A Brøndsted-Rockafellar Theorem for Diagonal Subdifferential Operators

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Dedicated to Jon Borwein on the occasion of his 60th birthday

Abstract. In this note we give a Brøndsted-Rockafellar Theorem for diagonal subdifferential operators in Banach spaces. To this end we apply an Ekeland-type variational principle for monotone bifunctions.

Key Words. monotone bifunction, Brøndsted-Rockafellar Theorem, Ekeland variational principle, diagonal subdifferential operator, subdifferential

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1 Introduction and preliminaries

Throughout this paper X denotes a real Banach space and X^* its topological dual space endowed with the dual norm. Since there is no danger of confusion, we use $\|\cdot\|$ as notation for the norms of both spaces X and X^*. We denote by $\langle x^*, x\rangle$ the value of the linear and continuous functional $x^* \in X^*$ at $x \in X$.

A function $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is called proper if the set $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$, called *effective domain* of f, is nonempty and $f(x) > -\infty$ for all $x \in X$. We consider also the *epigraph* of f, which is the set $\operatorname{epi} f = \{(x, r) \in X \times \mathbb{R} : f(x) \le r\}$. For a set $C \subseteq X$, let $\delta_C : X \to \overline{\mathbb{R}}$ be its *indicator function*, which is the function taking the values 0 on C and $+\infty$ otherwise.

The (convex) subdifferential of f at an element $x \in X$ such that $f(x) \in \mathbb{R}$ is defined as $\partial f(x) := \{x^* \in X^* : f(y) - f(x) \ge \langle x^*, y - x\rangle \ \forall y \in X\}$, while in case $f(x) \notin \mathbb{R}$ one takes by convention $\partial f(x) := \emptyset$. For every $\varepsilon \ge 0$, the *ε-subdifferential* of f, defined as $\partial_\varepsilon f(x) = \{x^* \in X^* : f(y) - f(x) \ge \langle x^*, y - x\rangle - \varepsilon \ \forall y \in X\}$ for $x \in X$ such that $f(x) \in \mathbb{R}$, and $\partial_\varepsilon f(x) := \emptyset$ otherwise, represents an enlargement of its subdifferential. Let us notice that in contrast to the classical subdifferential, the ε-subdifferential of a proper, convex

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and lower semicontinuous function at each point of its effective domain is in general a nonempty set, provided that $\varepsilon > 0$ (cf. [12, Proposition 3.15], see also [15, Theorem 2.4.4(iii)]).

For $\varepsilon \geq 0$, the *ε-normal set* of C at $x \in X$ is defined by $N_C^\varepsilon(x) := \partial_\varepsilon \delta_C(x)$, that is $N_C^\varepsilon(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq \varepsilon \ \forall y \in C\}$ if $x \in C$, and $N_C^\varepsilon(x) = \emptyset$ otherwise. The *normal cone* of the set C at $x \in X$ is $N_C(x) := N_C^0(x)$, that is $N_C(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0 \ \forall y \in C\}$ if $x \in C$, and $N_C(x) = \emptyset$ otherwise.

For the following characterizations of the ε-subdifferential via the ε-normal set we refer, for instance, to [13] (the extension from finite to infinite dimensional spaces is straightforward). If $x \in X$ is such that $f(x) \in \mathbb{R}$, then for all $\varepsilon \geq 0$ it holds $x^* \in \partial_\varepsilon f(x)$ if and only if $(x^*, -1) \in N_{\operatorname{epi} f}^\varepsilon(x, f(x))$. Moreover, for $r \in \mathbb{R}$ with $f(x) \leq r$, the relation $(x^*, -1) \in N_{\operatorname{epi} f}(x, r)$ implies $r = f(x)$. Furthermore, if $(x^*, -s) \in N_{\operatorname{epi} f}(x, r)$, then $s \geq 0$ and, if, additionally, $s \neq 0$, then $r = f(x)$ and $(1/s)x^* \in \partial f(x)$.

The celebrated Brøndsted-Rockafellar Theorem [7], which we recall as follows, emphasizes the fact that the ε-subdifferential of a proper, convex and lower semicontinuous function can be seen as an approximation of its subdifferential.

Theorem 1 *(Brøndsted-Rockafellar Theorem [7]) Let $f : X \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function and $x_0 \in \operatorname{dom} f$. Take $\varepsilon > 0$ and $x_0^* \in \partial_\varepsilon f(x_0)$. Then for all $\lambda > 0$ there exist $x \in X$ and $x^* \in X^*$ such that*

$$x^* \in \partial f(x), \ \|x - x_0\| \leq \frac{\varepsilon}{\lambda} \text{ and } \|x^* - x_0^*\| \leq \lambda.$$

Let us mention that a method for proving this result is by applying the Ekeland variational principle (see [12, Theorem 3.17]). For a more elaborated version of Theorem 1 we refer the interested reader to a result given by Borwein in [5] (see, also, [15, Theorem 3.1.1].

The aim of this note is to provide a Brøndsted-Rockafellar Theorem for so-called *diagonal subdifferential operators*. These are set-valued operators $A^F : X \rightrightarrows X^*$ defined by (see [1,6,9–11])

$$A^F(x) = \begin{cases} \{x^* \in X^* : F(x,y) - F(x,x) \geq \langle x^*, y - x\rangle \ \forall y \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{otherwise,} \end{cases}$$

where C is a nonempty subset of X and $F : C \times C \to \mathbb{R}$ is a so-called *bifunction*. The term *diagonal subdifferential operator* is justified by the formula $A^F(x) = \partial(F(x,\cdot) + \delta_C)(x)$ for all $x \in X$.

Bifunctions have been intensively studied in connection with equilibrium problems since the publication of the seminal work of Blum and Oettli [4] and, recently, in the context of diagonal subdifferential operators, when characterizing properties like local boundedness [1], monotonicity and maximal monotonicity in both reflexive [9, 10] and non-reflexive Banach spaces [6, 11].

A further operator of the same type, which has been considered in the literature, is ${}^F\!A : X \rightrightarrows X^*$, defined by

$${}^F\!A(x) = \begin{cases} \{x^* \in X^* : F(x,x) - F(y,x) \geq \langle x^*, y - x\rangle \ \forall y \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Notice that when F is *monotone*, namely, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$, (see [4]) and $F(x,x) = 0$ for all $x \in C$, then $A^F(x) \subseteq {}^F\!A(x)$ for all $x \in C$. Furthermore, if C is convex and closed, $F(x,x)=0$, $F(x,\cdot)$ is convex and $F(\cdot,y)$ is upper hemicontinuous, i.e. upper semicontinuous along segments, for all $x,y\in C$, then ${}^F\!A(x)\subseteq A^F(x)$ for all $x\in C$ (cf. [6, Lemma 5]). Under these hypotheses on can transfer properties from ${}^F\!A$ to A^F and viceversa.

In the following we will concentrate ourselves on A^F and consider, in analogy to the definition of the ε-subdifferential, what we call to be the *ε-diagonal subdifferential operator* of F, $A^F_\varepsilon : X \rightrightarrows X^*$, defined by,

$$A^F_\varepsilon(x)=\begin{cases}\{x^*\in X^* : F(x,y)-F(x,x)\geq \langle x^*, y-x\rangle-\varepsilon\ \forall y\in C\}, & \text{if } x\in C,\\ \emptyset, & \text{otherwise.}\end{cases}$$

If C is a nonempty, convex and closed set and $x\in C$ is such that $F(x,\cdot)$ is convex and lower semicontinuous, then $A^F_\varepsilon(x)\neq\emptyset$ for all $\varepsilon>0$.

The main result of this paper is represented by a Brøndsted-Rockafellar Theorem for the diagonal subdifferential operator A^F, the proof of which relies on the Ekeland variational principle for bifunctions given in [3].

For a generalization of the Brøndsted-Rockafellar Theorem for maximal monotone operators we refer to [14, Theorem 29.9], whereby, as pointed out in [14, pages 152–153] this result holds only in reflexive Banach spaces. Later, a special formulation of this theorem in the non-reflexive case was given in [2].

In contrast to this, our approach does not rely on the maximal monotonicity of the diagonal subdifferential operator, while the result holds in general Banach spaces. We present also some consequences of the given Brøndsted-Rockafellar Theorem concerning the density of the domain of diagonal subdifferential operators. We close the note by showing that a Brøndsted-Rockafellar-type Theorem for subdifferential operators can be obtained as a particular case of our main result.

2 A Brøndsted-Rockafellar Theorem

The following Ekeland variational principle for bifunctions was given in [3]. Although this result was stated there in Euclidian spaces, it is valid in general Banach spaces, too.

Theorem 2 *Assume that C is nonempty, convex and closed set and $f : C\times C\to\mathbb{R}$ satisfies:*

(i) $f(x,\cdot)$ is lower bounded and lower semicontinuous for every $x\in C$;

(ii) $f(x,x)=0$ for every $x\in C$;

(iii) $f(x,y)+f(y,z)\geq f(x,z)$ for every $x,y,z\in C$.

Then, for every $\varepsilon>0$ and for every $x_0\in C$, there exists $\overline{x}\in C$ such that

$$f(x_0,\overline{x})+\varepsilon\|x_0-\overline{x}\|\leq 0$$

and

$$f(\overline{x},x)+\varepsilon\|\overline{x}-x\|>0\ \forall x\in C,\ x\neq\overline{x}.$$

Remark 1 *By taking $z=x$, the assumptions (iii) and (ii) in the above theorem imply that $f(x,y)+f(y,x)\geq 0$ for all $x,y\in C$, which means that $-f$ is monotone.*

Theorem 2 will be an essential ingredient in proving the following Brøndsted-Rockafellar Theorem for diagonal subdifferential operators.

Theorem 3 *Assume that C is a nonempty, convex and closed set and $F : C \times C \to \mathbb{R}$ satisfies:*

(i) $F(x, \cdot)$ is a convex and lower semicontinuous function for every $x \in C$;

(ii) $F(x, x) = 0$ for every $x \in C$;

(iii) $F(x, y) + F(y, z) \geq F(x, z)$ for every $x, y, z \in C$.

Take $\varepsilon > 0$, $x_0 \in C$ and $x_0^* \in A_\varepsilon^F(x_0)$. Then for all $\lambda > 0$ there exist $x^* \in X^*$ and $x \in C$ such that

$$x^* \in A^F(x), \ \|x - x_0\| \leq \frac{\varepsilon}{\lambda} \text{ and } \|x^* - x_0^*\| \leq \lambda.$$

Proof. We fix $\varepsilon > 0$, $x_0 \in C$ and $x_0^* \in A_\varepsilon^F(x_0)$. According to the definition of the operator A_ε^F we have

$$F(x_0, y) \geq \langle x_0^*, y - x_0 \rangle - \varepsilon \ \forall y \in C. \tag{1}$$

Let us define the bifunction $f : C \times C \to \mathbb{R}$ by

$$f(x, y) = F(x, y) - \langle x_0^*, y - x \rangle \text{ for all } (x, y) \in C \times C.$$

We want to apply Theorem 2 to f and show to this aim that the assumptions (i)-(iii) in Theorem 2 are verified. Indeed, the lower semicontinuity of the function $f(x, \cdot)$ and the relation $f(x, x) = 0$, for all $x \in C$, are inherited from the corresponding properties of F. One can easily see that (iii) is fulfilled, too: for $x, y, z \in C$ it holds

$$f(x, y) + f(y, z) = F(x, y) + F(y, z) - \langle x_0^*, z - x \rangle \geq F(x, z) - \langle x_0^*, z - x \rangle = f(x, z).$$

It remains to prove that $f(x, \cdot)$ is lower bounded for all $x \in C$. Take an arbitrary $x \in C$. By using (1) we get for all $y \in C$

$$f(x, y) \geq f(x_0, y) - f(x_0, x) = F(x_0, y) - \langle x_0^*, y - x_0 \rangle - f(x_0, x) \geq -\varepsilon - f(x_0, x)$$

and the desired property follows.

Take now $\lambda > 0$. A direct application of Theorem 2 guarantees the existence of $\overline{x} \in C$ such that

$$f(x_0, \overline{x}) + \lambda \|x_0 - \overline{x}\| \leq 0 \tag{2}$$

and

$$f(\overline{x}, x) + \lambda \|\overline{x} - x\| > 0 \ \forall x \in C, \ x \neq \overline{x}. \tag{3}$$

From (2) we obtain

$$F(x_0, \overline{x}) - \langle x_0^*, \overline{x} - x_0 \rangle + \lambda \|x_0 - \overline{x}\| \leq 0,$$

which combined with (1) ensures

$$\lambda\|x_0-\overline{x}\| \leq \langle x_0^*, \overline{x}-x_0\rangle - F(x_0,\overline{x}) \leq \varepsilon,$$

hence $\|x_0-\overline{x}\| \leq \frac{\varepsilon}{\lambda}$.

Further, notice that (3) implies

$$0 \in \partial\big(f(\overline{x},\cdot)+\delta_C+\lambda\|\overline{x}-\cdot\|\big)(\overline{x}).$$

Since the functions in the above statement are convex and $\|\overline{x}-\cdot\|$ is continuous, we obtain via the subdifferential sum formula (cf. [15, Theorem 2.8.7])

$$0 \in \partial\big(f(\overline{x},\cdot)+\delta_C\big)(\overline{x}) + \partial\big(\lambda\|\overline{x}-\cdot\|\big)(\overline{x}). \tag{4}$$

Taking into account the definition of the bifunction f, we get (cf. [15, Theorem 2.4.2(vi)]) $\partial\big(f(\overline{x},\cdot)+\delta_C\big)(\overline{x}) = \partial\big(F(\overline{x},\cdot)+\delta_C\big)(\overline{x}) - x_0^* = A^F(\overline{x}) - x_0^*$. Moreover, $\partial\big(\lambda\|\overline{x}-\cdot\|\big)(\overline{x}) = \lambda B_{X^*}$, where B_{X^*} denotes the closed unit ball of the dual space X^* (see, for instance, [15, Corollary 2.4.16]). Hence (4) is nothing else than

$$0 \in A^F(\overline{x}) - x_0^* + \lambda B_{X^*},$$

from which we conclude that there exists $x^* \in A^F(\overline{x})$ with $\|x^*-x_0^*\| \leq \lambda$ and the proof is complete. □

For a similar result like the one given in Theorem 3, but formulated in reflexive Banach spaces and by assuming (Blum-Oettli-) maximal monotonicity for the bifunction F (see [4] for the definition of this notion), we refer the reader to [8, Theorem 1.1].

A direct consequence of the above Brøndsted-Rockafellar Theorem is the following result concerning the density of $D(A^F)$ in C, where $D(A^F) = \{x \in X : A^F(x) \neq \emptyset\}$ is the *domain* of the operator A^F.

Corollary 4 *Assume that the hypotheses of Theorem 3 are fulfilled. Then* $\overline{D(A^F)} = C$, *hence* $\overline{D(A^F)}$ *is a convex set.*

Proof. The implication $D(A^F) \subseteq C$ is obvious. Take now an arbitrary $x_0 \in C$. For all $n \in \mathbb{N}$ we have that $A^F_{1/n}(x_0) \neq \emptyset$, hence we can choose $x_n^* \in A^F_{1/n}(x_0)$. Theorem 3 guarantees the existence of $u_n^* \in X^*$ and $u_n \in C$ such that

$$u_n^* \in A^F(u_n),\ \|u_n - x_0\| \leq \sqrt{1/n} \text{ and } \|u_n^* - x_n^*\| \leq \sqrt{1/n} \text{ for all } n \in \mathbb{N}.$$

Since $u_n \in D(A^F)$ for all $n \in \mathbb{N}$, we get from above that $x_0 \in \overline{D(A^F)}$. □

Remark 2 *Similar statements to the one in Corollary 4 were furnished in [9, Section 4] in reflexive Banach spaces and by assuming maximal monotonicity for* A^F.

Let us show how Theorem 3 can be used in order to derive the classical Brøndsted-Rockafellar theorem for the subdifferential operator in case the domain of the function is closed.

Corollary 5 *Let $f : X \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function such that* $\operatorname{dom} f$ *is closed. Take $x_0 \in \operatorname{dom} f$, $\varepsilon > 0$ and $x_0^* \in \partial_\varepsilon f(x_0)$. Then for all $\lambda > 0$ there exist $x^* \in X^*$ and $x \in X$ such that*

$$x^* \in \partial f(x), \ \|x - x_0\| \leq \frac{\varepsilon}{\lambda} \text{ and } \|x^* - x_0^*\| \leq \lambda.$$

Proof. The result follows by applying Theorem 3 for $C = \operatorname{dom} f$ and the bifunction $F : \operatorname{dom} f \times \operatorname{dom} f \to \mathbb{R}$ defined by $F(x, y) = f(y) - f(x)$. □

The restriction "$\operatorname{dom} f$ closed" comes from the fact that in Theorems 2 and 3 the set C is assumed to be a closed set. In the following Brøndsted-Rockafellar-type Theorem for subdifferential operators, which we obtain as a consequence of Corollary 5, we abandon this assumption.

Corollary 6 *Let $f : X \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function. Take $x_0 \in \operatorname{dom} f$, $\varepsilon > 0$ and $x_0^* \in \partial_\varepsilon f(x_0)$. Then for all $\lambda > 0$ there exist $x^* \in X^*$ and $x \in X$ such that*

$$x^* \in \partial f(x), \ \|x - x_0\| \leq \varepsilon \left(\frac{1}{\lambda} + 1 \right) \text{ and } \|x^* - x_0^*\| \leq \lambda.$$

Proof. Take $x_0 \in \operatorname{dom} f$, $\varepsilon > 0$, $x_0^* \in \partial_\varepsilon f(x_0)$ and $\lambda > 0$. We consider $X \times \mathbb{R}$ endowed with the norm defined for all $(x, r) \in X \times \mathbb{R}$ as being $\|(x, r)\| = (\|x\|^2 + r^2)^{1/2}$. We divide the proof in two steps.

(I) Consider the case $x_0^* = 0$. We have $0 \in \partial_\varepsilon f(x_0)$, hence $(0, -1) \in N^\varepsilon_{\operatorname{epi} f}(x_0, f(x_0)) = \partial_\varepsilon \delta_{\operatorname{epi} f}(x_0, f(x_0))$. By applying Corollary 5 for the function $\delta_{\operatorname{epi} f}$ and $\lambda := \lambda/(\lambda + 1)$ we obtain the existence of $(x, r) \in \operatorname{epi} f$ and $(x^*, -s) \in \partial \delta_{\operatorname{epi} f}(x, r) = N_{\operatorname{epi} f}(x, r)$ such that

$$\|(x, r) - (x_0, f(x_0))\| \leq \varepsilon \frac{1 + \lambda}{\lambda} \text{ and } \|(x^*, -s) - (0, -1)\| \leq \frac{\lambda}{1 + \lambda}.$$

From here, it follows

$$\|x - x_0\| \leq \varepsilon/\lambda + \varepsilon, s \geq 0, \|x^*\| \leq \frac{\lambda}{1 + \lambda} \text{ and } |s - 1| \leq \frac{\lambda}{1 + \lambda}.$$

The last inequality ensures $0 < \frac{1}{1+\lambda} \leq s$, hence $r = f(x)$ and $(1/s)x^* \in \partial f(x)$. Moreover, $\|(1/s)x^*\| \leq \frac{\lambda}{1+\lambda} \cdot (1 + \lambda) = \lambda$.

(II) Let us consider now the general case, when $x_0^* \in \partial_\varepsilon f(x_0)$ is an arbitrary element. Define the function $g : X \to \overline{\mathbb{R}}$, $g(x) = f(x) - \langle x_0^*, x \rangle$, for all $x \in X$. Notice that $\partial_\alpha g(x) = \partial_\alpha f(x) - x_0^*$ for all $\alpha \geq 0$, hence the condition $x_0^* \in \partial_\varepsilon f(x_0)$ guarantees $0 \in \partial_\varepsilon g(x_0)$. Applying the statement obtained in the first part of the proof for g, we obtain that there exist $x^* \in X^*$ and $x \in X$ such that

$$x^* \in \partial g(x), \ \|x - x_0\| \leq \varepsilon \left(\frac{1}{\lambda} + 1 \right) \text{ and } \|x^*\| \leq \lambda.$$

Thus $x^* + x_0^* \in \partial f(x)$, $\|x - x_0\| \leq \varepsilon \left(\frac{1}{\lambda} + 1 \right)$ and $\|(x^* + x_0^*) - x_0^*\| = \|x^*\| \leq \lambda$, hence the proof is complete. □

The bounds in Corollary 6 differ from the ones in Theorem 1, nevertheless, by taking $\lambda = \sqrt{\varepsilon}$, they become $\sqrt{\varepsilon} + \varepsilon$ and, respectively, $\sqrt{\varepsilon}$, and allow one to derive (by letting $\varepsilon \searrow 0$) the classical density result regarding the domain of the subdifferential.

However, it remains an open question if Theorem 1 can be deduced from Theorem 3.

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