A new regularity condition for subdifferential calculus and Fenchel duality in infinite dimensional spaces. Applications for maximal monotone operators

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Abstract. In this survey we present some of our recent results concerning regularity conditions for subdifferential calculus and Fenchel duality in infinite dimensional spaces. As an application we deliver the maximal monotonicity of the operator $A^* \circ T \circ A$, where A is a linear continuous mapping between two reflexive Banach spaces and T is a maximal monotone operator, under the weakest constraint qualification known so far in this framework. As a special case follows the weakest condition that guarantees the maximal monotonicity of the sum of two maximal monotone operators on a reflexive Banach space.

Keywords. Conjugate functions, Fenchel duality, subdifferentials, regularity conditions, maximal monotone operators

1 Introduction

One of the most fruitful challenges in convex analysis and optimization is to provide weaker hypotheses that yield certain important results than the already existing ones. In a given framework the differences are usually made by the so-called regularity conditions which ensure the fulfillment of some statements, without being implied by them, though.

In this survey paper we focus on some of our recent results regarding regularity conditions for subdifferential calculus and Fenchel duality in infinite dimensional

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spaces. The literature on these topics is very rich and there are many sufficient conditions considered for such problems, let us mention here just [9, 15, 17, 22]. We also refer to [4] and the references therein for more on this subject.

The regularity conditions we give belong to the class of closedness conditions and they are weaker than all the so-called generalized interior-point regularity conditions known in the literature. Working in locally convex vector spaces, we give such weak regularity conditions that guarantee the so-called subdifferential sum formula and Fenchel duality, providing moreover examples which prove that they are indeed weaker than the other mentioned conditions.

Especially in the recent years, strong connections between convex analysis and monotone operators were discovered and studied. The cornerstone of the recent advances in connecting these fields is the rediscovery in [6, 12, 13] of the function introduced by Fitzpatrick (cf. [7]), which received his name. This function allows the characterization of maximal monotone operators by using the tools of convex analysis. We refer to [14, 18] for more on monotone operators.

Finding a weaker sufficient condition under which the sum of two maximal monotone operators on a reflexive Banach space is maximal monotone has been an older challenge for many mathematicians, the problem having more than four decades. Since its rediscovery, the Fitzpatrick function played an important role in dealing with this problem, let us mention here just the works [1, 2, 19, 22], the regularity conditions obtained in this way being weaker than the older ones. A problem closely related to this concerns the maximal monotonicity of $A^* \circ T \circ A$, where T is a maximal monotone operator and A is a linear continuous mapping. We give the weakest regularity conditions known so far for both these problems by using the results presented earlier in the paper.

Some words about the organization of the paper follow. The next section presents some preliminary notions and results in convex analysis, then the main part where the new regularity conditions are introduced in order to ensure the subdifferential sum formula and, respectively, Fenchel duality follows. In the fourth part we apply these results in maximal monotonicity, providing weaker constraint qualifications for the maximal monotonicity of $A^* \circ T \circ A$, introduced before, respectively of the sum of two maximal monotone operators. A short conclusive section follows.

2 Preliminary notions and results

2.1 Some elements of convex analysis

We consider two nontrivial locally convex vector spaces X and Y together with their continuous dual spaces X^* and, respectively, Y^*, endowed with the weak* topologies $w(X^*, X)$ and, respectively, $w(Y^*, Y)$. By $\langle x^*, x\rangle$ we denote the value of the linear continuous functional $x^* \in X^*$ at $x \in X$.

When C is a subset of X, its *indicator* function $\delta_C : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is defined as $\delta_C(x) = 0$, if $x \in C$ and $\delta_C(x) = +\infty$, otherwise, and we denote by $\mathrm{int}(C)$, $\mathrm{cl}(C)$ and $\mathrm{cone}(C)$ its *interior*, its *closure* in the corresponding topology, respectively its *conical hull*. The *core* of C is defined by $\mathrm{core}(C) = \{c \in C : \forall x \in X\ \exists \varepsilon > 0 : c + \lambda x \in C\ \forall \lambda \in [-\varepsilon, \varepsilon]\}$. We call *intrinsic core* of C its core relative to its affine hull $\mathrm{aff}(C)$ and we write it $\mathrm{icr}(C)$. For a convex subset $C \subseteq X$ we denote by ${}^{ic}C$, the *intrinsic relative algebraic interior* of C. One has $x \in {}^{ic}C$ if and only if $\cup_{\lambda > 0} \lambda(C - x)$ is a closed linear subspace of X. Let us remind moreover that a set $C \subseteq X$ is said to be *closed regarding the subspace* $Z \subseteq X$ if $C \cap Z = \mathrm{cl}(C) \cap Z$ (cf. [2]).

The *identity* function on X is $\mathrm{id}_X : X \to X$, $\mathrm{id}_X(x) = x\ \forall x \in X$, while the *first projection* is $\mathrm{pr}_1 : X \times Y \to X$, $\mathrm{pr}_1(x, y) = x\ \forall (x, y) \in X \times Y$. Denote also $\Delta_X = \{(x, x) : x \in X\}$.

Given a funcion $f : X \to \overline{\mathbb{R}}$ we consider the following notions

- *domain*: $\mathrm{dom}(f) = \{x \in X : f(x) < +\infty\}$;
- f is *proper*: $f(x) > -\infty\ \forall x \in X$ and $\mathrm{dom}(f) \neq \emptyset$;
- *epigraph*: $\mathrm{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$;
- *lower semicontinuous envelope of* f: the function $\mathrm{cl}(f) : X \to \overline{\mathbb{R}}$ defined by $\mathrm{epi}(\mathrm{cl}(f)) = \mathrm{cl}(\mathrm{epi}(f))$;
- *conjugate function*: $f^* : X^* \to \overline{\mathbb{R}}$, $f^*(x^*) = \sup\big\{\langle x^*, x\rangle - f(x) : x \in X\big\}$;
- *subdifferential of f at x (when $f(x) \in \mathbb{R}$)*:

$$\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle\ \forall y \in X\}.$$

We have also the so-called Fenchel-Young inequality

$$f^*(x^*) + f(x) \geq \langle x^*, x\rangle\ \forall x \in X\ x^* \in X^*.$$

When $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$, we define $f \times g : X \times Y \to \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ through $f \times g(x, y) = (f(x), g(y))$, $(x, y) \in X \times Y$.

When $f_i : X \to \overline{\mathbb{R}}$, $i = 1, ..., m$, are proper functions, we have their *infimal convolution* defined by $f_1 \Box \cdots \Box f_m : X \to \overline{\mathbb{R}}$, $f_1 \Box \cdots \Box f_m(x) = \inf\{\sum_{i=1}^m f_i(x_i) : \sum_{i=1}^m x_i = x\}$. We say that $f_1 \Box \cdots \Box f_m$ is *exact at* $x \in X$ if there exist some $x_i \in X, i = 1, ..., m$, such that $f_1 \Box \cdots \Box f_m(x) = f_1(x_1) + ... + f_m(x_m)$. We call $f_1 \Box \cdots \Box f_m$ *exact* if it is exact at every $x \in X$.

Given a linear continuous mapping $A : X \to Y$ one has its *range* $\mathrm{Im}(A) = \{Ax : x \in X\}$ and its *adjoint* $A^* : Y^* \to X^*$ given by $\langle A^* y^*, x\rangle = \langle y^*, Ax\rangle$ for any $(x, y^*) \in X \times Y^*$. If $f : X \to \overline{\mathbb{R}}$ is a proper function, the *infimal function of f through A* is $Af : Y \to \overline{\mathbb{R}}$, $Af(y) = \inf\big\{f(x) : x \in X, Ax = y\big\}$, $y \in Y$. If $U \subseteq Y$, denote also $A^{-1}(U) = \{x \in X : Ax \in U\}$.

2.2 Fenchel-Moreau-type statements

After introducing the most important notions needed in this paper, we present now some known results on which the ones in the following sections are based.

Theorem 1. (see [4, 20]) *Let $f, g : X \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions such that* $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$. *Then the following relation holds*

$$\mathrm{epi}((f+g)^*) = \mathrm{cl}(\mathrm{epi}(f^* \Box g^*)) = \mathrm{cl}(\mathrm{epi}(f^*) + \mathrm{epi}(g^*)). \tag{1}$$

Remark 1. One may notice that the second equality in (1) remains true even considering the closure in the product topology of $(X^*, \tau) \times \mathbb{R}$, where τ is an arbitrary compatible topology on X^*.

Proposition 1. (cf. [4]) *Let $f, g : X \to \overline{\mathbb{R}}$ be proper functions such that* $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$. *Then the following statements are equivalent*

(i) $\mathrm{epi}((f+g)^*) = \mathrm{epi}(f^*) + \mathrm{epi}(g^*)$,

(ii) $(f+g)^* = f^* \Box g^*$ *and* $f^* \Box g^*$ *is exact.*

Theorem 2. (cf. [8]) *Let a linear continuous mapping $A : X \to Y$ and a proper, convex and lower semicontinuous function $g : Y \to \overline{\mathbb{R}}$ be such that $g \circ A$ is proper on X. Then*

$$\mathrm{epi}((g \circ A)^*) = \mathrm{cl}(\mathrm{epi}(A^* g^*)), \tag{2}$$

where the closure is taken in the product topology of $(X^*, \tau) \times \mathbb{R}$, for every locally convex topology τ on X^* giving X as dual.

Remark 2. Significant choices for τ are the weak* topology $w(X^*, X)$ on X^* or the norm topology of X^* in case X is a reflexive Banach space.

Theorem 3. (cf. [4]) *Let τ a compatible topology on X^*, $A : X \to Y$ a linear continuous mapping and $g : Y \to \overline{\mathbb{R}}$ a proper function. Then*

$$\mathrm{cl}(\mathrm{epi}(A^* g^*)) = \mathrm{cl}(A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*))), \tag{3}$$

where the closure is taken in the product topology of $(X^*, \tau) \times \mathbb{R}$.

As noticed in [4], taking in (2) and (3) the closure in the product topology of $(X^*, w(X^*, X)) \times \mathbb{R}$ we obtain the following equality

$$\mathrm{epi}((g \circ A)^*) = \mathrm{cl}(\mathrm{epi}(A^* g^*)) = \mathrm{cl}(A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*))). \tag{4}$$

Considering $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ proper, convex and lower semicontinuous functions such that $A(\mathrm{dom}(f)) \cap \mathrm{dom}(g) \neq \emptyset$, we have, by Theorems 1 and 3,

$$\mathrm{epi}((f+g\circ A)^*) = \mathrm{cl}(\mathrm{epi}(f^*)+\mathrm{epi}((g\circ A)^*)) = \mathrm{cl}(\mathrm{epi}(f^*)+\mathrm{cl}(A^*\times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*)))),$$

which is nothing else than

$$\mathrm{epi}((f+g\circ A)^*) = \mathrm{cl}(\mathrm{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*))).$$

Inspired by the last relation we introduce the following regularity condition (cf. [4])

(RC_A) $\mathrm{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*))$ is closed in the product topology of $(X^*, w(X^*, X)) \times \mathbb{R}$.

Remark 3. One can note that the regularity condition (RC_A) is equivalent to

$$\mathrm{epi}((f+g\circ A)^*) = \mathrm{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*)).$$

3 New regularity conditions for conjugate duality

3.1 The subdifferential sum formula

We are ready to state now one of the main results in this paper, namely that the subdifferential sum formula holds under the regularity condition introduced above. Its proof can be found in [4].

Theorem 4. *Let $A : X \to Y$ be a linear continuous mapping, $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ proper, convex and lower semicontinuous functions such that $A(\mathrm{dom}(f)) \cap \mathrm{dom}(g) \neq \emptyset$. Then*

(i) (RC_A) is fulfilled if and only if $\forall x^* \in X^*$,

$$(f+g\circ A)^*(x^*) = \inf\{f^*(x^* - A^*y^*) + g^*(y^*) : y^* \in Y^*\}$$

and the infimum is attained.

(ii) If (RC_A) is fulfilled, then $\forall x \in \mathrm{dom}(f) \cap A^{-1}(\mathrm{dom}(g))$,

$$\partial(f+g\circ A)(x) = \partial f(x) + A^*\partial g(Ax).$$

In case $X = Y$ and $A = \mathrm{id}_X$, $A^* \times \mathrm{id}_{\mathbb{R}}$ becomes the identity mapping on $X^* \times \mathbb{R}$ and the regularity condition (RC_A) can be rewritten in the following way

(RC) $\mathrm{epi}(f^*) + \mathrm{epi}(g^*)$ is closed in the product topology of $(X^*, w(X^*, X)) \times \mathbb{R}$.

Theorem 5. (see [4]) *Let the proper convex lower semicontinuous functions* $f, g : X \to \overline{\mathbb{R}}$ *fulfilling* $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$. *Then*

(i) (RC) *is fulfilled if and only if* $\forall x^* \in X^*$,

$$(f+g)^*(x^*) = \inf\{f^*(x^* - y^*) + g^*(y^*) : y^* \in Y^*\} \tag{5}$$

and the infimum is attained.

(ii) If (RC) *is fulfilled, then* $\forall x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$,

$$\partial(f+g)(x) = \partial f(x) + \partial g(x). \tag{6}$$

Remark 4. The statement "(5) $\Rightarrow$ (6)" has been given for the first time in [10], while the implication "$(RC) \Rightarrow$ (6)" has been obtained first in [5].

Consider now the following generalized interior-point regularity conditions:

(i) $\exists x' \in \mathrm{dom}(f)$ such that $Ax' \in \mathrm{int}(\mathrm{dom}(g))$,

(ii) $0 \in \mathrm{core}(\mathrm{dom}(g) - A(\mathrm{dom}(f)))$ (cf. [15]),

(iii) $0 \in {}^{ic}(\mathrm{dom}(g) - A(\mathrm{dom}(f)))$ (cf. [17]),

(iv) $0 \in \mathrm{icr}(\mathrm{dom}(g) - A(\mathrm{dom}(f)))$ and $\mathrm{aff}(\mathrm{dom}(g) - A(\mathrm{dom}(f)))$ is a closed subspace (cf. [9]).

Remark 5. The following relation holds between these regularity conditions

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Leftrightarrow (iv) \Rightarrow (RC_A).$$

The implications between the four above introduced constraint qualifications were investigated in [9]. According to [17], (iii), which is actually the well-known Attouch-Brézis regularity condition, yields

$$(f + g \circ A)^*(x^*) = \inf\{f^*(x^* - A^*y^*) + g^*(y^*) : y^* \in Y^*\} \ \forall x^* \in X^*,$$

with the infimum attained.

By Theorem 4(i) this is equivalent to (RC_A), therefore (iii) implies (RC_A). Next we give an example to show that (RC_A) is indeed weaker than the mentioned interior-point regularity conditions.

Example 1. Let $X = Y = \mathbb{R}$, $A = \mathrm{id}_{\mathbb{R}}$, $f(x) = \frac{1}{2}x^2$, if $x \geq 0$, and $f(x) = +\infty$, otherwise, and $g = \delta_{(-\infty,0]}$. As $\mathbb{R}_+[\mathrm{dom}(g) - A(\mathrm{dom}(f))] = \mathbb{R}_+$, (iii) is not fulfilled, so neither are $(i), (ii)$ and (iv). On the other hand, $\mathrm{epi}(f^*) + \mathrm{epi}(g^*) = \mathbb{R} \times \mathbb{R}_+$, which is closed, so (RC_A) is fulfilled.

Taking in Theorem 4 $f \equiv 0$, one gets the following result.

Theorem 6. *Assume that $g \circ A$ is proper. Then*

(a) $A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*))$ *is closed in the product topology of* $(X^*, w(X^*, X)) \times \mathbb{R}$ *if and only if* $\forall x^* \in X^*$,

$$(g \circ A)^*(x^*) = \inf\{g^*(y^*) : A^*y^* = x^*\}$$

and the infimum is attained.

(b) If $A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*))$ *is closed in the product topology of* $(X^*, w(X^*, X)) \times \mathbb{R}$, *then* $\forall x \in A^{-1}(\mathrm{dom}(g))$,

$$\partial(g \circ A)(x) = A^* \partial g(Ax).$$

We conclude the subsection by stating another result which generalizes Theorem 6 and will be needed later. Its proof is available in [2].

Proposition 2. *Let X, Y and U be non-trivial locally convex spaces, $A : X \to Y$ a linear continuous mapping and $f : Y \to \overline{\mathbb{R}}$ a proper, convex and lower semicontinuous function such that $f \circ A$ is proper on X. Consider moreover the linear mapping $M : U \to X^*$. Let τ be any locally convex topology on X^* giving X as dual. The following statements are equivalent*

(a) $A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(f^*))$ *is closed regarding the subspace* $\mathrm{Im}(M) \times \mathbb{R}$ *in the product topology of* $(X^*, \tau) \times \mathbb{R}$,

(b) $(f \circ A)^*(Mu) = \inf\left\{f^*(y^*) : A^*y^* = Mu\right\}$ *and the infimum is attained, for all* $u \in U$.

3.2 A regularity condition for Fenchel duality

Given $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ proper, convex and lower semicontinuous functions such that $A(\mathrm{dom}(f)) \cap \mathrm{dom}(g) \neq \emptyset$, we consider the following convex optimization problem

$$(P_A) \qquad \inf_{x \in X} \{f(x) + g(Ax)\}.$$

The Fenchel dual problem to (P_A) is

$$(D_A) \qquad \sup_{y^* \in Y^*} \{-f^*(-A^*y^*) - g^*(y^*)\}.$$

Assuming that (RC_A) is fulfilled, Theorem 4(*i*) (taking $x^* = 0$) guarantees strong duality between (P_A) and (D_A), i.e. the situation when their optimal objective values, denoted by $v(P_A)$ and, respectively, $v(D_A)$, coincide and the dual has an optimal solution.

Actually this condition is too strong, as it is equivalent to the so-called *stable strong duality* between (P_A) and (D_A) (see [3]). That is why we consider another regularity condition, namely (cf. [4])

(FRC_A) $f^*\Box A^*g^*$ is a lower semicontinuous function and $\mathrm{epi}(f^*\Box A^*g^*) \cap (\{0\} \times \mathbb{R}) = (\mathrm{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*))) \cap (\{0\} \times \mathbb{R})$.

Using Proposition 1 and (4) one can prove that (RC_A) implies (FRC_A). The opposite implication does not hold in general, a counter-example will be given later. The Fenchel duality statement follows, its proof being given in [4].

Theorem 7. *If* (FRC_A) *is fulfilled, then* $v(P_A) = v(D_A)$ *and* (D_A) *has an optimal solution.*

Remark 6. Assume that (RC_A) is fulfilled, i.e.

$$\mathrm{epi}((f + g \circ A)^*) = \mathrm{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*)).$$

Thus $\mathrm{epi}(f^*\Box A^*g^*)$ is closed and (see [4])

$$\mathrm{epi}(f^*\Box A^*g^*) = \mathrm{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*))).$$

This means that (FRC_A) is also fulfilled.

In case $X = Y$ and $A = \mathrm{id}_X$, the identity mapping of X, the problems (P_A) and (D_A) turn into

$$(P) \qquad \inf_{x \in X} \{f(x) + g(x)\},$$

and, respectively,

$$(D) \qquad \sup_{y^* \in X^*} \{-f^*(-y^*) - g^*(y^*)\}.$$

In this situation, the regularity condition (FRC_A) becomes

(FRC) $f^*\Box g^*$ is a lower semicontinuous function and $\mathrm{epi}(f^*\Box g^*) \cap (\{0\} \times \mathbb{R}) = (\mathrm{epi}(f^*) + \mathrm{epi}(g^*)) \cap (\{0\} \times \mathbb{R})$, equivalently written as

(FRC) $\qquad f^*\Box g^*$ is a lower semicontinuous function and is exact at 0.

Theorem 8. (see [4]) *If (FRC) is fulfilled, then $v(P) = v(D)$ and (D) has an optimal solution.*

The following situation shows that (FRC) is indeed weaker than (RC), thus (FRC_A) is weaker than (RC_A), too.

Example 2. Let $X = \mathbb{R}^2$, $C = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq 0\}$, $D = \{(x_1, x_2)^T \in \mathbb{R}^2 : 2x_1 + x_2^2 \leq 0\}$, $f = \delta_C$ and $g = \delta_D$.

For every $(x_1^*, x_2^*)^T \in \mathbb{R}^2$, $(f + g)^*(x_1^*, x_2^*) = 0$ and

$$f^*\Box g^*(x_1^*, x_2^*) = \inf_{\substack{v_1^* \geq x_1^* \\ v_2^* = x_2^*}} \begin{cases} \frac{(v_2^*)^2}{v_1^*}, & \text{if } v_1^* > 0, \\ 0, & \text{if } v_1^* = v_2^* = 0, \end{cases} = 0.$$

As $f^*\Box g^*$ is lower semicontinuous on $\mathbb{R}^2$ and exact at $(0, 0)^T$, (FRC) is fulfilled. On the other hand, $f^*\Box g^*$ is not exact at every point of $\mathbb{R}^2$ and, so, the sets $\operatorname{epi}(f^*) + \operatorname{epi}(g^*)$ and $\operatorname{epi}((f + g)^*)$ are not equal. Thus $\operatorname{epi}(f^*) + \operatorname{epi}(g^*)$ cannot be closed and (RC) fails.

4 Applications for maximal monotone operators

4.1 Preliminary notions on monotone operators and Fitzpatrick functions

Consider further X a Banach space equipped with the norm $\|\cdot\|$, while the norm on X^* is $\|\cdot\|_*$.

Definition 1. A mapping (generally multivalued) $T : X \to 2^{X^*}$ is called *monotone operator* provided that for any $x, y \in X$ one has $\langle y^* - x^*, y - x\rangle \geq 0$ whenever $x^* \in T(x)$ and $y^* \in T(y)$.

Definition 2. For any monotone operator $T : X \to 2^{X^*}$ we have

- its *effective domain* $D(T) = \{x \in X : T(x) \neq \emptyset\}$,
- its *range* $R(T) = \cup\{T(x) : x \in X\}$,
- its *graph* $G(T) = \{(x, x^*) : x \in X, x^* \in T(x)\}$.

Definition 3. A monotone operator $T : X \to 2^{X^*}$ is called *maximal* when its graph is not properly included in the graph of any other monotone operator $T' : X \to 2^{X^*}$.

The most prominent example of a maximal monotone operator is (cf. [16]) the subdifferential of a proper convex lower semicontinuous function on X.

The *duality map* $J : X \to 2^{X^*}$ is defined as follows

$$J(x) = \frac{1}{2}\partial\|x\|^2 = \Big\{x^* \in X^* : \|x\|^2 = \|x^*\|^2 = \langle x^*, x\rangle\Big\} \ \forall x \in X.$$

Proposition 3. (cf. [1, 18]) *A monotone operator T on a reflexive Banach space X is maximal if and only if the mapping $T(x+\cdot) + J(\cdot)$ is surjective for all $x \in X$.*

To a monotone operator $T : X \to 2^{X^*}$ one can attach the following so-called *Fitzpatrick function* (cf. [7]) defined as follows

$$\varphi_T : X \times X^* \to \overline{\mathbb{R}},\ \varphi_T(x, x^*) = \sup\big\{\langle y^*, x\rangle + \langle x^*, y\rangle - \langle y^*, y\rangle : y^* \in T(y)\big\}.$$

For any monotone operator T the Fitzpatrick function φ_T is convex and lower semicontinuous as it is the supremum of a family of continuous affine functions.

Proposition 4. (cf. [19]) *Let T be a maximal monotone operator on a reflexive Banach space X. Then for any pair $(x, x^*) \in X \times X^*$ we have*

$$\varphi_T^*(x^*, x) \geq \varphi_T(x, x^*) \geq \langle x^*, x\rangle.$$

Moreover, $\varphi_T^*(x^*, x) = \varphi_T(x, x^*) = \langle x^*, x\rangle$ if and only if $(x, x^*) \in G(T)$.

4.2 Maximal monotonicity for T_A

Further we take X and Y reflexive Banach spaces. Given the maximal monotone operator T on Y and the linear continuous mapping $A : X \to Y$, such that $A\big(\mathrm{pr}_1(\mathrm{dom}(\varphi_T))\big) \neq \emptyset$, consider the operator $T_A : X \to 2^{X^*}$ defined by $T_A(x) = A^* \circ T \circ A(x)$, $x \in X$. It is known that T_A is monotone, but not always maximal monotone. That is why we consider the following constraint qualification (cf. [2]) inspired from Proposition 2

(CQ) $\ A^* \times \mathrm{id}_Y \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(\varphi_T^*))$ is closed regarding the subspace $X^* \times \mathrm{Im}(A) \times \mathbb{R}$.

Theorem 9. *If (CQ) is fulfilled, then T_A is a maximal monotone operator.*

Proof. Let $z \in X$ and $z^* \in X^*$ and consider $f,\, g : X \times X^* \to \overline{\mathbb{R}}$, defined by

$$f(x, x^*) = \inf\{\varphi_T(A(x+z), y^*) - \langle y^*, Az\rangle : A^* y^* = x^* + z^*\}$$

and $\qquad g(x,x^*) = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|_*^2 - \langle z^*, x\rangle,\ (x,x^*) \in X \times X^*.$
As f and g are convex and the latter is continuous, Fenchel's duality theorem (cf. [22]) yields the existence of a pair $(\bar{x}^*, \bar{x}) \in X^* \times X$ such that

$$\begin{aligned}\inf_{(x,x^*)\in X\times X^*}\{f(x,x^*)+g(x,x^*)\} &= \max_{(x^*,x)\in X^*\times X}\{-f^*(x^*,x)-g^*(-x^*,-x)\}\\ &= -f^*(\bar{x}^*,\bar{x})-g^*(-\bar{x}^*,-\bar{x}). \qquad (7)\end{aligned}$$

To calculate the conjugate of f we consider the linear continuous operator $B = A \times \mathrm{id}_{Y^*}$. For any $(w^*, w) \in X^* \times X$ we have

$$\begin{aligned} f^*(w^*,w) &= \sup_{\substack{x\in X,\\ x^*\in X^*}}\{\langle w^*,x\rangle+\langle x^*,w\rangle - \inf_{A^*y^*=x^*+z^*}\{\varphi_T(A(x+z),y^*)-\langle y^*,Az\rangle\}\}\\ &= \sup_{\substack{u\in X,\\ y^*\in Y^*}}\{\langle w^*,u\rangle+\langle y^*,A(w+z)\rangle-(\varphi_T\circ B)(u,y^*)\}-\langle w^*,z\rangle\\ &- \langle z^*,w\rangle = (\varphi_T\circ B)^*(w^*,A(w+z))-\langle w^*,z\rangle-\langle z^*,w\rangle.\end{aligned}$$

The conjugate of g is $g^*(w^*,w) = \frac{1}{2}\|w^*+z^*\|_*^2 + \frac{1}{2}\|w\|^2\ \forall (w^*,w) \in X^* \times X$.

By Proposition 2, (CQ) is equivalent to the fact that $\forall (w^*,w) \in X^* \times X$

$$(\varphi_T\circ B)^*(w^*,Aw) = \min_{(y^*,y)\in Y^*\times Y}\{\varphi_T^*(y^*,y) : B^*(y^*,y) = (w^*,Aw)\}.$$

Proposition 4 yields for all $(x,x^*) \in X\times X^*$ and $y^* \in Y^*$ such that $A^*y^* = x^*+z^*$

$$\varphi_T(A(x+z),y^*)-\langle y^*,Az\rangle \geq \langle y^*,Ax+Az\rangle-\langle y^*,Az\rangle = \langle A^*y^*,x\rangle = \langle x^*,x\rangle+\langle z^*,x\rangle$$

and so $f(x,x^*) \geq \langle x^*,x\rangle + \langle z^*,x\rangle$. Since $g(x,x^*) \geq -\langle x^*,x\rangle - \langle z^*,x\rangle$ we get $f(x,x^*)+g(x,x^*) \geq 0$. Thus $\inf_{(x,x^*)\in X\times X^*}\{f(x,x^*)+g(x,x^*)\} \geq 0$ and taking it into (7) one gets $f^*(\bar{x}^*,\bar{x})+g^*(-\bar{x}^*,-\bar{x}) \leq 0$, i.e.

$$(\varphi_T\circ B)^*(\bar{x}^*,A(\bar{x}+z))-\langle\bar{x}^*,z\rangle-\langle z^*,\bar{x}\rangle+\frac{1}{2}\|-\bar{x}^*+z^*\|_*^2+\frac{1}{2}\|-\bar{x}\|^2 \leq 0. \quad (8)$$

From Proposition 2 we have

$$(\varphi_T\circ B)^*(\bar{x}^*,A(\bar{x}+z)) = \min_{(y^*,y)\in Y^*\times Y}\{\varphi_T^*(y^*,y) : B^*(y^*,y) = (\bar{x}^*,A(\bar{x}+z))\},$$

with the minimum attained at some $(\bar{y}^*,\bar{y}) \in Y^*\times Y$. As the adjoint operator of B is $B^* : Y^*\times Y \to X^*\times Y$, $B^*(y^*,y) = (A^*y^*,y)$, it follows $B^*(\bar{y}^*,\bar{y}) = (A^*\bar{y}^*,\bar{y}) = (\bar{x}^*,A(\bar{x}+z))$. Taking the last two relations into (8) we have

$$\begin{aligned} 0 &\geq \varphi_T^*(\bar{y}^*,\bar{y})-\langle\bar{x}^*,z\rangle-\langle z^*,\bar{x}\rangle+\frac{1}{2}\|\bar{x}^*-z^*\|_*^2+\frac{1}{2}\|\bar{x}\|^2\\ &= \varphi_T^*(\bar{y}^*,A(\bar{x}+z))-\langle\bar{y}^*,Az\rangle-\langle\bar{y}^*,A\bar{x}\rangle+\langle\bar{y}^*,A\bar{x}\rangle-\langle z^*,\bar{x}\rangle+\frac{1}{2}\|\bar{x}\|^2\\ &+ \frac{1}{2}\|A^*\bar{y}^*-z^*\|_*^2 = \big(\varphi_T^*(\bar{y}^*,A(\bar{x}+z))-\langle\bar{y}^*,A(\bar{x}+z)\rangle\big)\\ &+ \big(\langle A^*\bar{y}^*-z^*,\bar{x}\rangle+\frac{1}{2}\|A^*\bar{y}^*-z^*\|_*^2+\frac{1}{2}\|\bar{x}\|^2\big) \geq 0,\end{aligned}$$

where the last inequality results from Proposition 4. Thus the inequalities above must be fulfilled as equalities, so

$$\varphi_T^*(\bar{y}^*, A(\bar{x}+z)) - \langle \bar{y}^*, A(\bar{x}+z)\rangle = 0,$$

i.e., by Proposition 4, $\bar{y}^* \in T \circ A(\bar{x}+z)$ and

$$\langle A^*\bar{y}^* - z^*, \bar{x}\rangle + \frac{1}{2}\|A^*\bar{y}^* - z^*\|_*^2 + \frac{1}{2}\|\bar{x}\|^2 = 0,$$

i.e. $z^* - A^*\bar{y}^* \in \partial \frac{1}{2}\|\cdot\|^2(\bar{x})$. Further one has $A^*\bar{y}^* \in A^* \circ T \circ A(z+\bar{x}) = T_A(z+\bar{x})$ and $z^* - A^*\bar{y}^* \in J(\bar{x})$, so $z^* \in T_A(z+\bar{x}) + J(\bar{x})$. As z and z^* have been arbitrarily chosen, Proposition 3 yields the conclusion. $\square$

Until [2], the weakest constraint qualification known so far in the literature for the maximal monotonicity of T_A was (cf. [21])

(CQ_Z) $\quad \underset{\lambda>0}{\cup} \lambda(D(T) - \mathrm{Im}(A))$ is a closed linear subspace.

Assuming (CQ_Z), one has that $0 \in^{ic} (\mathrm{dom}(\varphi_T) - \mathrm{Im}(B))$. Theorem 2.3.8$(vii)$ in [22] yields

$$(\varphi_T \circ B)^*(w^*, Aw) = \min_{B^*(y^*,y)=(w^*,Aw)} \{\varphi_T^*(y^*, y)\},$$

which is equivalent to (CQ). Therefore $(CQ_Z) \Rightarrow (CQ)$.

As we have remarked in [2], the maximal monotonicity of T_A is valid also when imposing the constraint qualification

$(\widetilde{CQ})$ $\quad A^* \times \mathrm{id}_Y \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(\varphi_T^*))$ is closed,

considered then also in [11]. One may notice that we have $(CQ_Z) \Rightarrow (\widetilde{CQ}) \Rightarrow (CQ)$. The following example shows that these conditions are indeed weaker than (CQ_Z).

Example 3. Let $X = \mathbb{R}$ and $Y = \mathbb{R}\times\mathbb{R}$. Then $X^* = \mathbb{R}$ and $Y^* = \mathbb{R}\times\mathbb{R}$. Let $T : \mathbb{R}\times\mathbb{R} \to 2^{\mathbb{R}\times\mathbb{R}}$ be defined $\forall (x,y) \in \mathbb{R}\times\mathbb{R}$ by

$$T(x,y) = \begin{cases} (-\infty, 0] \times \{0\}, & \text{if } x = 0, y < 0, \\ (-\infty, 0] \times [0, +\infty), & \text{if } x = y = 0, \\ \{x\} \times \{0\}, & \text{if } x > 0, y < 0, \\ \{x\} \times [0, +\infty), & \text{if } x > 0, y = 0, \\ \emptyset, & \text{otherwise,} \end{cases}$$

Considering the proper, convex and lower semicontinuous functions $f, g : \mathbb{R} \to \overline{\mathbb{R}}$, $f(x) = (1/2)x^2 + \delta_{[0,+\infty)}(x)$ and $g = \delta_{(-\infty,0]}$, for any $(x,y) \in \mathbb{R}\times\mathbb{R}$ we have $T(x,y) = \big(\partial f(x), \partial g(y)\big)$, thus T is a maximal monotone operator.

Taking $A : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, $Ax = (x, x)$, one gets, for any $x \in \mathbb{R}$,

$$T_A(x) = A^* \circ T \circ A(x) = \partial f(x) + \partial g(x) = \begin{cases} \mathbb{R}, & \text{if } x = 0, \\ \emptyset, & \text{otherwise,} \end{cases}$$

thus T_A is a maximal monotone operator, too.

The epigraph of the conjugate of φ_T is

$$\operatorname{epi}(\varphi_T^*) = \underset{x \geq 0}{\cup} \Big((-\infty, x] \times [0, +\infty) \times \{x\} \times (-\infty, 0] \times [x^2, +\infty) \Big),$$

so

$$A^* \times \mathrm{id}_{\mathbb{R}\times\mathbb{R}} \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(\varphi_T^*)) = \mathbb{R} \times \underset{x \geq 0}{\cup} \Big(\{x\} \times (-\infty, 0] \times [x^2, +\infty) \Big),$$

which is closed, i.e. $(\widetilde{CQ})$ is valid. Consequently, the constraint qualification (CQ) is satisfied for the chosen T and A.

On the other hand, we have $D(T) - \operatorname{Im}(A) = \{[x, +\infty) \times (-\infty, x] : x \in \mathbb{R}\}$, so

$$\underset{\lambda > 0}{\cup} \lambda(D(T) - \operatorname{Im}(A)) = \{[x, +\infty) \times (-\infty, x] : x \in \mathbb{R}\} = \{(x, y) \in \mathbb{R}^2 : x \geq y\},$$

which is not a subspace, thus (CQ_Z) is violated.

4.3 Maximal monotonicity for the sum of two maximal monotone operators

Take now $Y = X \times X$, $A(x) = (x, x)$ for any $x \in X$ and $T : X \times X \to X^* \times X^*$, $T(x, y) = (T_1(x), T_2(y))$ when $(x, y) \in X \times X$, where T_1 and T_2 are maximal monotone operators on X.

One can prove that T is maximal monotone and $T_A(x) = T_1(x) + T_2(x)\ \forall x \in X$. The condition on the domain of φ_T becomes $\mathrm{pr}_1(\operatorname{dom}(\varphi_{T_1})) \cap \mathrm{pr}_1(\operatorname{dom}(\varphi_{T_2})) \neq \emptyset$. The constraint qualification (CQ) becomes in this case

(CQ^s) $\quad \{(x^* + y^*, x, y, r) : \varphi_{T_1}^*(x^*, x) + \varphi_{T_2}^*(y^*, y) \leq r\}$ is closed regarding the subspace $X^* \times \Delta_X \times \mathbb{R}$.

We are ready to formulate the statement that gives the weakest constraint qualification known so far which guarantees the maximal monotonicity of the sum of two maximal monotone operators on a reflexive Banach space.

Theorem 10. Let T_1 and T_2 be maximal monotone operators on X such that $\mathrm{pr}_1(\operatorname{dom}(\varphi_{T_1})) \cap \mathrm{pr}_1(\operatorname{dom}(\varphi_{T_2})) \neq \emptyset$. If (CQ^s) is fulfilled, then $T_1 + T_2$ is a maximal monotone operator on X.

5 Conclusions

We gathered in this survey some of our recent results concerning regularity conditions in convex analysis, namely we give the weakest such conditions known at the moment that guarantee the validity of the subdifferential sum formula and, respectively, Fenchel duality. These are also applied for maximal monotone operators. Using the Fitzpatrick function, we give the weakest regularity conditions known so far that guarantee the maximal monotonicity of $A^* \circ T \circ A$, where T is a maximal monotone operator on a reflexive Banach space and A a linear continuous mapping, respectively of the sum of two maximal monotone operators on a reflexive Banach space.

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