On the Brézis - Haraux - type approximation in nonreflexive Banach spaces

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Abstract. We give Brézis - Haraux - type approximation results for the range of the monotone operator $S + A^* \circ T \circ A$ when A is a linear continuous mapping between two Banach spaces and S and T are star - monotone operators. These lead to Brézis - Haraux - type approximation results for the range of the subdifferential of the sum between a proper convex lower - semicontinuous function and the precomposition to A of another proper convex lower - semicontinuous function defined on a Banach space. This is proven to hold under a weak sufficient condition. The results in this paper extend some existing ones in the literature.

Keywords. monotone operator, range of an operator, subdifferential, Brézis - Haraux - type approximation

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1 Introduction

The sum of the ranges of two monotone operators defined on Banach spaces is usually larger than the range of their sum. Under some additional conditions these sets are almost equal, i.e. their interiors and closures coincide. Brézis and Haraux wrote the first papers on the subject, namely [8, 9] and since then determining when the sum of the ranges of two monotone operators is almost equal in the sense mentioned above to the range of their sum is known as the *Brézis - Haraux approximation* problem. Given in Hilbert spaces, their original result has been extended and generalized in more general frameworks and also for monotone composite operators like $A^* \circ T \circ A$, where A is a linear continuous mapping and T is a monotone operator. We recall here the works due to Reich

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([20]), Chu ([13, 14]), Pennanen ([17]) and Simons ([24]), who treated the problem in reflexive Banach spaces, respectively of Chbani and Riahi ([12]), of Riahi ([21]) and of the authors of the present paper ([5]), who dealt with the problem in general Banach spaces.

There is a rich literature on the applications of the Brézis - Haraux approximation. From the large class of areas where these results are applied in we mention variational inequality problems ([1]), Hammerstein equations and Neumann problem ([8, 9]), generalized equations of maximal monotone type ([16]), Kruzkov's solutions of the Burger - Carleman's system ([11]), projection algorithms ([2]), Bregman algorithms ([3]), Fenchel - Rockafellar - Moreau generalized duality model ([16, 17]), optimization problems, Hammerstein differential inclusions and complementarity problems ([12]), and the list could go on.

In this paper we work in non - reflexive Banach spaces and we give Brézis - Haraux - type approximation statements for the range of $S + A^* \circ T \circ A$, where S and T are monotone operators and A is a linear continuous mapping. Then we use a weak constraint qualification that ensures a Brézis - Haraux - type approximation assertion for the range of the subdifferential of $f + g \circ A$, where f and g are proper convex lower semicontinuous functions. These results generalize our previous ones in [5, 6] and also those in [12, 21]. An application of our new results concerning the Brézis - Haraux - type approximation is also brought into attention, namely the existence of a solution to an optimization problem and the so - called locally stable total generalized Fenchel duality.

We have structured the paper as follows. The next section contains some necessary preliminaries, definitions and results used later. The third part is the core of the paper and here we give our new Brézis - Haraux - type approximation results for $S + A^* \circ T \circ A$ and then for $\partial(f + g \circ A)$ and some special cases. After the mentioned application a comprehensive list of references closes the paper.

2 Preliminaries

In the following we introduce the context we work within and we recall the necessary notions and results, in order to make the paper self - contained. Consider the locally convex spaces X and Y and their continuous dual spaces X^* and Y^*, endowed with the weak* topologies $w(X^*, X)$ and $w(Y^*, Y)$, respectively. By $\langle x^*, x\rangle$ we denote the value of the linear continuous functional $x^* \in X^*$ at $x \in X$. Having a subset M of X, we denote by $\text{int}(M)$ and $\text{cl}(M)$ its *interior*, respectively its *closure* in the corresponding topology. We call it *closed regarding the subspace* $Z \subseteq X$ if $M \cap Z = \text{cl}(M) \cap Z$ and we have its *indicator* function $\delta_M : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, defined by

$$\delta_M(x) = \begin{cases} 0, & \text{if } x \in M, \\ +\infty, & \text{otherwise.} \end{cases}$$

For a function $f : X \to \overline{\mathbb{R}}$, we have

- the *domain*: $\mathrm{dom}(f) = \{x \in X : f(x) < +\infty\}$,
- the *epigraph*: $\mathrm{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \le r\}$,
- the *conjugate*: $f^* : X^* \to \overline{\mathbb{R}}$ given by $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\}$,
- the *subdifferential* of f at $x \in X$ where $f(x) \in \mathbb{R}$: $\partial f(x) = \{x^* \in X^* : f(y) - f(x) \ge \langle x^*, y - x\rangle\ \forall y \in X\}$,
- f is *proper*: $f(x) > -\infty\ \forall x \in X$ and $\mathrm{dom}(f) \neq \emptyset$.

When $f, g : X \to \overline{\mathbb{R}}$ are proper functions, their *infimal convolution* is defined by

$$f\Box g : X \to \overline{\mathbb{R}},\ f\Box g(a) = \inf\{f(x) + g(a - x) : x \in X\}.$$

For $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$, we define the *product function*

$$(f \times g) : X \times Y \to \overline{\mathbb{R}} \times \overline{\mathbb{R}},\ (f \times g)(x, y) = (f(x), g(y))\ \forall (x, y) \in X \times Y.$$

Given a linear continuous mapping $A : X \to Y$, its *adjoint* is

$$A^* : Y^* \to X^*,\ \langle A^*y^*, x\rangle = \langle y^*, Ax\rangle\ \forall (x, y^*) \in X \times Y^*.$$

For a proper function $f : X \to \overline{\mathbb{R}}$ we recall also the definition of the *infimal function of f through A* as being

$$Af : Y \to \overline{\mathbb{R}},\ Af(y) = \inf\big\{f(x) : x \in X, Ax = y\big\}\ \forall y \in Y.$$

Consider also the *identity* function on X defined by

$$\mathrm{id}_X : X \to X,\ \mathrm{id}_X(x) = x\ \forall x \in X.$$

Let us mention moreover that all around this paper we write min (max) instead of inf (sup) when the infimum (supremum) is attained.

Lemma 1. ([7]) *Let $A : X \to Y$ be a linear continuous mapping and $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ two proper, convex and lower semicontinuous functions such that $A(\mathrm{dom}(f)) \cap \mathrm{dom}(g) \neq \emptyset$. Then*

(a) $\mathrm{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}\big(\mathrm{epi}(g^*)\big)$ *is closed in the product topology of* $\big(X^*, w(x^*, X)\big) \times \mathbb{R}$ *if and only if for any $x^* \in X^*$ it holds*

$$(f + g \circ A)^*(x^*) = \min_{y^* \in Y^*} \big[f^*(x^* - A^*y^*) + g^*(y^*)\big].$$

(b) If $\mathrm{epi}(f^*)+A^*\times \mathrm{id}_{\mathbb{R}}\big(\mathrm{epi}(g^*)\big)$ *is closed in the product topology of* $\big(X^*, w(x^*, X)\big)\times\mathbb{R}$*, then for any* $x\in \mathrm{dom}(f)\cap A^{-1}(\mathrm{dom}(g))$ *one has*

$$\partial(f+g\circ A)(x)=\partial f(x)+A^*\partial g(Ax).$$

The second part of this section is devoted to monotone operators and some of their properties. Until the end of the paper X and Y are considered Banach spaces, unless otherwise specified. We denote by $\|\cdot\|$ the norm on X.

Definition 1. ([23]) A multifunction $T:X\to 2^{X^*}$ is called *monotone operator* provided that for any $x,y\in X$ one has

$$\langle y^*-x^*, y-x\rangle\geq 0 \ \text{ whenever } x^*\in T(x) \text{ and } y^*\in T(y).$$

Definition 2. ([23]) For any monotone operator $T:X\to 2^{X^*}$ we consider

· its *effective domain* $D(T)=\{x\in X: T(x)\neq\emptyset\}$,

· its *range* $R(T)=\cup\{T(x):x\in X\}$,

· its *graph* $G(T)=\{(x,x^*):x\in X, x^*\in T(x)\}$.

Definition 3. ([23]) A monotone operator $T:X\to 2^{X^*}$ is called *maximal* when its graph is not properly included in the graph of any other monotone operator $T':X\to 2^{X^*}$.

Like in [15] let τ_1 be the weakest topology on X^{**} which renders continuous the following real functions

$$\begin{array}{ll} X^{**}\to\mathbb{R}: & x^{**}\mapsto\langle x^{**},x^*\rangle \quad \forall x^*\in X^*,\\ X^{**}\to\mathbb{R}: & x^{**}\mapsto\|x^{**}\|. \end{array}$$

The topology τ in $X^{**}\times X^*$ is the product topology of τ_1 and the strong (norm) topology of X^* ([15]).

Definition 4. ([15]) A monotone operator $T:X\to 2^{X^*}$ is called of *dense type* provided that its *closure* operator $\overline{T}:X^{**}\to 2^{X^*}$,

$$G(\overline{T})=\Big\{(x^{**},x^*)\in X^{**}\times X^*:\exists (x_i,x_i^*)_i\in G(T) \text{ with } (\hat{x}_i,x_i^*)\xrightarrow{\tau}(x^{**},x^*)\Big\}$$

is maximal monotone, where $\hat{y}$ denotes the canonical image of y in X^{**}.

Different to Riahi ([21]) and Chbani and Riahi ([12]), where these operators are called *densely maximal monotone*, respectively *densely monotone*, we decided to name them as Gossez ([15]) did when he introduced them. By Lemme 2.1

in [15], whenever the monotone operator $T : X \to 2^{X^*}$ is of dense type one has $(x^{**}, x^*) \in G(\overline{T})$ if and only if $\langle x^{**} - \hat{y}, x^* - y^* \rangle \geq 0 \ \forall (y, y^*) \in G(T)$.

The monotone operators belonging to the following class are known as *star - monotone operators* ([17]), but also as 3^* - *monotone operators* ([12, 21]) and (BH) *operators* ([13, 14]), being first introduced in [9].

Definition 5. ([14, 12, 17, 21]) A monotone operator $T : X \to 2^{X^*}$ is called *star - monotone* if for all $x^* \in R(T)$ and $x \in D(T)$ there is some $\beta(x^*, x) \in \mathbb{R}$ such that $\inf_{(y,y^*)\in G(T)} \langle x^* - y^*, x - y \rangle \geq \beta(x^*, x)$.

Remark 1. The subdifferential of a proper convex lower semicontinuous function on X is a classical example for all these classes of monotone operators. We refer to [15, 17, 19, 21, 22, 24, 25, 26] for proofs and more on these subjects.

Lemma 2. ([15]) *In reflexive Banach spaces every maximal monotone operator is of dense type and coincides with its closure operator.*

Lemma 3. ([21]) *Given the dense type operator* $T : X \to 2^{X^*}$ *and the nonempty subset* $E \subseteq X^*$ *such that for any* $x^* \in E$ *there is some* $x \in X$ *fulfilling* $\inf_{(y,y^*)\in G(T)} \langle x^* - y^*, x - y \rangle > -\infty$, *one has* $E \subseteq \operatorname{cl}(R(T))$ *and* $\operatorname{int}(E) \subseteq R(\overline{T})$.

Remark 2. Let E be a nonempty subset of X^*. In [14] an operator $T : X \to 2^{X^*}$ for which for every $x^* \in E$ there is some $x \in X$ fulfilling $\inf_{(y,y^*)\in G(T)} \langle x^* - y^*, x - y \rangle > -\infty$ is called E - *operator.*

3 Brézis - Haraux - type approximation of the range of the sum between a monotone operator and a monotone operator composed with a linear mapping

We give in this section the main results concerning the so - called Brézis - Haraux - type approximation (cf. [5, 9, 24]) of the range of the sum between a monotone operator and a monotone operator composed with a linear mapping. These results are then particularized by taking for the monotone operators the subdifferentials of some proper, convex and lower semicontinuous functions. We extend here our earlier results from [5] and let us notice that some results related to ours were obtained by Pennanen in [17], but in reflexive spaces, while we work in general Banach spaces.

Consider two monotone operators $S : X \to 2^{X^*}$ and $T : Y \to 2^{Y^*}$ and a linear continuous mapping $A : X \to Y$. It is known that $S + A^* \circ T \circ A$ is a monotone operator and under certain conditions it is maximal monotone (see [18, 17], for instance). Before presenting our main results let us mention that the corollaries given after each theorem arise easily by choosing S the zero operator defined as $S(x) = \{0\}\ \forall x \in X$, respectively for A the identity mapping on X. We show first that $S+A^*\circ T\circ A$ is star - monotone when S and T are star - monotone.

Theorem 1. *If $S : X \to 2^{X^*}$ and $T : Y \to 2^{Y^*}$ are star - monotone operators and $A : X \to Y$ is a linear continuous mapping, then $S + A^* \circ T \circ A$ is star - monotone, too.*

Proof. If $D(S + A^* \circ T \circ A) = \emptyset$, the conclusion arises trivially. Elsewise take $w^* \in R(S + A^* \circ T \circ A)$, i.e. there are some $w \in X$ and $x^*, z^* \in X^*$ such that $x^* \in S(w)$, $z^* \in A^* \circ T \circ A(w)$ and $w^* = x^* + z^*$. Let $x \in D(S + A^* \circ T \circ A)$. We have

$$\inf_{(y,y^*)\in G(S+A^*\circ T\circ A)} \langle w^* - y^*, x - y\rangle = \inf_{\substack{(y,u^*)\in G(S),\\ (y,v^*)\in G(A^*\circ T\circ A),\\ u^*+v^*=y^*}} \langle x^* + z^* - (u^* + v^*), x - y\rangle$$

$$\geq \inf_{(y,u^*)\in G(S)} \langle x^* - u^*, x - y\rangle + \inf_{(y,v^*)\in G(A^*\circ T\circ A)} \langle z^* - v^*, x - y\rangle. \qquad (1)$$

As $z^* \in A^* \circ T \circ A(w)$, there is some $r^* \in T \circ A(w)$ such that $z^* = A^*r^*$. Clearly, $r^* \in R(T)$. Denote $u = Ax \in D(T)$. When $v^* \in A^* \circ T \circ A(y)$ there is some $s^* \in T \circ A(y)$ such that $v^* = A^*s^*$. We have

$$\begin{aligned}
\inf_{(y,v^*)\in G(A^*\circ T\circ A)} \langle z^* - y^*, x - y\rangle &= \inf_{(y,s^*)\in G(T\circ A)} \langle A^*r^* - A^*s^*, x - y\rangle \\
&= \inf_{(y,s^*)\in G(T\circ A)} \langle r^* - s^*, A(x - y)\rangle \\
&\geq \inf_{(v,s^*)\in G(T)} \langle r^* - s^*, u - v\rangle \geq \beta(r^*, u) \in \mathbb{R},
\end{aligned}$$

since T is star - monotone. As S is also star - monotone, (1) yields that $S + A^* \circ T \circ A$ is star - monotone, too. □

Corollary 1. *If $T : Y \to 2^{Y^*}$ is a star - monotone operator and $A : X \to Y$ is a linear continuous mapping, then $A^* \circ T \circ A$ is star - monotone, too.*

Remark 3. The statement in Corollary 1 rediscovers the result given in Proposition 5 in [5].

Corollary 2. *If $S, T : X \to 2^{X^*}$ are star - monotone operators, then $S + T$ is star - monotone, too.*

Lemma 4. *If $S : X \to 2^{X^*}$ and $T : Y \to 2^{Y^*}$ are star - monotone operators and $A : X \to Y$ is a linear continuous mapping such that $S + A^* \circ T \circ A$ is of dense type, then*

(i) $R(S) + A^*(R(T)) \subseteq \mathrm{cl}(R(S + A^* \circ T \circ A))$, *and*

(ii) $\mathrm{int}(R(S) + A^*(R(T))) \subseteq R(\overline{S + A^* \circ T \circ A})$.

Proof. The operator $S + A^* \circ T \circ A$ being of dense type implies that $D(S + A^* \circ T \circ A) \neq \emptyset$, thus $D(S) \cap D(A^* \circ T \circ A) \neq \emptyset$.

Let $x^* \in R(S) + A^*(R(T))$, thus there are some $x_1^* \in R(S)$, $x_2^* \in R(A^* \circ T \circ A)$ and $z^* \in R(T)$ such that $x^* = x_1^* + x_2^*$ and $x_2^* = A^* z^*$. Taking some $x \in D(S + A^* \circ T \circ A)$ there holds

$$\inf_{(y,y^*) \in G(S + A^* \circ T \circ A)} \langle x^* - y^*, x - y \rangle = \inf_{\substack{(y,u^*) \in G(S), \\ (y,v^*) \in G(A^* \circ T \circ A), \\ u^* + v^* = y^*}} \langle x_1^* + x_2^* - (u^* + v^*), x - y \rangle$$

$$\geq \inf_{(y,u^*) \in G(S)} \langle x_1^* - u^*, x - y \rangle + \inf_{(y,v^*) \in G(A^* \circ T \circ A)} \langle x_2^* - v^*, x - y \rangle > -\infty,$$

as both S and $A^* \circ T \circ A$ are star - monotone. Applying Lemma 3 for $E = R(S) + A^*(R(T))$ and the dense type monotone operator $S + A^* \circ T \circ A$, we obtain (i) and (ii). $\square$

Using this intermediate assertion we give the main result in the paper, the Brézis - Haraux - type approximation of the range of the monotone operator $S + A^* \circ T \circ A$.

Theorem 2. *If $S : X \to 2^{X^*}$ and $T : Y \to 2^{Y^*}$ are star - monotone operators and $A : X \to Y$ is a linear continuous mapping such that $S + A^* \circ T \circ A$ is of dense type, then*

(i) $\mathrm{cl}(R(S) + A^*(R(T))) = \mathrm{cl}(R(S + A^* \circ T \circ A))$, *and*

(ii) $\mathrm{int}(R(S + A^* \circ T \circ A)) \subseteq \mathrm{int}(R(S) + A^*(R(T))) \subseteq \mathrm{int}(R(\overline{S + A^* \circ T \circ A}))$.

Proof. The operator $S + A^* \circ T \circ A$ being of dense type implies that $D(S + A^* \circ T \circ A) \neq \emptyset$. Take some $x^* \in R(S + A^* \circ T \circ A)$. Then there are some $x \in D(S + A^* \circ T \circ A)$, $y^*, z^* \in X^*$ such that $x^* = y^* + z^*$, $y^* \in S(x)$ and $z^* \in A^* \circ T \circ A(x)$. Obviously $z^* \in A^*(R(T))$, thus $x^* = y^* + z^* \in R(S) + A^*(R(T))$. Consequently $R(S + A^* \circ T \circ A) \subseteq R(S) + A^*(R(T))$ and the same inclusion exists also between the closures, respectively the interiors, of these sets. By Lemma 4 we obtain immediately (i) and (ii). $\square$

Corollary 3. *If $T : Y \to 2^{Y^*}$ is star - monotone and $A : X \to Y$ is a linear continuous mapping such that $A^* \circ T \circ A$ is of dense type, then* *(i)* $\mathrm{cl}(A^*(R(T))) = \mathrm{cl}(R(A^* \circ T \circ A))$, *and*

(ii) $\mathrm{int}(R(A^* \circ T \circ A)) \subseteq \mathrm{int}(A^*(R(T))) \subseteq \mathrm{int}(R(\overline{A^* \circ T \circ A}))$.

Corollary 4. *If $S : X \to 2^{X^*}$ and $T : X \to 2^{X^*}$ are star - monotone operators such that $S + T$ is of dense type, then*

(i) $\mathrm{cl}(R(S) + R(T)) = \mathrm{cl}(R(S + T))$, *and*

(ii) $\mathrm{int}(R(S + T)) \subseteq \mathrm{int}(R(S) + (R(T))) \subseteq \mathrm{int}(R(\overline{S + T}))$.

Remark 4. Corollary 3 rediscovers as a special case of Theorem 2 the result given in Theorem 1 in [5], while Corollary 4 does the same with Theorem 3.1 in [12] and Theorem 1 in [21].

When X is moreover reflexive the inequalities in Theorem 2(ii) turn into equalities and we get a more accurate Brézis - Haraux approximation of the range of $S + A^* \circ T \circ A$.

Theorem 3. *If the Banach space X is moreover reflexive, $S : X \to 2^{X^*}$ and $T : Y \to 2^{Y^*}$ are star - monotone operators and $A : X \to Y$ is a linear continuous mapping such that $S + A^* \circ T \circ A$ is maximal monotone, then*

(i) $\mathrm{cl}(R(S) + A^*(R(T))) = \mathrm{cl}(R(S + A^* \circ T \circ A))$, *and*

(ii) $\mathrm{int}(R(S + A^* \circ T \circ A)) = \mathrm{int}(R(S) + A^*(R(T)))$.

Proof. As X is reflexive, Lemma 2 yields that $S + A^* \circ T \circ A$ is maximal monotone of dense type and $\overline{S + A^* \circ T \circ A}$ and $S + A^* \circ T \circ A$ coincide. Theorem 2 delivers the conclusion. □

Corollary 5. *If the Banach space X is moreover reflexive, $T : Y \to 2^{Y^*}$ is a star - monotone operator and $A : X \to Y$ is a linear continuous mapping such that $A^* \circ T \circ A$ is maximal monotone, then*

$$\mathrm{cl}(A^*(R(T))) = \mathrm{cl}(R(A^* \circ T \circ A)) \quad \textit{and} \quad \mathrm{int}(R(A^* \circ T \circ A)) = \mathrm{int}(A^*(R(T))).$$

Corollary 6. *If X is a reflexive Banach space, $S, T : X \to 2^{X^*}$ are star - monotone operators such that $S + T$ is maximal monotone, then*

$$\mathrm{cl}(R(S) + R(T)) = \mathrm{cl}(R(S + T)) \quad \textit{and} \quad \mathrm{int}(R(S) + R(T)) = \mathrm{int}((R(S + T))).$$

Remark 5. Corollary 5 rediscovers as a special case of Theorem 3 the result given in Corollary 1 in [5], while Corollary 6 does the same with Corollary 3.1 in

[12] and Corollary 1 in [21].

Now we turn our attention to the most usual example for many classes of monotone operators, namely the subdifferentials of proper convex lower semicontinuous functions. In [5] we have generalized and corrected the statement given in both Corollary 3.2 in [12] and Corollary 2 in [21], providing a Brézis - Haraux - type approximation of the range of the subdifferential of the precomposition of a proper convex lower semicontinuous function with a linear continuous mapping. We generalize here this result, too. The mentioned statements appear bellow as corollaries of the following theorem. Before stating it we introduce the following constraint qualification (cf. [7])

(CQ) $\quad$ $\mathrm{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}\big(\mathrm{epi}(g^*)\big)$ is closed in the product topology of $\big(X^*, w(x^*, X)\big) \times \mathbb{R}$.

Theorem 4. *Let the proper convex lower semicontinuous functions* $f : X \to \overline{\mathbb{R}}$ *and* $g : Y \to \overline{\mathbb{R}}$, *and the linear continuous mapping* $A : X \to Y$ *such that* $f + g \circ A$ *is proper,* $A(\mathrm{dom}(f)) \cap \mathrm{dom}(g) \neq \emptyset$ *and* (CQ) *is valid. Then one has*

(i) $\mathrm{cl}(R(\partial f) + A^*(R(\partial g))) = \mathrm{cl}(R(\partial(f + g \circ A)))$, *and*

(ii) $\mathrm{int}(R(\partial(f + g \circ A))) \subseteq \mathrm{int}(R(\partial f) + A^*(R(\partial g))) \subseteq \mathrm{int}(D(\partial(f^* \Box A^* g^*)))$.

Proof. As $f + g \circ A$ is proper, convex and lower semicontinuous, by Théoréme 3.1 in [15] we know that $\partial(f + g \circ A)$ is an operator of dense type, while according to Theorem B in [22] (see also [17, 21]) ∂f and ∂g are star - monotone.

By Lemma 1(ii) we know that (CQ) implies $\partial f + A^* \circ \partial g \circ A = \partial(f + g \circ A)$. Therefore $\partial f + A^* \circ \partial g \circ A$ is an operator of dense type, too.
Applying Theorem 2 for $S = \partial f$ and $T = \partial g$ we get

$$\mathrm{cl}(R(\partial f) + A^*(R(\partial g))) = \mathrm{cl}(R(\partial f + A^* \circ \partial g \circ A)) = \mathrm{cl}(R(\partial(f + g \circ A)))$$

and

$$\mathrm{int}(R(\partial f + A^* \circ \partial g \circ A)) \subseteq \mathrm{int}(R(\partial f) + A^*(R(\partial f))) \subseteq \mathrm{int}(R(\overline{\partial f + A^* \circ \partial g \circ A})).$$

The relation above that involves closures yields (i), while the other becomes

$$\mathrm{int}(R(\partial(f + g \circ A))) \subseteq \mathrm{int}(R(\partial f) + A^*(R(\partial g))) \subseteq \mathrm{int}(R(\overline{\partial(f + g \circ A)})). \quad (2)$$

As from Lemma 1(i) one may deduce that under (CQ) the equality $(f + g \circ A)^* = f^* \Box A^* g^*$ holds, by Théoréme 3.1 in [15] we get $R(\overline{\partial(f + g \circ A)}) = D(\partial(f + g \circ A)^*) = D(\partial(f^* \Box\, A^* g^*))$. Putting this into (2) we get (ii). $\Box$

Corollary 7. (see [5]) *Let the proper convex lower semicontinuous function* $g : Y \to \overline{\mathbb{R}}$ *and the linear continuous mapping* $A : X \to Y$ *be such that* $g \circ A$ *is* *proper and assume the constraint qualification*

(CQ_A) $A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*))$ *is closed in the product topology of* $(X^*, w(X^*, X)) \times \mathbb{R}$

valid. Then one has

(i) $\mathrm{cl}(A^*(R(\partial g))) = \mathrm{cl}(R(\partial(g \circ A)))$, *and*

(ii) $\mathrm{int}(R(\partial(g \circ A))) \subseteq \mathrm{int}(A^*(R(\partial g))) \subseteq \mathrm{int}(D(\partial(A^* g^*)))$.

Corollary 8. (see [5, 6]) *Let f and g be two proper convex lower semicontinuous functions on the Banach space X with extended real values such that* $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$. *Assume the constraint qualification*

(CQ^s) $\mathrm{epi}(f^*) + \mathrm{epi}(g^*)$ *is closed in the product topology of* $(X^*, w(X^*, X)) \times \mathbb{R}$

satisfied. Then one has

(i) $\mathrm{cl}(R(\partial f) + R(\partial g)) = \mathrm{cl}(R(\partial(f + g)))$, *and*

(ii) $\mathrm{int}(R(\partial f + \partial g)) \subseteq \mathrm{int}(R(\partial f) + R(\partial g)) \subseteq \mathrm{int}(D(\partial(f^* \Box g^*))) = \mathrm{int}(D(\partial((f + g)^*)))$.

Remark 6. Similar results to the ones in the last statement have been obtained by Riahi in Corollary 2 in [21] and by Chbani and Riahi in Corollary 3.2 in [12], under the constraint qualification

$$(CQ_R) \qquad \bigcup_{t>0} t(\mathrm{dom}(f) - \mathrm{dom}(g)) \text{ is a closed linear subspace of } X.$$

In [21] (CQ_R) is said to imply (i) in Corollary 8 and $\mathrm{int}(R(\partial f) + R(\partial g)) = \mathrm{int}(D(\partial(f^* \Box g^*)))$, while according to [12] it yields (i) in Corollary 8 and $\mathrm{int}(R(\partial f) + R(\partial g)) = \mathrm{int}(D(\partial(f + g)^*))$.

We have proved in [5, 6] that the equalities involving interiors from above are not always true in non - reflexive Banach spaces. We have shown it by using Example 2.21 in [19], originally given by Fitzpatrick.

Remark 7. As proven in Proposition 3.1 in [10] (see also [7]), (CQ_R) implies (CQ^s), but the converse is not true, as shown by Example 3.1 in [10]. Therefore our Corollary 8 extends, by weakening the constraint qualification, and corrects Corollary 3.2 in [12] and Corollary 2 in [21].

4 Application: existence of a solution to an optimization problem

We work within the framework of Theorem 4, i.e. consider the proper convex lower semicontinuous functions $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ and the linear continuous mapping $A : X \to Y$ such that $f + g \circ A$ is proper and $A(\mathrm{dom}(f)) \cap \mathrm{dom}(g) \neq \emptyset$.

Theorem 5. *Assume (CQ) satisfied and moreover that $0 \in \mathrm{int}(R(\partial f) + A^*(R(\partial g)))$. Then there is a neighborhood V of 0 in X^* such that $\forall x^* \in V$ there exists an $\bar{x} \in \mathrm{dom}(f) \cap A^{-1}(\mathrm{dom}(g))$ for which*

$$f(\bar{x}) + g(A(\bar{x})) - \langle x^*, \bar{x} \rangle = \min_{x \in X} [f(x) + g(A(x)) - \langle x^*, x \rangle].$$

Proof. By Theorem 4 we have $\mathrm{int}(R(\partial f) + A^*(R(\partial g))) \subseteq \mathrm{int}(D(\partial(f^* \Box A^* g^*)))$, thus $0 \in \mathrm{int}(D(\partial(f^* \Box A^* g^*)))$, i.e. there is a neighborhood V of 0 in X^* such that $V \subseteq D(\partial(f^* \Box A^* g^*)) = D(\partial((f + g \circ A)^*))$

Fix some $x^* \in V$. The properties of the subdifferential yield that there is an $\bar{x} \in \mathrm{dom}(f) \cap A^{-1}(\mathrm{dom}(g))$ such that $(f + g \circ A)^*(x^*) + (f + g \circ A)^{**}(\bar{x}) = \langle x^*, \bar{x} \rangle$. As $f + g \circ A$ is a proper convex lower semicontinuous function we have $(f + g \circ A)^{**} = f + g \circ A$, thus the equality above becomes

$$f(\bar{x}) + g(A(\bar{x})) - \langle x^*, \bar{x} \rangle = -(f + g \circ A)^*(x^*) = -\max_{x \in X} \{\langle x^*, x \rangle - f(x) - g(A(x))\}.$$

This means actually that the conclusion is valid. $\Box$

Remark 8. Remaining in the hypotheses of Theorem 5, we know (cf. [7]) that (CQ) is equivalent to

$$\inf_{x \in X} [f(x) + g(A(x)) - \langle x^*, x \rangle] = \max_{y^* \in Y^*} \{-f^*(x^* - A^* y^*) - g^*(y^*)\} \ \forall x^* \in X^*.$$

Thus one may notice that under the assumptions of the problem we obtain something that may be called locally stable total Fenchel duality, i.e. the situation where both problems, the primal on the left - hand side and the dual on the right - hand side, have optimal solutions and their values coincide for small enough linear perturbations of the objective function of the primal problem. Let us notice moreover that as $0 \in V$, for $x^* = 0$ we obtain also the Fenchel duality statement, but where moreover the primal problem has a solution, too. Taking also A the identity mapping in X we obtain the classical Fenchel duality statement where both the primal and the dual have solutions.

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