \qquad \sup_{(\alpha,\beta)\geqq 0} \inf_{(x,y)\in X\times\mathbb{R}^k} \left\{ \widetilde{f}(x,y) + (\alpha,\beta)^T \widetilde{g}(x,y) \right\},$$

which is nothing else than

$$(D_c) \qquad \sup_{\substack{\alpha\geqq 0,\\ \beta\geqq 0}} \inf_{\substack{x\in X,\\ y\in\mathbb{R}^k}} \left\{ f(y) + \alpha^T g(x) + \beta^T \big(F(x) - y\big) \right\}.$$

Regarding the inner infimum concerning $(x, y) \in X \times \mathbb{R}^k$, by using the definition of the conjugate relative to a set, we have

$$\begin{aligned}
&\inf_{\substack{x\in X,\\ y\in\mathbb{R}^k}} \left\{ f(y) + \alpha^T g(x) + \beta^T \big(F(x) - y\big) \right\} \\
=\ & \inf_{x\in X} \left\{ \alpha^T g(x) + \beta^T F(x) \right\} + \inf_{y\in\mathbb{R}^k} \left\{ f(y) - \beta^T y \right\} \\
=\ & -\sup_{x\in X} \left\{ -\alpha^T g(x) - \beta^T F(x) \right\} - \sup_{y\in\mathbb{R}^k} \left\{ \beta^T y - f(y) \right\} \\
=\ & -\left(\alpha^T g + \beta^T F \right)^*_X (0) - f^*(\beta).
\end{aligned}$$

Since X is a nonempty convex set we have for all $(\alpha, \beta) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$

$$\operatorname{ri}\big(\operatorname{dom}(\beta^T F)\big) \cap \operatorname{ri}\big(\operatorname{dom}(\alpha^T g + \delta_X)\big) = \mathbb{R}^n \cap \operatorname{ri}(X) = \operatorname{ri}(X) \neq \emptyset$$

and thus, by Theorem 2.1,

$$\begin{aligned}
\left(\alpha^T g + \beta^T F \right)^*_X (0) &= \left(\beta^T F + \alpha^T g + \delta_X \right)^* (0) \\
&= \inf_{p\in\mathbb{R}^n} \left\{ \big(\beta^T F\big)^*(p) + \big(\alpha^T g + \delta_X(x)\big)^*(-p) \right\} \\
&= \inf_{p\in\mathbb{R}^n} \left\{ \big(\beta^T F\big)^*(p) + \big(\alpha^T g\big)^*_X(-p) \right\},
\end{aligned}$$

and the infimum is attained at some $p \in \mathbb{R}^n$.

The latter allow us to reformulate the dual problem (D_c) in the following way

$$(D_c) \qquad \sup_{\substack{p\in\mathbb{R}^n,\\ \alpha\geqq 0,\beta\geqq 0}} \Big\{ -f^*(\beta) - \big(\beta^T F\big)^*(p) - \big(\alpha^T g\big)^*_X(-p)\Big\}.$$

Before going further, we would like to mention that the same dual problem can be obtained employing the theory presented in the previous section for the primal problem

$$(P_c') \qquad \inf_{\substack{(x,y)\in X\times\mathbb{R}^k,\\ \widetilde{g}(x,y)\leqq 0}} \widetilde{f}(x,y).$$

Since the conditions imposed at the beginning of the previous section are fulfilled (relation (3) allow us to prove that (1) is fulfilled for $\widetilde{f}$ and $\widetilde{G}$), and the conjugate of the functions $\widetilde{f}$ and $(\alpha,\beta)^T\widetilde{G}$ can be easily calculated, a detailed verification is left to the reader.

It is well-known that the optimal objective value of the problem (P_c') is always greater than or equal to the optimal objective value of its dual, i.e. $v(P_c') \geq v(D_c)$. As the equality $v(P_c) = v(P_c')$ holds, the problem (D_c) is a dual of (P_c), too, and the following assertion arises easily (see also [3]).

Theorem 4.1 Between the primal problem (P_c) and the dual problem (D_c) weak duality always hold, namely $v(P_c) \geq v(D_c)$.

In order to have equality between the optimal objective values of the problems (P_c) and (D_c), we consider the constraint qualification

$$(CQ_c) \quad \exists x' \in \operatorname{ri}(X) \text{ such that } \begin{cases} F(x') \in \operatorname{ri}\big(\operatorname{dom}(f)\big) - \operatorname{int}\big(\mathbb{R}^k_+\big), & \\ g_i(x') \leq 0, & i \in L,\\ g_i(x') < 0, & i \in N. \end{cases}$$

Theorem 4.2 If (CQ_c) is fulfilled, then between (P_c) and (D_c) strong duality holds, i.e. $v(P_c) = v(D_c)$ and the dual problem has an optimal solution.

Theorem 4.1 and Theorem 4.2 are the backbone in the proof of the next result (see [3]).

Theorem 4.3 Suppose that (CQ_c) holds. Then the following assertions are equivalent:

(i) $x \in X,\ g(x) \leqq 0 \Rightarrow f\big(F(x)\big) \geq 0$;

(ii) there exist $p \in \mathbb{R}^n$, $\alpha \geqq 0$ and $\beta \geqq 0$ such that

$$f^*(\beta) + \big(\beta^T F\big)^*(p) + \big(\alpha^T g\big)_X^*(-p) \leq 0. \tag{4}$$

The theorem of the alternative which follows is an immediate consequence of the theorem above.

Theorem 4.4 Assume that the hypothesis of Theorem 4.3 is fulfilled. Then either the inequality system

$$(I) \quad x \in X, g(x) \leqq 0, f\big(F(x)\big) < 0$$

has a solution or the system

$$\begin{array}{ll}(II) & f^*(\beta) + \big(\beta^T F\big)^*(p) + \big(\alpha^T g\big)_X^*(-p) \leq 0, \\ & p \in \mathbb{R}^n, \alpha \geqq 0, \beta \geqq 0\end{array}$$

has a solution, but never both.

As a remark, we would like to mention that Theorem 3.3 and Theorem 3.4 can be easily derived as special cases of the previous results if we consider $k = 1$ and the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$ for all $x \in \mathbb{R}$ (see also [3]).

5 Applications

In the following we give two applications of Theorem 4.3 and of Theorem 4.4, respectively. More precisely, we prove first that the containment of a polyhedral set in a reverse open polyhedral set is actually a special case of Theorem 4.3. Then we rediscover Mozkin's theorem of the alternative as special instance of Theorem 4.4. Since the function

$$f : \mathbb{R}^k \to \mathbb{R}, \quad f(x) = \max\{x_1, ..., x_k\}, \quad x = (x_1, ..., x_k)^T \in \mathbb{R}^k$$

and its conjugate are used within the proof of both results, we recall the formula of the latter (see [9])

$$f^* : \mathbb{R}^k \to \overline{\mathbb{R}}, \quad f^*(\beta) = \begin{cases} 0, & \beta \geqq 0, \sum\limits_{j=1}^{k} \beta_j = 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

We give now a dual characterization of the containment of a polyhedral set in a reverse open polyhedral set (see also [6]).

Theorem 5.1 Let $A \in \mathbb{R}^{k \times n}$, $B \in \mathbb{R}^{m \times n}$, $a \in \mathbb{R}^k$, $b \in \mathbb{R}^m$ and the sets $\mathcal{A} := \{x \in \mathbb{R}^n : Ax > a\}$ and $\mathcal{B} := \{x \in \mathbb{R}^n : Bx \leqq b\}$ be such that $\mathcal{B} \neq \emptyset$. Then the following statements are equivalent

(i) $\mathcal{B} \subseteq \mathbb{R}^n \setminus \mathcal{A}$;

(ii) there exists $\beta \in \mathbb{R}^k_+ \setminus \{0\}$, $\alpha \in \mathbb{R}^m_+$ such that $B^T \alpha = A^T \beta$ and $b^T \alpha \leq a^T \beta$.

Proof. In order to prove that this result arises as a consequence of Theorem 4.3 we consider $X = \mathbb{R}^n$ and the functions

$$g : \mathbb{R}^n \to \mathbb{R}^m, \quad g(x) = Bx - b$$

and

$$F : \mathbb{R}^n \to \mathbb{R}^k, \quad F(x) = (a_1 - A_1^T x, ..., a_k - A_k^T x)^T,$$

where by A_i^T, $i = 1, ..., k$, we have denoted the i-th row of the matrix A and $a = (a_1, ..., a_k)^T$. It is not hard to see that the statement (i) can be equivalently written as

$$x \in \mathbb{R}^n, g(x) \leqq 0 \Rightarrow f(F(x)) \geq 0. \tag{5}$$

The constraint qualification (CQ) is fulfilled (since the set $\mathcal{B}$ is nonempty) and we can apply Theorem 4.3. Thus we have $\mathcal{B} \subseteq \mathbb{R}^n \setminus \mathcal{A}$ if and only if there exist $p \in \mathbb{R}^n$, $\alpha \geqq 0$ and $\beta \geqq 0$ such that

$$f^*(\beta) + \left(\beta^T F\right)^*(p) + \left(\alpha^T g\right)^*(-p) \leq 0.$$

Because of the special form of the function f^* relation (5) holds if and only if there exist $p \in \mathbb{R}^n$, $\alpha \geqq 0$ and $\beta \geqq 0$, $\sum_{i=1}^k \beta_i = 1$ such that

$$\left(\beta^T F\right)^*(p) + \left(\alpha^T g\right)^*(-p) \leq 0.$$

Further we have $\left(\beta^T F\right)^*(p) = -\beta^T a$ if $p^T = -\sum_{i=1}^k \beta_i A_i^T = \beta^T A$ and it is equal to $+\infty$ otherwise, and that $\left(\alpha^T g\right)^*(-p) = \alpha^T b$ if $-p^T = \alpha^T B$ and it is equal to $+\infty$ otherwise. Therefore we have $\mathcal{B} \subseteq \mathbb{R}^n \setminus \mathcal{A}$ if and only if

$$-\beta^T a + \alpha^T b \leq 0 \quad \text{ and } \quad -\alpha^T B + \beta^T A = 0,$$

and, since $\beta \neq 0$, the conclusion follows. $\square$

In the following we prove that Motzkin's theorem of the alternative is actually a special instance of Theorem 4.4 (see also [6]).

Theorem 5.2 Let $A \in \mathbb{R}^{k\times n}$, $C \in \mathbb{R}^{s\times n}$ and $D \in \mathbb{R}^{t\times n}$ be given matrices with $A \neq 0$. Then either the inequality system

(i) $Ax > 0$, $Cx \geqq 0$, $Dx = 0$
has a solution $x \in \mathbb{R}^n$ or the system

(ii) $A^T y_1 + C^T y_2 + D^T y_3 = 0$, $y_1 \geqq 0$, $y_1 \neq 0$, $y_2 \geqq 0$
has a solution $y_1 \in \mathbb{R}^k$, $y_2 \in \mathbb{R}^s$, $y_3 \in \mathbb{R}^t$, but never both.

Proof. The above theorem arises as a consequence of Theorem 4.4 for $X = \mathbb{R}^n$ and

$$g : \mathbb{R}^n \to \mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^t, g(x) = (-Cx, Dx, -Dx)^T$$

and

$$F : \mathbb{R}^n \to \mathbb{R}^k, F(x) = (-A_1^T x, ..., -A_k^T x)^T,$$

where by A_i^T, $i = 1, ..., k$, is denoted the i-th row of the matrix A. Since the constraint qualification (CQ) is fulfilled (for $x' = 0$), Theorem 4.4 allows us to affirm that either the system (i) has a solution or

$$f^*(\beta) + \left(\beta^T F\right)^*(p) + \left(\alpha^T g\right)^*(-p) \leq 0, p \in \mathbb{R}^n, \alpha \geqq 0, \beta \geqq 0, \tag{6}$$

has a solution, but never both. Our aim is to prove that the statement (ii) of the theorem is equivalent to (6). Following a reasoning similar to the one presented within the proof of the previous theorem, it can be proved that the previous inequality holds if and only if there exists $\alpha \geqq 0$ and $\beta \geqq 0$, $\sum_{i=1}^k \beta_i = 1$, such that

$$\left(\alpha^T g\right)^* \left(\beta^T A\right) \leq 0.$$

Since $\alpha^T g$ is linear, it is binding to have

$$\beta^T A = \alpha_1^T(-C) + \alpha_2^T D + \alpha_3^T(-D),$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^s_+ \times \mathbb{R}^t_+ \times \mathbb{R}^t_+$. Obviously the previous equality can be rewritten as

$$\beta^T A + \alpha_1^T C + (\alpha_3 - \alpha_2)^T D = 0.$$

Take $y_1 = \beta \in \mathbb{R}^k_+$, $y_2 = \alpha_1 \in \mathbb{R}^s_+$ and $y_3 = \alpha_3 - \alpha_2 \in \mathbb{R}^t$. Since $y_1 = \beta \neq 0$ the proof is complete. □

Let us mention that in [6] other set containment characterizations as well as other theorems of the alternative have been derived from general Farkas-type results.

6 Conclusions

In this paper we present some Farkas-type results for inequality systems with convex and composed convex functions. We also give a dual characterization of the containment of a polyhedral set in a reverse open polyhedral set. Moreover, we rediscover the classical theorem of the alternative of Motzkin as a special instance of the general results we present.

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