Optimal Investment in Incomplete Financial Markets

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Abstract

We give a review of classical and recent results on maximization of expected utility for an investor who has the possibility of trading in a financial market. Emphasis will be given to the duality theory related to this convex optimization problem.

For expository reasons we first consider the classical case where the underlying probability space Ω is finite. This setting has the advantage that the technical difficulties of the proofs are reduced to a minimum, which allows for a clearer insight into the basic ideas, in particular the crucial role played by the Legendre-transform. In this setting we state and prove an existence and uniqueness theorem for the optimal investment strategy, and its relation to the dual problem; the latter consists in finding an equivalent martingale measure optimal with respect to the conjugate of the utility function. We also discuss economic interpretations of these theorems.

We then pass to the general case of an arbitrage-free financial market modeled by an $\mathbb{R}^d$-valued semi-martingale. In this case some regularity conditions have to be imposed in order to obtain an existence result for the primal problem of finding the optimal investment, as well as for a proper duality theory. It turns out that one may give a *necessary and sufficient* condition, namely a mild condition on the asymptotic behavior of the utility function, its so-called *reasonable asymptotic elasticity*. This property allows for an economic interpretation motivating the term "reasonable". The remarkable fact is that this regularity condition only pertains to the behavior of the utility function, while we do not have to impose any regularity conditions on the stochastic process modeling the financial market (to be precise: of course, we have to require the arbitrage-freeness of this process in a proper sense; also we have to assume in one of the cases considered below that this process is locally bounded; but otherwise it may be an arbitrary $\mathbb{R}^d$-valued semi-martingale).

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We state two general existence and duality results pertaining to the setting of optimizing expected utility of terminal consumption. We also survey some of the ramifications of these results allowing for intermediate consumption, state-dependent utility, random endowment, non-smooth utility functions and transaction costs.

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1 Introduction

A basic problem of mathematical finance is the problem of an economic agent, who invests in a financial market so as to maximize the expected utility of her terminal wealth. As we shall see in (16) below, this problem can be written in an abstract way as

$$\mathbb{E}\left[U\left(x+\int_0^T H_u dS_u\right)\right] \longrightarrow \max!, \tag{1}$$

where we optimize over all "admissible" trading strategies H. In the framework of a continuous-time model the problem was studied for the first time by R. Merton in two seminal papers [M 69] and [M 71] (see also [M 90] as well as [S 69] for a treatment of the discrete time case). Using the methods of stochastic optimal control Merton derived a non-linear partial differential equation (Bellman equation) for the value function of the optimization problem. He also produced the closed-form solution of this equation, when the utility function is a power function, the logarithm, or of the form $-e^{-\gamma x}$ for $\gamma > 0$.

The Bellman equation of stochastic programming is based on the assumption of Markov state processes. The modern approach to the problem of expected utility maximization, which permits us to avoid the assumption of Markovian asset prices, is based on duality characterizations of portfolios provided by the set of martingale measures. For the case of a *complete* financial market, where the set of martingale measures is a singleton, this "martingale" methodology was developed by Pliska [P 86], Cox and Huang [CH 89], [CH 91] and Karatzas, Lehoczky and Shreve [KLS 87]. It was shown that the marginal utility of the terminal wealth of the optimal portfolio is proportional to the density of the martingale measure; this key result naturally extends the classical Arrow-Debreu theory of an optimal investment derived in a one-step, finite probability space model.

Considerably more difficult is the case of incomplete financial models. It was studied in a discrete-time, finite probability space model by He and Pearson [HP 91], and, in a continuous-time diffusion model, by He and Pearson

[HP 91a], and by Karatzas, Lehoczky, Shreve and Xu in their seminal paper [KLSX 91]. The central idea here is to solve a *dual* variational problem and then to find the solution of the original problem by convex duality, the latter step being similar as in the case of a complete model.

We now formally assemble the ingredients of the optimization problem.

We consider a model of a security market which consists of $d+1$ assets. We denote by $S = ((S_t^i)_{0\le t\le T})_{0\le i\le d}$ the price process of the d stocks and suppose that the price of the asset S^0, called the "bond" or "cash account", is constant, $S_t^0 \equiv 1$. The latter assumption does not restrict the generality of the model as we always may choose the bond as numéraire (c.f., [DS 95]). In other words, $((S_t^i)_{0\le t\le T})_{1\le i\le d}$, is an $\mathbb{R}^d$-valued semi-martingale modeling the discounted price process of d risky assets.

The process S is assumed to be a semimartingale based on and adapted to a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{P})$ satisfying the usual conditions of saturatedness and right continuity. As usual in mathematical finance, we consider a finite horizon T, but we remark that our results can also be extended to the case of an infinite horizon.

In section 2 we shall consider the case of finite Ω, in which case the paths of S are constant except for jumps at a finite number of times. We then can write S as $(S_t)_{t=0}^T = (S_0, S_1, \ldots, S_T)$, for some $T \in \mathbb{N}$.

The assumption that the bond is constant is mainly chosen for notational convenience as it allows for a compact description of self-financing portfolios: a self-financing portfolio Π is defined as a pair (x, H), where the constant x is the initial value of the portfolio and $H = (H^i)_{1\le i\le d}$ is a predictable S-integrable process specifying the amount of each asset held in the portfolio. The value process $X = (X_t)_{0\le t\le T}$ of such a portfolio Π at time t is given by

$$X_t = X_0 + \int_0^t H_u dS_u, \qquad 0 \le t \le T, \tag{2}$$

where $X_0 = x$ and the integral refers to stochastic integration in $\mathbb{R}^d$.

In order to rule out doubling strategies and similar schemes generating arbitrage-profits (by going deeply into the red) we follow Harrison and Pliska ([HP 81], see also [DS 94]), calling a predictable, S-integrable process *admissible*, if there is a constant $C \in \mathbb{R}_+$ such that, almost surely, we have

$$(H \cdot S)_t := \int_0^t H_u dS_u \ge -C, \quad \text{for } 0 \le t \le T. \tag{3}$$

Let us illustrate these general concepts in the case of an $\mathbb{R}^d$-valued process $S = (S_t)_{t=0}^T$ in finite, discrete time adapted to the filtration $(\mathcal{F}_t)_{t=0}^T$. In this case each $\mathbb{R}^d$-valued process $(H_t)_{t=1}^T$, which is predictable (i.e. each H_t is $\mathcal{F}_{t-1}$-measurable), is S-integrable, and the stochastic integral reduces to a finite sum

$$(H \cdot S)_t = \int_0^t H_u dS_u \tag{4}$$

$$= \sum_{u=1}^{t} H_u \Delta S_u \tag{5}$$

$$= \sum_{u=1}^{t} H_u (S_u - S_{u-1}), \tag{6}$$

where $H_u \Delta S_u$ denotes the inner product of the vectors H_u and $\Delta S_u = S_u - S_{u-1}$ in $\mathbb{R}^d$. Of course, each such trading strategy H is admissible if the underlying probability space Ω is finite.

Passing again to the general setting of an $\mathbb{R}^d$-valued semi-martingale $S = (S_t)_{0 \le t \le T}$ we denote as in [KS 99] by $\mathcal{M}^e(S)$ (resp. $\mathcal{M}^a(S)$) the set of probability measures Q equivalent to $\mathbf{P}$ (resp. absolutely continuous with respect to $\mathbf{P}$) such that for each admissible integrand H, the process $H \cdot S$ is a local martingale under Q.

Throughout the paper we assume the following version of the no-arbitrage condition on S:

Assumption 1.1 *The set $\mathcal{M}^e(S)$ is not empty.*[1]

We note that in this paper we shall mainly be interested in the case when $\mathcal{M}^e(S)$ is not reduced to a singleton, i.e., the case of an *incomplete* financial market.

After having specified the process S modeling the financial market we now define the function $U(x)$ modeling the utility of an agent's wealth x at the terminal time T.

We make the classical assumptions that $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is *increasing on* $\mathbb{R}$, *continuous* on $\{U > -\infty\}$, *differentiable and strictly concave* on the interior of $\{U > -\infty\}$, and that marginal utility tends to zero when wealth tends to infinity, i.e.,

$$U'(\infty) := \lim_{x \to \infty} U'(x) = 0. \tag{7}$$

[1] If follows from [DS 94] and [DS 98a] that Assumption 1.1 is equivalent to the condition of "no free lunch with vanishing risk". This property can also be equivalently characterised in terms of the existence of a measure $Q \sim \mathbf{P}$ such that the process S itself (rather than the integrals $H \cdot S$ for admissible integrands) is "something like a martingale". The precise notion in the general semi-martingale setting is that S is a sigma-martingale under Q (see [DS 98a]); in the case when S is locally bounded (resp. bounded) the term "sigma-martingale" may be replaced by the more familiar term "local martingale" (resp. "martingale").

Readers who are not too enthusiastic about the rather subtle distinctions between martingales, local martingales and sigma-martingales may find some relief by noting that, in the case of finite Ω, or, more generally, for bounded processes, these three notions coincide. Also note that in the general semi-martingale case, when S is locally bounded (resp. bounded), the set $\mathcal{M}^e(S)$ as defined above coincides with the set of equivalent measures $Q \sim \mathbf{P}$ such that S is a local martingale (resp. martingale) under Q (see [E 80] and [AS 94]).

These assumptions make good sense economically and it is clear that the requirement (7) of marginal utility decreasing to zero, as x tends to infinity, is necessary, if one is aiming for a general existence theorem for optimal investment. Indeed, if $U'(\infty) > 0$, then even in the case of the Black-Scholes model the solution to the optimization problem (1) fails to exist.

As regards the behavior of the (marginal) utility at the other end of the wealth scale we shall distinguish throughout the paper two cases.

Case 1 (negative wealth not allowed): in this setting we assume that U satisfies the conditions $U(x) = -\infty$, for $x < 0$, while $U(x) > -\infty$, for $x > 0$, and that

$$U'(0) := \lim_{x \searrow 0} U'(x) = \infty. \tag{8}$$

Case 2 (negative wealth allowed): in this case we assume that $U(x) > -\infty$, for all $x \in \mathbb{R}$, and that

$$U'(-\infty) := \lim_{x \searrow -\infty} U'(x) = \infty. \tag{9}$$

Typical examples for case 1 are

$$U(x) = \ln(x) \tag{10}$$

or

$$U(x) = \frac{x^\alpha}{\alpha}, \quad 0 < \alpha < 1, \tag{11}$$

whereas a typical example for case 2 is

$$U(x) = -e^{-\gamma x}, \quad \gamma > 0. \tag{12}$$

We again note that it is natural from economic considerations to require that the marginal utility tends to infinity when the wealth x tends to the infimum of its allowed values.

For later reference we summarize our assumptions on the utility function:

Assumption 1.2 *Throughout the paper the utility function* $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ *is increasing on* $\mathbb{R}$, *continuous on* $\{U > -\infty\}$, *differentiable and strictly concave on the interior of* $\{U > -\infty\}$, *and satisfies*

$$U'(\infty) := \lim_{x \to \infty} U'(x) = 0. \tag{13}$$

Denoting by $\mathrm{dom}(U)$ *the interior of* $\{U > -\infty\}$, *we assume that we have one of the two following cases.*

Case 1: $\mathrm{dom}(U) =]0, \infty[$ *in which case* U *satisfies the condition*

$$U'(0) := \lim_{x \searrow 0} U'(x) = \infty. \tag{14}$$

Case 2: $\mathrm{dom}(U) = \mathbb{R}$ *in which case* U *satisfies*

$$U'(-\infty) := \lim_{x \searrow -\infty} U'(x) = \infty. \tag{15}$$

We now can give a precise meaning to the expression (1) at the beginning of this section. Define the value function

$$u(x) := \sup_{H \in \mathcal{H}} \mathbb{E}\left[U(x + (H \cdot S)_T)\right], \quad x \in \mathrm{dom}(U), \tag{16}$$

where H ranges through the admissible S-integrable trading strategies. To exclude trivial cases we shall assume throughout the paper that the value function u is not degenerate:

Assumption 1.3

$$u(x) < \sup_{\xi} U(\xi), \quad \textit{for some} \quad x \in \mathrm{dom}(U). \tag{17}$$

One easily verifies that this assumption implies that

$$u(x) < \sup_{\xi} U(\xi), \quad \text{for all} \quad x \in \mathrm{dom}(U), \tag{18}$$

and that, in the case of finite Ω, Assumptions 1.1 and 1.2 already imply Assumption 1.3.

2 Utility Maximization on Finite Probability Spaces

In this section we consider an $\mathbb{R}^{d+1}$-valued process $(S_t)_{t=0}^T = (S_t^0, S_t^1, \ldots, S_t^d)_{t=0}^T$ with $S_t^0 \equiv 1$, based on and adapted to the *finite* filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbf{P})$, which we write as $\Omega = \{\omega_1, \ldots, \omega_N\}$. Without loss of generality we assume that $\mathcal{F}_0$ is trivial, that $\mathcal{F}_T = \mathcal{F}$ is the power set of Ω, and that $\mathbf{P}[\omega_n] > 0$, for all $1 \leq n \leq N$.

Assumption 1.1 is the existence a measure $Q \sim \mathbf{P}$, i.e., $Q[\omega_n] > 0$, for $1 \leq n \leq N$, such that S is a Q-martingale.

2.1 The complete case (Arrow-Debreu)

As a first case we analyze the situation of a financial market which is *complete*, i.e., the set $\mathcal{M}^e(S)$ of equivalent probability measures under which S is a martingale is reduced to a singleton $\{Q\}$. In this setting consider the *Arrow-Debreu assets* $\mathbf{1}_{\{\omega_n\}}$, which pay 1 unit of the numéraire at time T, when ω_n turns out to be the true state of the world, and 0 otherwise. In view of our normalization of the numéraire $S_t^0 \equiv 1$, we get for the price of the Arrow-Debreu assets at time $t = 0$ the relation

$$\mathbb{E}_Q\left[\mathbf{1}_{\{\omega_n\}}\right] = Q[\omega_n], \tag{19}$$

and each Arrow-Debreu asset $\mathbf{1}_{\{\omega_n\}}$ may be represented as $\mathbf{1}_{\{\omega_n\}} = Q[\omega_n] + (H \cdot S)_T$, for some predictable trading strategy $H \in \mathcal{H}$.

Hence, for fixed initial endowment $x \in \operatorname{dom}(U)$, the utility maximization problem (16) above may simply be written as

$$\mathbb{E}_{\mathbf{P}}\left[U(X_T)\right] = \sum_{n=1}^{N} p_n U(\xi_n) \to \max! \tag{20}$$

$$\mathbb{E}_Q[X_T] = \sum_{n=1}^{N} q_n \xi_n \quad \leq \; x. \tag{21}$$

To verify that (20) and (21) indeed are equivalent to the original problem (16) above (in the present finite, complete case), note that a random variable $X_T(\omega_n) = \xi_n$ can be dominated by a random variable of the form $x + (H \cdot S)_T = x + \sum_{t=1}^{T} H_t \Delta S_t$ iff $\mathbb{E}_Q[X_T] = \sum_{n=1}^{N} q_n \xi_n \leq x$. This basic relation has a particularly evident interpretation in the present setting, as q_n is simply the price of the Arrow-Debreu asset $\mathbf{1}_{\{\omega_n\}}$.

Let us fix some notation for the domain over which the problem (20) is optimized:

$$C(x) = \left\{ X_T \in L^0(\Omega, \mathcal{F}_T, \mathbf{P}) : \mathbb{E}_Q[X_T] \leq x \right\}. \tag{22}$$

The notation $L^0(\Omega, \mathcal{F}_T, \mathbf{P})$ only serves to indicate that X_T is an $\mathcal{F}_T$-measurable random variable at this stage, as for finite Ω all the L^p-spaces coincide. But we have chosen the notation to be consistent with that of the general case below.

We have written ξ_n for $X_T(\omega_n)$ to stress that (20) simply is a concave maximization problem in $\mathbb{R}^N$ with one linear constraint. To solve it, we form the Lagrangian

$$\begin{aligned} L(\xi_1, \ldots, \xi_N, y) &= \sum_{n=1}^{N} p_n U(\xi_n) - y \left(\sum_{n=1}^{N} q_n \xi_n - x \right) && (23) \\ &= \sum_{n=1}^{N} p_n \left(U(\xi_n) - y \tfrac{q_n}{p_n} \xi_n \right) + yx. && (24) \end{aligned}$$

We have used the letter $y \geq 0$ instead of the usual $\lambda \geq 0$ for the Lagrange multiplier; the reason is the dual relation between x and y which will become apparent in a moment.

Writing

$$\Phi(\xi_1, \ldots, \xi_N) = \inf_{y>0} L(\xi_1, \ldots, \xi_N, y), \quad \xi_n \in \operatorname{dom}(U), \tag{25}$$

and

$$\Psi(y) = \sup_{\xi_1, \ldots, \xi_N} L(\xi_1, \ldots, \xi_N, y), \quad y \geq 0, \tag{26}$$

it is straight forward to verify that we have

$$\sup_{\xi_1, \ldots, \xi_N} \Phi(\xi_1, \ldots, \xi_N) = \sup_{\substack{\xi_1, \ldots, \xi_N \\ \sum_{n=1}^{N} q_n \xi_n \leq x}} \sum_{n=1}^{N} p_n U(\xi_n) = u(x). \tag{27}$$

As regards the function $\Psi(y)$ we make the following pleasant observation which is the basic reason for the efficiency of the duality approach: using the form (24) of the Lagrangian and fixing $y > 0$, the optimization problem appearing in (26) splits into N independent optimization problems over $\mathbb{R}$

$$U(\xi_n) - y\tfrac{q_n}{p_n}\xi_n \mapsto \max!, \quad \xi_n \in \mathbb{R}. \tag{28}$$

In fact, these one-dimensional optimization problems are of a very convenient form: recall (see, e.g., [R 70], [ET 76] or [KLSX 91]) that, for a concave function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, the *conjugate function* V (which — up to the sign — is just the Legendre-transform) is defined by

$$V(\eta) = \sup_{\xi \in \mathbb{R}} \left[U(\xi) - \eta\xi \right], \quad \eta > 0. \tag{29}$$

The following facts are well known (and easily verified by one-dimensional calculus): if U satisfies Assumption 1.2, we have that V is finitely valued, differentiable, strictly convex on $]0, \infty[$, and satisfies

$$V'(0) := \lim_{y \searrow 0} V'(y) = -\infty, \qquad V(0) := \lim_{y \searrow 0} V(y) = U(\infty). \tag{30}$$

As regards the behavior of V at infinity, we have to distinguish between case 1 and case 2 in Assumption 1.2 above:

$$\text{case 1:} \qquad \lim_{y \to \infty} V(y) = \lim_{x \to 0} U(x) \quad \text{and} \quad \lim_{y \to \infty} V'(y) = 0 \tag{31}$$

$$\text{case 2:} \qquad \lim_{y \to \infty} V(y) = \infty \qquad\qquad \text{and} \quad \lim_{y \to \infty} V'(y) = \infty \tag{32}$$

We also note that these properties of the conjugate function V are, in fact, equivalent to the properties of U listed in Assumption 1.2. We also have the inversion formula to (29)

$$U(\xi) = \inf_{\eta} \left[V(\eta) + \eta\xi \right], \quad \xi \in \mathrm{dom}(U) \tag{33}$$

and that $-V'(y)$, denoted by $I(y)$ for "inverse function" in [KLSX 91], is the inverse function of $U'(x)$; of course, U' has a good economic interpretation as the *marginal utility* of an economic agent modeled by the utility function U.

Here are some concrete examples of pairs of conjugate functions:

$$U(x) = \ln(x), \quad x > 0, \qquad V(y) = -\ln(y) - 1, \tag{34}$$

$$U(x) = \tfrac{x^\alpha}{\alpha}, \quad x > 0, \qquad V(y) = \tfrac{1-\alpha}{\alpha} y^{\frac{\alpha}{\alpha-1}}, \quad 0 < \alpha < 1, \tag{35}$$

$$U(x) = -\tfrac{e^{-\gamma x}}{\gamma}, \quad x \in \mathbb{R}, \qquad V(y) = \tfrac{y}{\gamma}(\ln(y) - 1), \quad \gamma > 0. \tag{36}$$

We now apply these general facts about the Legendre transformation to calculate $\Psi(y)$. Using definition (29) of the conjugate function V and (24), formula (26) becomes

$$\Psi(y) = \sum_{n=1}^{N} p_n V\left(y\tfrac{q_n}{p_n} \right) + yx \tag{37}$$

$$= \mathbb{E}_{\mathbf{P}} \left[V\left(y\tfrac{dQ}{d\mathbf{P}} \right) \right] + yx. \tag{38}$$

Denoting by $v(y)$ the dual value function

$$v(y) := \mathbb{E}_{\mathbf{P}}\left[V\left(y\tfrac{dQ}{d\mathbf{P}}\right)\right] = \sum_{n=1}^{N} p_n V\left(y\tfrac{q_n}{p_n}\right), \quad y > 0, \tag{39}$$

the function v clearly has the same qualitative properties as the function V listed above. Hence by (30), (31), and (32) we find, for fixed $x \in \mathrm{dom}(U)$, a unique $\widehat{y} = \widehat{y}(x) > 0$ such that $v'(\widehat{y}(x)) = -x$, which therefore is the unique minimizer to the dual problem

$$\Psi(y) = \mathbb{E}_{\mathbf{P}}\left[V\left(y\tfrac{dQ}{d\mathbf{P}}\right)\right] + yx = \min! \tag{40}$$

Fixing the critical value $\widehat{y}(x)$ of the Lagrange multiplier, the concave function

$$(\xi_1, \ldots, \xi_N) \mapsto L(\xi_1, \ldots, \xi_N, \widehat{y}(x)) \tag{41}$$

defined in (24) assumes its unique maximum at the point $(\widehat{\xi}_1, \ldots, \widehat{\xi}_N)$ satisfying

$$U'(\widehat{\xi}_n) = \widehat{y}(x)\tfrac{q_n}{p_n} \quad \text{or, equivalently,} \quad \widehat{\xi}_n = I\left(\widehat{y}(x)\tfrac{q_n}{p_n}\right), \tag{42}$$

so that we have

$$\begin{aligned} \inf_{y>0} \Psi(y) &= \inf_{y>0} \left(v(y) + xy\right) && (43)\\ &= v(\widehat{y}(x)) + x\widehat{y}(x) && (44)\\ &= L(\widehat{\xi}_1, \ldots, \widehat{\xi}_N, \widehat{y}(x)). && (45) \end{aligned}$$

Note that $\widehat{\xi}_n$ are in $\mathrm{dom}(U)$, for $1 \le n \le N$, so that L is continuously differentiable at $(\widehat{\xi}_1, \ldots, \widehat{\xi}_N, \widehat{y}(x))$, which implies that $\frac{\partial}{\partial y} L(\xi_1, \ldots, \xi_N, y)|_{(\widehat{\xi}_1, \ldots, \widehat{\xi}_N, \widehat{y}(x))} = 0$; hence we infer from (23) and the fact that $\widehat{y}(x) > 0$ that the constraint (21) is binding, i.e.,

$$\sum_{n=1}^{N} q_n \widehat{\xi}_n = x, \tag{46}$$

and that

$$\sum_{n=1}^{N} p_n U(\widehat{\xi}_n) = L(\widehat{\xi}_1, \ldots, \widehat{\xi}_N, \widehat{y}(x)). \tag{47}$$

In particular, we obtain that

$$u(x) = \sum_{n=1}^{N} p_n U(\widehat{\xi}_n). \tag{48}$$

Indeed, the inequality $u(x) \ge \sum_{n=1}^{N} p_n U(\widehat{\xi}_n)$ follows from (46) and (27), while the reverse inequality follows from (47) and the fact that for all $\xi_1, \ldots, \xi_N$ verifying the constraint (21)

$$\sum_{n=1}^{N} p_n U(\xi_n) \le L(\xi_1, \ldots, \xi_N, \widehat{y}(x)) \le L(\widehat{\xi}_1, \ldots, \widehat{\xi}_N, \widehat{y}(x)). \tag{49}$$

We shall write $\widehat{X}_T(x) \in C(x)$ for the optimizer $\widehat{X}_T(x)(\omega_n) = \widehat{\xi}_n$, $n = 1, \dots, N$.

Combining (43), (47) and (48) we note that the value functions u and v are conjugate:

$$\inf_{y>0} (v(y) + xy) = v(\widehat{y}(x)) + x\widehat{y}(x) = u(x), \quad x \in \mathrm{dom}(U), \tag{50}$$

which, by the remarks after equations (32) and (39), implies that u inherits the properties of U listed in Assumption 1.2. The relation $v'(\widehat{y}(x)) = -x$ which was used to define $\widehat{y}(x)$, therefore translates into

$$u'(x) = \widehat{y}(x), \quad \text{for } x \in \mathrm{dom}(U). \tag{51}$$

Let us summarize what we have proved:

Theorem 2.1 (finite Ω, complete market) *Let the financial market $S = (S_t)_{t=0}^T$ be defined over the finite filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t=0}^T, \mathbf{P})$ and satisfy $\mathcal{M}^e(S) = \{Q\}$, and let the utility function U satisfy Assumption 1.2.*

Denote by $u(x)$ and $v(y)$ the value functions

$$u(x) = \sup_{X_T \in C(x)} \mathbb{E}[U(X_T)], \qquad x \in \mathrm{dom}(U), \tag{52}$$

$$v(y) = \mathbb{E}\left[V\left(y\tfrac{dQ}{d\mathbf{P}}\right)\right], \qquad y > 0. \tag{53}$$

We then have:

(i) The value functions $u(x)$ and $v(y)$ are conjugate and u inherits the qualitative properties of U listed in Assumption 1.2.

(ii) The optimizer $\widehat{X}_T(x)$ in (52) exists, is unique and satisfies

$$\widehat{X}_T(x) = I(y\tfrac{dQ}{d\mathbf{P}}), \quad \textit{or, equivalently,} \quad y\tfrac{dQ}{d\mathbf{P}} = U'(\widehat{X}_T(x)), \tag{54}$$

where $x \in \mathrm{dom}(U)$ and $y > 0$ are related via $u'(x) = y$ or, equivalently, $x = -v'(y)$.

(iii) The following formulae for u' and v' hold true:

$$u'(x) = \mathbb{E}_{\mathbf{P}}[U'(\widehat{X}_T(x))], \qquad v'(y) = \mathbb{E}_Q\left[V'\left(y\tfrac{dQ}{d\mathbf{P}}\right)\right] \tag{55}$$

$$xu'(x) = \mathbb{E}_{\mathbf{P}}\left[\widehat{X}_T(x)U'(\widehat{X}_T(x))\right], \qquad yv'(y) = \mathbb{E}_{\mathbf{P}}\left[y\tfrac{dQ}{d\mathbf{P}}V'\left(y\tfrac{dQ}{d\mathbf{P}}\right)\right]. \tag{56}$$

Proof Items (i) and (ii) have been shown in the preceding discussion, hence we only have to show (iii). The formulae for $v'(y)$ in (55) and (56) immediately follow by differentiating the relation

$$v(y) = \mathbb{E}_{\mathbf{P}}\left[V\left(y\tfrac{dQ}{d\mathbf{P}}\right)\right] = \sum_{n=1}^N p_n V\left(y\frac{q_n}{p_n}\right). \tag{57}$$

Of course, the formula for v' in (56) is an obvious reformulation of the one in (55). But we write both of them to stress their symmetry with the formulae for $u'(x)$.

The formula for u' in (55) translates via the relations exhibited in (ii) into the identity

$$y = \mathbb{E}_{\mathbf{P}}\left[y\tfrac{dQ}{d\mathbf{P}}\right], \tag{58}$$

while the formula for $u'(x)$ in (56) translates into

$$v'(y)y = \mathbb{E}_{\mathbf{P}}\left[V'\left(y\tfrac{dQ}{d\mathbf{P}}\right)y\tfrac{dQ}{d\mathbf{P}}\right], \tag{59}$$

which we just have seen to hold true. ■

Remark 2.2 Firstly, let us recall the economic interpretation of (54)

$$U'\left(\widehat{X}_T(x)(\omega_n)\right) = y\frac{q_n}{p_n}, \quad n = 1, \dots, N. \tag{60}$$

This equality means that, in every possible state of the world ω_n, the *marginal utility* $U'(\widehat{X}_T(x)(\omega_n))$ of the wealth of an optimally investing agent at time T is *proportional to the ratio of the price* q_n *of the corresponding Arrow-Debreu security* $\mathbf{1}_{\{\omega_n\}}$ *and the probability of its success* $p_n = \mathbf{P}[\omega_n]$. This basic relation was analyzed in the fundamental work of K. Arrow and G. Debreu and allows for a convincing economic interpretation: considering for a moment the situation where this proportionality relation fails to hold true, one immediately deduces from a marginal variation argument that the investment of the agent cannot be optimal. Hence for the optimal investment the proportionality must hold true. The above result also identifies the proportionality factor as $y = u'(x)$, where x is the initial endowment of the investor.

Theorem 2.1 indicates an easy way to solve the utility maximization at hand: calculate $v(y)$ by (53), which reduces to a simple one-dimensional computation; once we know $v(y)$, the theorem provides easy formulae to calculate all the other quantities of interest, e.g., $\widehat{X}_T(x)$, $u(x)$, $u'(x)$ etc.

Another message of the above theorem is that the value function $x \mapsto u(x)$ may be viewed as a utility function as well, sharing all the qualitative features of the original utility function U. This makes sense economically, as $u(x)$ denotes the expected utility at time T of an agent with initial endowment x, after having optimally invested in the financial market S.

Let us also give an economic interpretation of the formulae for $u'(x)$ in item (iii) along these lines: suppose the initial endowment x is varied to $x + h$, for some small real number h. The economic agent may use the additional endowment h to finance, in addition to the optimal pay-off function $\widehat{X}_T(x)$, h units of the cash account, thus ending up with the pay-off function $\widehat{X}_T(x)+h$ at time T. Comparing this investment strategy to the optimal one corresponding to the initial endowment $x+h$, which is $\widehat{X}_T(x+h)$, we obtain

$$\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} = \lim_{h\to 0}\frac{\mathbb{E}[U(\widehat{X}_T(x+h))-U(\widehat{X}_T(x))]}{h} \tag{61}$$

$$\geq \lim_{h\to 0}\frac{\mathbb{E}[U(\widehat{X}_T(x)+h)-U(\widehat{X}_T(x))]}{h} \tag{62}$$

$$= \mathbb{E}[U'(\widehat{X}_T(x))]. \tag{63}$$

Using the fact that u is differentiable, and that h may be positive as well as negative, we have found another proof of formula (55) for $u'(x)$; the economic interpretation of this proof is that the economic agent, who is optimally investing, is indifferent of first order towards a (small) additional investment into the cash account.

Playing the same game as above, but using the additional endowment $h\in\mathbb{R}$ to finance an additional investment into the optimal portfolio $\widehat{X}_T(x)$ (assuming, for simplicity, $x\neq 0$), we arrive at the pay-off function $\frac{x+h}{x}\widehat{X}_T(x)$. Comparing this investment with $\widehat{X}_T(x+h)$, an analogous calculation as in (61) leads to the formula for $u'(x)$ displayed in (56). The interpretation now is, that the optimally investing economic agent is indifferent of first order towards a marginal variation of the investment into the optimal portfolio.

It now becomes clear that formulae (55) and (56) for $u'(x)$ are just special cases of a more general principle: for each $f\in L^\infty(\Omega,\mathcal{F},\mathbf{P})$ we have

$$\mathbb{E}_Q[f]u'(x) = \lim_{h\to 0}\frac{\mathbb{E}_{\mathbf{P}}[U(\widehat{X}_T(x)+hf)-U(\widehat{X}_T(x))]}{h}. \tag{64}$$

The proof of this formula again is along the lines of (61) and the interpretation is the following: by investing an additional endowment $h\mathbb{E}_Q[f]$ to finance the contingent claim hf, the increase in expected utility is of first order equal to $h\mathbb{E}_Q[f]u'(x)$; hence again the economic agent is of first order indifferent towards an additional investment into the contingent claim f.

2.2 The Incomplete Case

We now drop the assumption that the set $\mathcal{M}^e(S)$ of equivalent martingale measures is reduced to a singleton (but we still remain in the framework of a finite probability space Ω) and replace it by Assumption 1.1 requiring that $\mathcal{M}^e(S)\neq\emptyset$.

In this setting it follows from basic linear algebra that a random variable $X_T(\omega_n)=\xi_n$ may be dominated by a random variable of the form $x+(H\cdot S)_T$ iff $\mathbb{E}_Q[X_T]=\sum_{n=1}^N q_n\xi_n\leq x$, for each $Q=(q_1\ldots,q_N)\in\mathcal{M}^a(S)$ (or equivalently, for every $Q\in\mathcal{M}^e(S)$). This basic result is proved in [KQ 95], [J 92], [AS 94], [DS 94] and [DS 98a] in varying degrees of generality; in the present finite-dimensional case this fact is straightforward to prove, using elementary linear algebra (see, e.g, [S 00a]).

In order to reduce the infinitely many constraints, where Q runs through $\mathcal{M}^a(S)$, to a finite number, make the easy observation that $\mathcal{M}^a(S)$ is a bounded, closed, convex polytope in $\mathbb{R}^N$ and therefore the convex hull of its finitely many extreme points $\{Q^1, \ldots, Q^M\}$. Indeed, $\mathcal{M}^a(S)$ is given by finitely many linear constraints. For $1 \leq m \leq M$, we identify Q^m with its probabilites $(q_1^m, \ldots, q_N^m)$.

Fixing the initial endowment $x \in \mathrm{dom}(U)$, we therefore may write the utility maximization problem (16) similarly as in (20) as a concave optimization problem over $\mathbb{R}^N$ with finitely many linear constraints:

$$(\mathbf{P_x}) \quad \mathbb{E}_{\mathbf{P}}\left[U(X_T)\right] = \sum_{n=1}^{N} p_n U(\xi_n) \to \max! \tag{65}$$

$$\mathbb{E}_{Q^m}\left[X_T\right] = \sum_{n=1}^{N} q_n^m \xi_n \leq x, \quad \text{for } m = 1, \ldots, M. \tag{66}$$

Writing again

$$C(x) = \left\{X_T \in L^0(\Omega, \mathcal{F}, \mathbf{P}) : \mathbb{E}[X_T] \leq x, \quad \text{for all } Q \in \mathcal{M}^a(S)\right\} \tag{67}$$

we define the value function

$$u(x) = \sup_{H \in \mathcal{H}} \mathbb{E}\left[U\left(x + (H \cdot S)_T\right)\right] = \sup_{X_T \in C(x)} \mathbb{E}[U(X_T)], \quad x \in \mathrm{dom}(U). \tag{68}$$

The Lagrangian now is given by

$$L(\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_M) \tag{69}$$

$$= \sum_{n=1}^{N} p_n U(\xi_n) - \sum_{m=1}^{M} \eta_m \left(\sum_{n=1}^{N} q_n^m \xi_n - x\right) \tag{70}$$

$$= \sum_{n=1}^{N} p_n \left(U(\xi_n) - \sum_{m=1}^{M} \frac{\eta_m q_n^m}{p_n} \xi_n\right) + \sum_{m=1}^{M} \eta_m x, \tag{71}$$

$$\text{where } \ (\xi_1, \ldots, \xi_N) \in \mathrm{dom}(U)^N, \quad (\eta_1, \ldots, \eta_M) \in \mathbb{R}_+^M. \tag{72}$$

Writing $y = \eta_1 + \ldots + \eta_M$, $\mu_m = \frac{\eta_m}{y}$, $\mu = (\mu_1, \ldots, \mu_m)$ and

$$Q^\mu = \sum_{m=1}^{M} \mu_m Q^m, \tag{73}$$

note that, when $(\eta_1, \ldots, \eta_M)$ runs trough $\mathbb{R}_+^M$, the pairs (y, Q^μ) run through $\mathbb{R}_+ \times \mathcal{M}^a(S)$. Hence we may write the Lagrangian as

$$\begin{aligned} &L(\xi_1, \ldots, \xi_N, y, Q) = \\ &= \mathbb{E}_{\mathbf{P}}[U(X_T)] - y\left(\mathbb{E}_Q[X_T - x]\right) \\ &= \sum_{n=1}^{N} p_n \left(U(\xi_n) - \frac{y q_n}{p_n} \xi_n\right) + yx, \\ &\quad \text{where } \ \xi_n \in \mathrm{dom}(U), \quad y > 0, \quad Q = (q_1, \ldots, q_N) \in \mathcal{M}^a(S). \end{aligned} \tag{74}$$

This expression is entirely analogous to (24), the only difference now being that Q runs through the set $\mathcal{M}^a(S)$ instead of being a fixed probability measure. Defining again

$$\Phi(\xi_1, \ldots, \xi_n) = \inf_{y>0, Q\in\mathcal{M}^a(S)} L(\xi_1, \ldots, \xi_N, y, Q), \tag{75}$$

and

$$\Psi(y, Q) = \sup_{\xi_1,\ldots,\xi_N} L(\xi_1, \ldots, \xi_N, y, Q), \tag{76}$$

we obtain, just as in the complete case,

$$\sup_{\xi_1,\ldots,\xi_N} \Phi(\xi_1, \ldots, \xi_N) = u(x), \quad x \in \mathrm{dom}(U), \tag{77}$$

and

$$\Psi(y, Q) = \sum_{n=1}^{N} p_n V\left(\frac{yq_n}{p_n}\right) + yx, \qquad y > 0, \quad Q \in \mathcal{M}^a(S), \tag{78}$$

where $(q_1, \ldots, q_N)$ denotes the probabilities of $Q \in \mathcal{M}^a(S)$. The minimization of Ψ will be done in two steps: first we fix $y > 0$ and minimize over $\mathcal{M}^a(S)$, i.e.,

$$\Psi(y) := \inf_{Q\in\mathcal{M}^a(S)} \Psi(y, Q), \quad y > 0. \tag{79}$$

For fixed $y > 0$, the continuous function $Q \to \Psi(y, Q)$ attains its minimum on the compact set $\mathcal{M}^a(S)$, and the minimizer $\widehat{Q}(y)$ is unique by the strict convexity of V. Writing $\widehat{Q}(y) = (\widehat{q}_1(y), \ldots, \widehat{q}_N(y))$ for the minimizer, it follows from $V'(0) = -\infty$ that $\widehat{q}_n(y) > 0$, for each $n = 1, \ldots, N$; in other words, $\widehat{Q}(y)$ is an equivalent martingale measure for S.

Defining the dual value function $v(y)$ by

$$\begin{aligned} v(y) &= \inf_{Q\in\mathcal{M}^a(S)} \sum_{n=1}^{N} p_n V\left(y\frac{q_n}{p_n}\right) & (80)\\ &= \sum_{n=1}^{N} p_n V\left(y\frac{\widehat{q}_n(y)}{p_n}\right) & (81)\end{aligned}$$

we find ourselves in an analogous situation as in the complete case above: defining again $\widehat{y}(x)$ by $v'(\widehat{y}(x)) = -x$ and

$$\widehat{\xi}_n = I\left(\widehat{y}(x)\frac{\widehat{q}_n(y)}{p_n}\right), \tag{82}$$

similar arguments as above apply to show that $(\widehat{\xi}_1, \ldots, \widehat{\xi}_N, \widehat{y}(x), \widehat{Q}(y))$ is the unique saddle-point of the Lagrangian (74) and that the value functions u and v are conjugate.

Let us summarize what we have found in the incomplete case:

Theorem 2.3 (finite Ω, incomplete market) *Let the financial market $S = (S_t)_{t=0}^T$ defined over the finite filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t=0}^T, \mathbf{P})$ and let $\mathcal{M}^e(S) \neq \emptyset$, and the utility function U satisfies Assumptions 1.2.*

Denote by $u(x)$ and $v(y)$ the value functions

$$u(x) = \sup\nolimits_{X_T \in C(x)} \mathbb{E}[U(X_T)], \qquad x \in \mathrm{dom}(U), \tag{83}$$

$$v(y) = \inf\nolimits_{Q \in \mathcal{M}^a(S)} \mathbb{E}\left[V\left(y\tfrac{dQ}{d\mathbf{P}}\right)\right], \qquad y > 0. \tag{84}$$

We then have:

(i) The value functions $u(x)$ and $v(y)$ are conjugate and u shares the qualitative properties of U listed in Assumption 1.2.

(ii) The optimizers $\widehat{X}_T(x)$ and $\widehat{Q}(y)$ in (83) *and* (84) *exist, are unique, $\widehat{Q}(y) \in \mathcal{M}^e(S)$, and satisfy*

$$\widehat{X}_T(x) = I\left(y\frac{d\widehat{Q}(y)}{d\mathbf{P}}\right), \qquad y\frac{d\widehat{Q}(y)}{d\mathbf{P}} = U'(\widehat{X}_T(x)), \tag{85}$$

where $x \in \mathrm{dom}(U)$ and $y > 0$ are related via $u'(x) = y$ or, equivalently, $x = -v'(y)$.

(iii) The following formulae for u' and v' hold true:

$$u'(x) = \mathbb{E}_{\mathbf{P}}[U'(\widehat{X}_T(x))], \qquad v'(y) = \mathbb{E}_{\widehat{Q}}\left[V'\left(y\tfrac{d\widehat{Q}(y)}{d\mathbf{P}}\right)\right] \tag{86}$$

$$xu'(x) = \mathbb{E}_{\mathbf{P}}[\widehat{X}_T(x)U'(\widehat{X}_T(x))], \qquad yv'(y) = \mathbb{E}_{\mathbf{P}}\left[y\tfrac{d\widehat{Q}(y)}{d\mathbf{P}}V'\left(y\tfrac{d\widehat{Q}(y)}{d\mathbf{P}}\right)\right] \tag{87}$$

Remark 2.4 Let us again interpret the formulae (86), (87) for $u'(x)$ similarly as in Remark 2.2 above. In fact, the interpretations of these formulae as well as their derivations remain in the incomplete case exactly the same.

But a new and interesting phenomenon arises when we pass to the variation of the optimal pay-off function $\widehat{X}_T(x)$ by a small unit of an arbitrary pay-off function $f \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})$. Similarly as in (64) we have the formula

$$\mathbb{E}_{\widehat{Q}(y)}[f]u'(x) = \lim_{h \to 0} \frac{\mathbb{E}_{\mathbf{P}}[U(\widehat{X}_T(x) + hf) - U(\widehat{X}_T(x))]}{h}, \tag{88}$$

the only difference being that Q has been replaced by $\widehat{Q}(y)$ (recall that x and y are related via $u'(x) = y$).

The remarkable feature of this formula is that it does not only pertain to variations of the form $f = x + (H \cdot S)_T$, i.e, contingent claims attainable at price x, but to arbitrary contingent claims f, for which — in general — we cannot derive the price from no arbitrage considerations.

The economic interpretation of formula (88) is the following: the pricing rule $f \mapsto \mathbb{E}_{\widehat{Q}(y)}[f]$ yields precisely those prices, at which an economic agent with initial endowment x, utility function U and investing optimally, is indifferent of first order towards adding a (small) unit of the contingent claim f to her portfolio $\widehat{X}_T(x)$.

In fact, one may turn the view around, and this was done by M. Davis [D 97] (compare also the work of L. Foldes [F 90]): one may *define* $\widehat{Q}(y)$ by (88), verify that this indeed is an equivalent martingale measure for S, and interpret this pricing rule as "pricing by marginal utility", which is, of course, a classical and basic paradigm in economics.

Let us give a proof for (88) (under the hypotheses of Theorem 2.3). One possibility, which also has the advantage of a nice economic interpretation, is the idea of introducing "fictitious securities" as developed in [KLSX 91]: fix $x \in \mathrm{dom}(U)$ and $y = u'(x)$ and let $(f^1, \dots, f^k)$ be finitely many elements of $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ such that the space $K = \{(H \cdot S)_T : H \in \mathcal{H}\}$, the constant function $\mathbf{1}$, and $(f^1, \dots, f^k)$ linearly span $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$. Define the k processes

$$S_t^{d+j} = \mathbb{E}_{\widehat{Q}(y)}[f^j | \mathcal{F}_t], \quad j = 1, \dots, k, \quad t = 0, \dots, T. \tag{89}$$

Now extend the $\mathbb{R}^{d+1}$-valued process $S = (S^0, \dots, S^d)$ to the $\mathbb{R}^{d+k+1}$-valued process $\overline{S} = (S^0, \dots, S^d, S^{d+1}, \dots, S^{d+k})$ by adding these new coordinates. By (89) we still have that $\overline{S}$ is a martingale under $\widehat{Q}(y)$, which now is the unique probability under which $\overline{S}$ is a martingale, by our choice of $(f^1, \dots, f^k)$.

Hence we find ourselves in the situation of Theorem 2.1. By comparing (54) and (85) we observe that the optimal pay-off function $\widehat{X}_T(x)$ has not changed. Economically speaking this means that in the "completed" market $\overline{S}$ the optimal investment may still be achieved by trading only in the first $d+1$ assets and without touching the "fictitious" securities $S^{d+1}, \dots, S^{d+k}$.

In particular, we now may apply formula (64) to $Q = \widehat{Q}(y)$ to obtain (88).

Finally remark that the pricing rule induced by $\widehat{Q}(y)$ is precisely such that the interpretation of the optimal investment $\widehat{X}_T(x)$ defined in (85) (given in Remark 2.2 in terms of marginal utility and the ratio of Arrow-Debreu prices $\widehat{q}_n(y)$ and probabilities p_n) carries over to the present incomplete setting. The above completion of the market by introducing "fictious securities" allows for an economic interpretation of this fact.

3 The general case

In the previous section we have analyzed the duality theory of the optimization problem (1) in detail and with full proofs, for the case when the underlying probability space is finite.

We now pass to the question under which conditions the crucial features of the above Theorem 2.3 carry over to the general setting. In particular one is naturally led to ask: under which conditions

- are the optimizers $\widehat{X}_T(x)$ and $\widehat{Q}(y)$ of the value functions $u(x)$ and $v(y)$ attained?

- does the basic duality formula

$$U'\left(\widehat{X}_T(x)\right) = \widehat{y}(x)\frac{d\widehat{Q}(\widehat{y}(x))}{d\mathbf{P}} \tag{90}$$

  or, equivalently

$$\widehat{X}_T(x) = I\left(\widehat{y}(x)\frac{d\widehat{Q}(\widehat{y}(x))}{d\mathbf{P}}\right) \tag{91}$$

  hold true?

- are the value functions $u(x)$ and $v(y)$ conjugate?

- does the value function $u(x)$ still inherit the qualitative properties of U listed in Assumption 1.2?

- do the formulae for $u'(x)$ still hold true?

We shall see that we get affirmative answers to these questions under two provisos: firstly, one has to make an appropriate choice of the sets in which X_T and Q are allowed to vary. This choice will be different for case 1, where $\mathrm{dom}(U) = \mathbb{R}_+$, and case 2, where $\mathrm{dom}(U) = \mathbb{R}$. Secondly, the utility function U has to satisfy — in addition to Assumption 1.2 — a mild regularity condition, namely the property of *"reasonable asymptotic elasticity"*.

The essential message of the theorems below is that, assuming that U has *"reasonable asymptotic elasticity"*, the duality theory works just as well as in the case of finite Ω. Note that we do not have to impose any regularity conditions on the underlying stochastic process S, except for its arbitrage-freeness in the sense made precise by Assumption 1.1. On the other hand, the assumption of reasonable asymptotic elasticity on the utility function U cannot be relaxed, even if we impose very strong assumptions on the process S (e.g., having continuous paths and defining a complete financial market), as we shall see below.

Before passing to the positive results we first analyze the notion of "reasonable asymptotic elasticity" and sketch the announced counterexample.

Definition 3.1 *A utility function U satisfying Assumption 1.2 is said to have* "reasonable asymptotic elasticity" *if*

$$\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1, \tag{92}$$

and, in case 2 of Assumption 1.2, we also have

$$\liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1. \tag{93}$$

Let us discuss the economic meaning of this notion: as H.-U. Gerber observed, the quantity $\frac{xU'(x)}{U(x)}$ is the elasticity of the function U at x. We are interested in its asymptotic behaviour. It easily follows from Assumption 1.2 that the limits in (92) and (93) are less (resp. bigger) than or equal to one. What does it mean that $\frac{xU'(x)}{U(x)}$ tends to one, for $x \mapsto \infty$? It means that the ratio between the *marginal utility* $U'(x)$ and the *average utility* $\frac{U(x)}{x}$ tends to one. A typical example is a function $U(x)$ which equals $\frac{x}{\ln(x)}$, for x large enough; note however, that in this example Assumption 1.2 is not violated insofar as the marginal utility still decreases to zero for $x \to \infty$, i.e., $\lim_{x\to\infty} U'(x) = 0$.

If the marginal utility $U'(x)$ is approximately equal to the average utility $\frac{U(x)}{x}$ for large x, this means that for an economic agent, modeled by the utility function U, the increase in utility by varying wealth from x to $x+1$, when x is large, is approximately equal to the average of the increase of utility by changing wealth from n to $n+1$, where n runs through $1, 2, \ldots, x-1$ (we assume in this argument that x is a large natural number and, w.l.o.g., that $U(1) \approx 0$). We feel that the economic intuition behind decreasing marginal utility suggests that, for large x, the marginal utility $U'(x)$ should be substantially smaller than the average utility $\frac{U(x)}{x}$. Therefore we have denoted a utility function, where the ratio of $U'(x)$ and $\frac{U(x)}{x}$ tends to one, as being "unreasonable". Another justification for this terminology will be the results of Example 3.2 and Theorems 3.4 and 3.5 below.

P. Guasoni observed, that there is a close connection between the asymptotic behaviour of the *elasticity* of U, and the asymptotic behaviour of the *relative risk aversion* associated to U. Recall (see, e.g., [HL88]) that the relative risk aversion of an agent with endowment x, whose preferences are described by the utility function U, equals

$$RRA(U)(x) = -\frac{xU''(x)}{U'(x)}. \tag{94}$$

A formal application of de l'Hôpital's rule yields

$$\lim_{x\mapsto\infty} \frac{xU'(x)}{U(x)} = \lim_{x\mapsto\infty} \frac{U'(x) + xU''(x)}{U'(x)} = 1 - \lim_{x\mapsto\infty} \left(-\frac{xU''(x)}{U'(x)}\right) \tag{95}$$

which insinuates that the asymptotic elasticity of U is less than one iff the "*asymptotic relative risk aversion*" is strictly positive.

Turning the above formal argument into a precise statement, one easily proves the following result: if $\lim_{x\mapsto\infty}(-\frac{xU''(x)}{U'(x)})$ exists, then $\lim_{x\mapsto\infty} \frac{xU'(x)}{U(x)}$ exists too, and the former is strictly positive iff the latter is less than one. Hence "*essentially*" these two concepts coincide.

On the other hand, in general (i.e. without assuming that the above limit exists), there is no way to characterize the condition $\limsup_{x\mapsto\infty} \frac{xU'(x)}{U(x)} < 1$ in terms of the asymptotic behaviour of $-\frac{xU''(x)}{U'(x)}$, as $x \mapsto \infty$. Firstly, for the second expression to make sense, we have to assume that U is twice differentiable;

but, even doing so, does not help, as it is easy to construct examples where $\frac{xU'(x)}{U(x)}$ converges (to 1 or to a number less than 1), while $-\frac{xU''(x)}{U'(x)}$ oscillates wildly in $]0,\infty[$, as $x \mapsto \infty$.

We shall see that the assumption of reasonable asymptotic elasticity is *necessary and sufficient* for several key results in the duality theory of utility maximization to hold true. Hence in order to obtain these sharp results, we cannot reformulate things in terms of the asymptotic relative risk aversion. However, readers which are happy with sufficient conditions, may replace the assumption $\limsup_{x\mapsto\infty} \frac{xU'(x)}{U(x)} < 1$ by the assumption $\liminf_{x\mapsto\infty}(-\frac{xU''(x)}{U'(x)}) > 0$ below: it is easy to verify that the latter assumption implies the former (but not vice versa).

Similar reasoning applies to the asymptotic behaviour of $\frac{xU'(x)}{U(x)}$, as x tends to $-\infty$, in case 2. In this context the typical counter-example is $U(x) \sim x\ln(|x|)$, for $x < x_0$; in this case one finds similarly

$$\lim_{x\to-\infty} U'(x) = \infty, \quad \text{while} \quad \lim_{x\to-\infty} \frac{xU'(x)}{U(x)} = 1. \tag{96}$$

The message of Definition 3.1 above is — roughly speaking — that we want to exclude utility functions U which behave like $U(x) \sim \frac{x}{\ln(x)}$, as $x \to \infty$, or $U(x) \sim x\ln|x|$, as $x \to -\infty$. Similar (but not quite equivalent) notions comparing the behaviour of $U(x)$ with that of power functions in the setting of case 1, were defined and analyzed in [KLSX 91] (see [KS 99], lemma 6.5, for a comparison of these concepts).

We start with a sketch of a counterexample showing the relevance of the notion of asymptotic elasticity in the context of utility maximization: *whenever U fails to have reasonable asymptotic elasticity* the duality theory breaks down in a rather dramatic way. We only state the version of the counterexample where both assumptions (92) and (93) are violated and refer to [KS 99] and [S 00a] for the other cases.

Example 3.2 ([S 00], prop. 3.5) Let U be *any utility function* satisfying Assumption 1.2, case 2 and such that

$$\lim_{x\to-\infty} \frac{xU'(x)}{U(x)} = \lim_{x\to\infty} \frac{xU'(x)}{U(x)} = 1, \tag{97}$$

Then there is an $\mathbb{R}$-valued process $(S_t)_{0\le t\le T}$ of the form

$$S_t = \exp\left(B_t + \mu_t\right), \tag{98}$$

where $B = (B_t)_{0\le t\le T}$ is a standard Brownian motion, based on its natural filtered probability space, and μ_t a predictable process, such that the following properties hold true:

(i) $\mathcal{M}^e(S) = \{Q\}$, i.e., S defines a complete financial market.

(ii) The primal value function $u(x)$ fails to be strictly concave and to satisfy $u'(\infty) = 0$, $u'(-\infty) = \infty$ in a rather striking way: $u(x)$ is a straight line of the form $u(x) = c + x$, for some constant $c \in \mathbb{R}$.

(iii) The optimal investment $\widehat{X}_T(x)$ fails to exist, for all $x \in \mathbb{R}$, except for one point $x = x_0$. In particular, for $x \neq x_0$, the formula (91) does not define the optimal investment $\widehat{X}_T(x)$.

(iv) The dual value function v fails to be a finite, smooth, strictly convex function on $\mathbb{R}_+$ in a rather striking way: in fact, $v(1) < \infty$ while $v(y) = \infty$, for all $y > 0$, $y \neq 1$.

We do not give a rigorous proof for these assertions but refer to [S 00, Proposition 3.5], which in turn is a variant of [KS 99, Proposition 5.3].

We shall try to sketch the basic idea underlying the construction of the example, in mathematical as well as economic terms. Arguing mathematically, one starts by translating the assumptions (97) on the utility function U into equivalent properties of the conjugate function V: roughly speaking, the corresponding property of $V(y)$ is, that it increases very rapidly to infinity, as $y \to 0$ and $y \to \infty$ (see [KS 99, Corolary 6.1] and [S 00, Proposition 4.1]). Having isolated this property of V, it is an easy exercise to construct a function $f : [0, 1] \to]0, \infty[$, $\mathbb{E}[f] = 1$ such that

$$\mathbb{E}[V(f)] < \infty \quad \text{while} \quad \mathbb{E}[V(yf)] = \infty, \quad \text{for } y \neq 1, \tag{99}$$

where $\mathbb{E}$ denotes expectation with respect to Lebesque measure λ. In fact one may find such a function f taking only the values $(y_n)_{n=-\infty}^{\infty}$, for a suitable chosen increasing sequence $(y_n)_{n=-\infty}^{\infty}$, $\lim_{n\to-\infty} y_n = 0$, $\lim_{n\to\infty} y_n = \infty$.

Next we construct a measure Q on the sigma algebra $\mathcal{F} = \mathcal{F}_T$ generated by the Brownian motion $B = (B_t)_{0 \leq t \leq T}$ which is equivalent to Wiener measure $\mathbf{P}$, and such that the distribution of $\frac{dQ}{d\mathbf{P}}$ (under $\mathbf{P}$) equals that of f (under Lebesgue measure λ). There is no uniqueness in this part of the construction, but it is straightforward to find some appropriate measure Q with this property.

By Girsanov's theorem we know that we can find an adapted process $(\mu_t)_{0 \leq t \leq T}$, such that Q is the unique equivalent local martingale measure for the process defined in (98), hence we obtain assertion (i).

This construction makes sure that we obtain property (iv), i.e.

$$v(y) = \mathbb{E}_{\mathbf{P}}\left[V\left(y\frac{dQ}{d\mathbf{P}}\right)\right] = \mathbb{E}_\lambda\left[V(yf)\right] < \infty \text{ iff } y = 1. \tag{100}$$

Once this crucial property is established, most of the assertions made in (ii) and (iii) above easily follow (in fact, for the existence of $\widehat{X}_T(x)$ for precisely one $x = x_0$, some extra care is needed).

Instead of elaborating further on the mathematical details of the construction sketched above, let us try to give an economic interpretation of what is really happening in the above example. This is not easy, but we find it worth trying. We concentrate on the behaviour of U as $x \to \infty$, the case when $x \to -\infty$ being similar.

How is the "unreasonability" property of the utility function U used to construct the pathologies in the above example? Here is a rough indication of the underlying economic idea: the financial market S is constructed in such a way that one may find positive numbers $(x_n)_{n=1}^{\infty}$, disjoint sets $(A_n)_{n=1}^{\infty}$ in $\mathcal{F}_T$, with $\mathbf{P}[A_n] = p_n$ and $Q[A_n] = q_n$, such that for the contingent claims $x_n \mathbf{1}_{A_n}$ we approximately have

$$\mathbb{E}_Q \left[x_n \mathbf{1}_{A_n} \right] = q_n x_n \approx 1 \tag{101}$$

and

$$\mathbb{E}_{\mathbf{P}} \left[U(x_n) \mathbf{1}_{A_n} \right] = p_n U(x_n) \approx 1. \tag{102}$$

Hence $\frac{q_n}{p_n} \approx \frac{U(x_n)}{x_n}$.

It is easy to construct a complete, continuous market S over the Brownian filtration such that this situation occurs and this is, in fact, what is done in the above "mathematical" argument to define f and Q. We remark in passing that one might just as well construct S as a complete, discrete time model $S = (S_t)_{t=0}^{\infty}$ over a countable probability space Ω displaying sets A_n and real numbers x_n having the properties listed above. But for esthetical reasons we have prefered to do the construction in terms of an exponential Brownian motion with drift.

We claim that, for any $x \in \mathbb{R}$ and any investment strategy $X_T = x + (H \cdot S)_T$, we can find an investment strategy $\widetilde{X}_T = (x+1) + (\widetilde{H} \cdot S)_T$ such that

$$\mathbb{E} \left[U(\widetilde{X}_T) \right] \approx \mathbb{E} \left[U(X_T) \right] + 1. \tag{103}$$

The above relation should motivate why the value function $u(x)$ becomes a straight line with slope one, at least for x sufficiently large (for the corresponding behaviour of $u(x)$ on the left hand side of $\mathbb{R}$ one has to play in addition a similar game as above with $(x_n)_{n=1}^{\infty}$ tending to $-\infty$).

To present the idea behind (103), suppose that we have $\mathbb{E}[U(X_T)] < \infty$, so that $\lim_{n\to\infty} \mathbb{E}[U(X_T)\mathbf{1}_{A_n}] = 0$. Varying our initial endowment from x to $x + 1$ €, we may use the additional € to add to the pay-off function X_T the function $x_n \mathbf{1}_{A_n}$, for some large n; by (101) this may be financed (approximately) with the additional € and by (102) this will increase the expected utility (approximately) by 1

$$\begin{aligned} \mathbb{E}\left[U\left(X_T + x_n \mathbf{1}_{A_n}\right)\right] &\approx \mathbb{E}\left[U(X_T)\mathbf{1}_{\Omega \setminus A_n}\right] + \mathbb{E}\left[U(X_T + x_n)\mathbf{1}_{A_n}\right] \\ &\approx \mathbb{E}\left[U(X_T)\right] + p_n U(x_n) \\ &\approx \mathbb{E}\left[U(X_T)\right] + 1, \end{aligned} \tag{104}$$

which was claimed in (103).

The above argument also gives a hint why we cannot expect that the optimal strategy $\widehat{X}_T(x) = x + (\widehat{H} \cdot S)_T$ exists, as one cannot "pass to the limit as $n \to \infty$" in the above reasoning.

Observe that we have not yet used the assumption $\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} = 1$, as it always is possible to construct things in such a way that (101) and (102) hold true (provided only that $\lim_{x\to\infty} U(x) = \infty$, which we assume from now on). How does the "unreasonable asymptotic elasticity" come into play? The point is that we have to do the construction described in (101) and (102) without violating Assumption 1.3, i.e.,

$$
\begin{aligned}
u(x) \quad = \quad & \sup_{H\in\mathcal{H}} \mathbb{E}[U(x + (H\cdot S)_T)] < \infty, \\
& \qquad\qquad \text{for some (equivalently, for all)} \;\; x \in \mathbb{R}. \qquad (105)
\end{aligned}
$$

In order to satisfy Assumption 1.3 we have to make sure that

$$
\mathbb{E}\left[\sum_{n=1}^{\infty} U(\mu_n x_n)\mathbf{1}_{A_n}\right] = \sum_{n=1}^{\infty} p_n U(\mu_n x_n) \qquad (106)
$$

remains bounded, when $(\mu_n)_{n=1}^{\infty}$ runs through all convex weights $\mu_n \geq 0$, $\sum_{n=1}^{\infty} \mu_n = 1$, i.e., when we consider all investments into non-negative linear combinations of the contingent claims $x_n \mathbf{1}_{A_n}$, which can be financed with one €.

The message of Example 3.2 is that this is not possible, if and only if $\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} = 1$ (for this part of the construction we only use the asymptotic behaviour of $U(x)$, as $x \to \infty$). To motivate this claim, think for a moment of the "reasonable" case, e.g., $U(x) = \frac{x^\alpha}{\alpha}$, for some $0 < \alpha < 1$, in which case we have $\lim_{x\to\infty} \frac{xU'(x)}{U(x)} = \alpha < 1$. Letting $\mu_n \approx n^{-(1+\epsilon)}$, we get

$$
\begin{aligned}
\sum_{n=1}^{\infty} p_n U(\mu_n x_n) \quad &\approx \quad \sum_{n=1}^{\infty} n^{-(1+\epsilon)\alpha} p_n U(x_n) \qquad (107) \\
&\approx \quad \sum_{n=1}^{\infty} n^{-(1+\epsilon)\alpha}, \qquad (108)
\end{aligned}
$$

which equals infinity if $\epsilon > 0$ is small enough, that $(1+\epsilon)\alpha \leq 1$. This argument indicates that in the case of the power utility $U(x) = \frac{x^\alpha}{\alpha}$ it is impossible to reconcile the validity of (101) and (102) with the requirement (106). On the other hand, it turns out that in the "unreasonable" case, where we have $\lim_{x\to\infty} \frac{xU'(x)}{U(x)} = 1$, we can do the construction in such a way that $U(\mu_n x_n)$ is sufficiently close to $\mu_n U(x_n)$ such that we obtain a uniform bound on the sum in (106).

Let us now stop our attempt at an economic interpretation. We hope that the above informal arguments were of some use for the reader in developing her intuition for the concept of "reasonable asymptotic elasticity" and that she now has some background information to find her way through the corresponding formal arguments in [KS 99] and [S 00].

We now pass to the positive results in the spirit of Theorem 2.1 and Theorem 2.3 above. We first consider the case where U satisfies the Inada conditions (7) and (8), which was studied in [KS 99].

Case 1: $\operatorname{dom}(U) = \mathbb{R}_+$.

The heart of the argument in the proof of Theorem 2.3 (which we now want to extend to the general case) is the applicability of the minimax theorem, which underlies the theory of Lagrange multipliers. We want to extend the applicability of the minimax theorem to the situation. The infinite-dimensional versions of the minimax theorem available in the literature (see, e.g, [ET 76] or [St 85]) are along the following lines: Let $\langle E, F\rangle$ be a pair of locally convex vector spaces in separating duality, $C \subseteq E$, $D \subseteq F$ a pair of convex subsets, and $L(x, y)$ a function defined on $C \times D$, concave in the first and convex in the second variable, having some (semi-)continuity property compatible with the topologies of E and F (which in turn should be compatible with the duality between E and F). If one of the sets C and D is compact and the other is complete, then one may assert the existence of a saddle point $(\widehat{\xi}, \widehat{\eta}) \in C \times D$ such that

$$L(\widehat{\xi}, \widehat{\eta}) = \sup_{\xi\in C} \inf_{\eta\in D} L(\xi, \eta) = \inf_{\eta\in D} \sup_{\xi\in C} L(\xi, \eta). \tag{109}$$

We try to apply this theorem to the analogue of the Lagrangian encountered in the proof of Theorem 2.3 above. Fixing $x > 0$ and $y > 0$ let us formally write the Lagrangian (74) in the infinite-dimensional setting,

$$\begin{aligned} L^{x,y}(X_T, Q) &= \mathbb{E}_{\mathbf{P}}[U(X_T)] - y(\mathbb{E}_Q[X_T - x]) & (110)\\ &= \mathbb{E}_{\mathbf{P}}\left[U(X_T) - y\tfrac{dQ}{d\mathbf{P}}X_T\right] + yx, & (111)\end{aligned}$$

where X_T runs through "all" non-negative $\mathcal{F}_T$-measurable functions and Q through the set $\mathcal{M}^a(S)$ of absolutely continuous local martingale measures.

To restrict the set of "all" nonnegative functions to a more amenable one note that $\inf_{y>0, Q\in\mathcal{M}^a(S)} L^{x,y}(X_T, Q) > -\infty$ iff

$$\mathbb{E}_Q[X_T] \le x, \quad \text{for all} \quad Q \in \mathcal{M}^a(S). \tag{112}$$

Using the basic result on the super-replicability of the contingent claim X_T (see [KQ 95], [J 92], [AS 94], [DS 94], and [DS 98b]), we have — as encountered in the finite dimensional case — that a non-negative $\mathcal{F}_T$-measurable random variable X_T satisfies (112) iff there is an admissible trading strategy H such that

$$X_T \le x + (H \cdot S)_T. \tag{113}$$

Hence let

$$\begin{aligned} C(x) \;=\; \big\{&X_T \in L^0_+(\Omega, \mathcal{F}_T, \mathbf{P}) : \\ &X_T \le x + (H\cdot S)_T, \text{ for some admissible } H\big\} \end{aligned} \tag{114}$$

and simply write C for $C(1)$ (observe that $C(x) = xC$).

We thus have found a natural set $C(x)$ in which X_T should vary when we are mini-maxing the Lagrangian $L^{x,y}$. Dually, the set $\mathcal{M}^a(S)$ seems to be the natural domain where the measure Q is allowed to vary (in fact, we shall see later, that this set still has to be slightly enlarged). But what are the locally convex vector spaces E and F in separating duality into which C and $\mathcal{M}^a(S)$ are naturally embedded? As regards $\mathcal{M}^a(S)$ the natural choice seems to be $L^1(\mathbf{P})$ (by identifying a measure $Q \in \mathcal{M}^a(S)$ with its Radon-Nikodym derivative $\frac{dQ}{d\mathbf{P}}$); note that $\mathcal{M}^a(S)$ is a *closed* subset of $L^1(\mathbf{P})$, which is good news. On the other hand, there is no reason for C to be contained in $L^\infty(\mathbf{P})$, or even in $L^p(\mathbf{P})$, for any $p > 0$; the natural space in which C is embedded is just $L^0(\Omega, \mathcal{F}_T, \mathbf{P})$, the space of all real-valued $\mathcal{F}_T$-measurable functions endowed with the topology of convergence in probability.

The situation now seems hopeless (if we don't want to impose artificial $\mathbf{P}$-integrability assumptions on X_T and/or $\frac{dQ}{d\mathbf{P}}$), as $L^0(\mathbf{P})$ and $L^1(\mathbf{P})$ are not in any reasonable duality; in fact, $L^0(\mathbf{P})$ is not even a locally convex space, hence there seems to be no hope for a good duality theory, which could serve as a basis for the application of the minimax theorem. But the good news is that the sets C and $\mathcal{M}^a(S)$ are in the *positive orthant* of $L^0(\mathbf{P})$ and $L^1(\mathbf{P})$ respectively; the crucial observation is, that for $f \in L^0_+(\mathbf{P})$ and $g \in L^1_+(\mathbf{P})$, it is possible to well-define

$$\langle f, g \rangle := \mathbb{E}_{\mathbf{P}}[fg] \in [0, \infty]. \tag{115}$$

The spirit here is similar as in the very foundation of Lebesgue integration theory: For positive measurable functions the integral is always defined, but possibly $+\infty$. This does not cause any logical inconsistency.

Similarly the bracket $\langle \cdot, \cdot \rangle$ defined in (115) shares many of the usual properties of a scalar product. The difference is that $\langle f, g \rangle$ now may assume the value $+\infty$ and that the map $(f, g) \mapsto \langle f, g \rangle$ is not continuous on $L^0_+(\mathbf{P}) \times L^1_+(\mathbf{P})$, but only lower semi-continuous (this immediately follows from Fatou's lemma).

At this stage it becomes clear that the role of $L^1_+(\mathbf{P})$ is somewhat artificial, and it is more natural to define (115) in the general setting where f and g are both allowed to vary in $L^0_+(\mathbf{P})$. The pleasant feature of the space $L^0(\mathbf{P})$ in the context of Mathematical Finance is, that it is invariant under the passage to an equivalent measure Q, a property only shared by $L^\infty(\mathbf{P})$, but by no other $L^p(\mathbf{P})$, for $0 < p < \infty$.

We now can turn to the polar relation between the sets C and $\mathcal{M}^a(S)$. By (113) we have, for an element $X_T \in L^0_+(\Omega, \mathcal{F}, \mathbf{P})$,

$$X_T \in C \Leftrightarrow \mathbb{E}_Q[X_T] = \mathbb{E}_{\mathbf{P}}[X_T \tfrac{dQ}{d\mathbf{P}}] \leq 1, \quad \text{for} \quad Q \in \mathcal{M}^a(S). \tag{116}$$

Denote by D the closed, convex, solid hull of $\mathcal{M}^a(S)$ in $L^0_+(\mathbf{P})$. It is easy to show (using, e.g., Lemma 3.3 below), that D equals

$$\begin{aligned} D \;=\; & \{Y_T \in L^0_+(\Omega, \mathcal{F}_T, \mathbf{P}) : \text{ there is} \\ & \quad (Q_n)_{n=1}^\infty \in \mathcal{M}^a(S) \text{ s.t. } Y_T \leq \lim_{n\to\infty} \tfrac{dQ_n}{d\mathbf{P}}\}, \end{aligned} \tag{117}$$

where the $\lim_{n\to\infty} \frac{dQ_n}{d\mathbf{P}}$ is understood in the sense of almost sure convergence. We have used the letter Y_T for the elements of D to stress the dual relation to the elements X_T in C. In further analogy we write, for $y > 0$, $D(y)$ for yD, so that $D = D(1)$. By (117) and Fatou's lemma we again find that, for $X_T \in L^0_+(\Omega, \mathcal{F}, \mathbf{P})$

$$X_T \in C \Leftrightarrow \mathbb{E}_{\mathbf{P}}[X_T Y_T] \leq 1, \quad \text{for} \quad Y_T \in D. \tag{118}$$

Why did we pass to this enlargement D of the set $\mathcal{M}^a(S)$? The reason is that we now obtain a more symmetric relation between C and D: for $Y_T \in L^0_+(\Omega, \mathcal{F}, \mathbf{P})$ we have

$$Y_T \in D \Leftrightarrow \mathbb{E}_{\mathbf{P}}[X_T Y_T] \leq 1, \quad \text{for} \quad X_T \in C. \tag{119}$$

The proof of (119) relies on an adaption of the "bipolar theorem" from the theory of locally convex spaces (see, e.g., [Sch 66]) to the present duality $\langle L^0_+(\mathbf{P}), L^0_+(\mathbf{P})\rangle$, which was worked out in [BS 99].

Why is it important to define the enlargement D of $\mathcal{M}^a(S)$ in such a way that (119) holds true? After all, $\mathcal{M}^a(S)$ is a nice, convex, closed (w.r.t. the norm of $L^1(\mathbf{P})$) set and we also have that, for $g \in L^1(\mathbf{P})$ *such that* $\mathbb{E}_{\mathbf{P}}[g] = 1$,

$$g \in \mathcal{M}^a(S) \Leftrightarrow \mathbb{E}_{\mathbf{P}}[X_T g] \leq 1, \quad \text{for} \quad X_T \in C. \tag{120}$$

The reason is that, in general, the saddle point $(\widehat{X}_T, \widehat{Q})$ of the Lagrangian will *not* be such that $\widehat{Q}$ is a probability measure; it will only satisfy $\mathbb{E}\left[\frac{d\widehat{Q}}{d\mathbf{P}}\right] \leq 1$, the inequality possibly being strict. But it will turn out that $\widehat{Q}$, which we identify with $\frac{d\widehat{Q}}{d\mathbf{P}}$, is always in D. In fact, the passage from $\mathcal{M}^a(S)$ to D is the *crucial feature* in order to make the duality work in the present setting: we shall see below that even for nice utility functions U, such as the logarithm, and for nice processes, such as a continuous process $(S_t)_{0 \leq t \leq T}$ based on the filtration of two Brownian motions, the above described phenomenon can occur: the saddle point of the Lagrangian leads out of $\mathcal{M}^a(S)$.

The set D can be characterized in several equivalent manners. We have defined D above in the abstract way as the convex, closed, solid hull of $\mathcal{M}^a(S)$ and mentioned the description (117). Equivalently, one may define D as the set of random variables $Y_T \in L^0_+(\Omega, \mathcal{F}, \mathbf{P})$ such that there is a process $(Y_t)_{0 \leq t \leq T}$ starting at $Y_0 = 1$ with $(Y_t X_t)_{0 \leq t \leq T}$ a $\mathbf{P}$-supermartingale, for every non-negative process $(X_t)_{0 \leq t \leq T} = (x + (H \cdot S)_t)_{0 \leq t \leq T}$, where $x > 0$ and H is predictable and S-integrable. This definition was used in [KS 99]. Another equivalent characterization was used in [CSW 00]: Consider the convex, solid hull of $\mathcal{M}^a(S)$, and embed this subset of $L^1(\mathbf{P})$ into the bidual $L^1(\mathbf{P})^{**} = L^\infty(\mathbf{P})^*$; denote by $\overline{\mathcal{M}^a(S)}$ the weak-star closure of the convex solid hull of $\mathcal{M}^a(S)$ in $L^\infty(\mathbf{P})^*$. Each element of $\overline{\mathcal{M}^a(S)}$ may be decomposed into its regular part $\mu^r \in L^1(\mathbf{P})$ and its purely singular part $\mu^s \in L^\infty(\mathbf{P})^*$. It turns out that D equals the set $\{\mu^r \in L^1(\mathbf{P}) : \mu \in \overline{\mathcal{M}^a(S)}\}$, i.e. consists of the regular parts of the elements of $\overline{\mathcal{M}^a(S)}$. This description has the advantage that we may associate to the elements $\mu^r \in D$ a singular part μ^s, and it is this extra information which is crucial when extending the present results to the case of random endowment (see [CSW 00]).

Why are the sets C and D hopeful candidates for the minimax theorem to work out properly for a function L defined on $C \times D$? Both are closed, convex and bounded subsets of $L^0_+(\mathbf{P})$. But recall that we still need some compactness property to be able to localize the mini-maximizers (resp. maxi-minimizers) on C (resp. D). In general, neither C nor D is compact (w.r.t. the topology of convergence in measure), i.e., for a sequence $(f_n)_{n=1}^\infty$ in C (resp. $(g_n)_{n=1}^\infty$ in D) we cannot pass to a subsequence converging in measure. But C and D have a property which is close to compactness and in many applications turns out to serve just as well.

Lemma 3.3 *Let A be a closed, convex, bounded subset of $L^0_+(\Omega, \mathcal{F}, \mathbf{P})$. Then for each sequence $(h_n)_{n=1}^\infty \in A$ there exists a sequence of convex combinations $k_n \in \mathrm{conv}(h_n, h_{n+1}, \ldots)$ which converges almost surely to a function $k \in A$.*

This easy lemma (see, e.g., [DS 94, Lemma A.1.1], for a proof) is in the spirit of the celebrated theorem of Komlos [Kom 67], stating that for a bounded sequence $(h_n)_{n=1}^\infty$ in $L^1(\mathbf{P})$ there is a subsequence converging in Cesaro-mean almost surely. The methodology of finding pointwise limits by using convex combinations has turned out to be extremely useful as a surrogate for compactness. For an extensive discussion of more refined versions of the above lemma and their applications to Mathematical Finance we refer to [DS 99].

The application of the above lemma is the following: by passing to convex combinations of optimizing sequences $(f_n)_{n=1}^\infty$ in C (resp. $(g_n)_{n=1}^\infty$ in D), we can always find limits $f \in C$ (resp. $g \in D$) w.r.t. almost sure convergence. Note that the passage to convex combinations does not cost more than passing to a subsequence in the application to convex optimization.

We have now given sufficient motivation to state the central result of [KS 99], which is the generalization of Theorem 2.3 to the semi-martingale setting under Assumption 1.2, case 1, and having reasonable asymptotic elasticity.

Theorem 3.4 ([KS 99], th. 2.2) *Let the semi-martingale $S = (S_t)_{0 \le t \le T}$ and the utility function U satisfy Assumptions 1.1, 1.2 case 1 and 1.3; suppose in addition that U has reasonable asymptotic elasticity. Define*

$$u(x) = \sup_{X_T \in C(x)} \mathbb{E}[U(X_T)], \qquad v(y) = \inf_{Y_T \in D(y)} \mathbb{E}[V(Y_T)]. \tag{121}$$

Then we have:

(i) The value functions $u(x)$ and $v(y)$ are conjugate; they are continuously differentiable, strictly concave (resp. convex) on $]0, \infty[$ and satisfy

$$u'(0) = -v'(0) = \infty, \qquad u'(\infty) = v'(\infty) = 0. \tag{122}$$

(ii) The optimizers $\widehat{X}_T(x)$ and $\widehat{Y}_T(y)$ in (121) *exist, are unique and satisfy*

$$\widehat{X}_T(x) = I(\widehat{Y}_T(y)), \qquad \widehat{Y}_T(y) = U'(\widehat{X}_T(x)), \tag{123}$$

where $x > 0$, $y > 0$ are related via $u'(x) = y$ or equivalently $x = -v'(y)$.

(iii) We have the following relations between u', v' and $\widehat{X}_T, \widehat{Y}_T$ respectively:

$$u'(x) = \mathbb{E}\left[\frac{\widehat{X}_T(x)U'(\widehat{X}_T(x))}{x}\right], \; x > 0, \;\; v'(y) = \mathbb{E}\left[\frac{\widehat{Y}_T(y)V'(\widehat{Y}_T(y))}{y}\right], \; y > 0. \tag{124}$$

For the proof of the theorem we refer to [KS 99].

We finish the discussion of utility functions satisfying the Inada conditions (7) and (8) by briefly indicating an example, when the dual optimizer $\widehat{Y}_T(y)$ fails to be of the form $\widehat{Y}_T(y) = y\frac{d\widehat{Q}(y)}{d\mathbf{P}}$, for some probability measure $\widehat{Q}(y)$.

It suffices to consider a stock-price process of the form

$$\begin{aligned} S_t &= \left(\exp\left(B_t + \tfrac{t}{2}\right)\right)^{\tau} \\ &= \exp\left(B_{t\wedge\tau} + \tfrac{t\wedge\tau}{2}\right), \qquad t \geq 0, \end{aligned} \tag{125}$$

where $(B_t)_{t\geq 0}$ is Brownian motion based on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t>0}, \mathbf{P})$ and τ a suitably chosen finite stopping time (to be discussed below) with respect to the filtration $(\mathcal{F}_t)_{t>0}$, after which the process S remains constant.

The usual way to find a risk-neutral measure Q for the process S above is to use Girsanov's formula, which amounts to considering

$$Z_\tau = \exp(-B_\tau - \tfrac{\tau}{2}) \tag{126}$$

as a candidate for the Radon-Nikodym derivative $\frac{dQ}{d\mathbf{P}}$.

It turns out that one may construct τ in such a way that the density process given by Girsanov's theorem

$$Z_t = \exp(-B_{t\wedge\tau} - \tfrac{t\wedge\tau}{2}), t > 0 \tag{127}$$

fails to be a uniformly integrable martingale: Then in particular

$$\mathbb{E}[Z_\tau] < 1. \tag{128}$$

The trick is to choose the filtration $(\mathcal{F}_t)_{t\geq 0}$ to be generated by two independent Brownian motions $(B_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$. Using the information of *both* $(B_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ one may define τ in a suitable way such that (128) holds true and nevertheless we have that $\mathcal{M}^e(S) \neq \emptyset$. In other words, there are equivalent martingale measures Q for the process S, but Girsanov's theorem fails to produce one.

This example is known for quite some time ([DS 98a]) and served as a kind of "universal counterexample" to several questions arising in Mathematical Finance.

How can one use this example in the present context? Consider the logarithmic utility $U(x) = \ln(x)$ and recall that its conjugate function V equals $V(y) = -\ln(y) - 1$. Hence the dual optimization problem — formally — is given by

$$\begin{aligned} \mathbb{E}\left[V\left(y\tfrac{dQ}{d\mathbf{P}}\right)\right] &= \mathbb{E}\left[-\ln\left(y\tfrac{dQ}{d\mathbf{P}}\right) - 1\right] = \\ &= -\mathbb{E}\left[\ln\left(\tfrac{dQ}{d\mathbf{P}}\right)\right] - (\ln(y) + 1) \longmapsto \min!, \quad Q \in \mathcal{M}^a(S). \end{aligned} \tag{129}$$

It is well known (see, e.g., the literature on the "numéraire portfolio" [L 90], [J 96], [A 97] and [B 00]), that for a process $(S_t)_{t\geq 0}$ based, e.g., on the filtration generated by an n-dimensional Brownian motion, the martingale measure obtained from applying Girsanov's theorem (which equals the "minimal martingale measure" investigated by Föllmer and Schweizer [FS 91]) is the minimizer for (129), *provided it exists*.

In the present example we have seen that the candidate for the density of the minimal martingale measure Z_τ obtained from a formal application of Girsanov's theorem fails to have full measure; but nevertheless one may show that Z_τ is the optimizer of the dual problem (125), which shows in particular that we have to pass from $\mathcal{M}^a(S)$ to the larger set D to find the dual optimizer in (129).

Passing again to the general setting of Theorem 3.4 one might ask: how severe is the fact that the dual optimizer $\widehat{Y}_T(1)$ may fail to be the density of a probability measure (or that $\mathbb{E}[\widehat{Y}_T(y)] < y$, for $y > 0$, which amounts to the same thing)? In fact, in many respects it does not bother us at all: we still have the basic duality relation between the primal and the dual optimizer displayed in Theorem 3.4 (ii). Even more is true: using the terminology from [KS 99] the product $(\widehat{X}_t(x)\widehat{Y}_t(y))_{0\leq t\leq T}$, where x and y satisfy $u'(x) = y$, is a uniformly integrable martingale. This fact can be interpreted in the following way: by taking the optimal portfolio $(\widehat{X}_t(x))_{0\leq t\leq T}$ as numéraire instead of the original cash account, the pricing rule obtained from the dual optimizer $\widehat{Y}_T(y)$ then is induced by an equivalent martingale measure. We refer to ([KS 99], p. 912) for a thorough discussion of this argument.

Finally we want to draw the attention of the reader to the fact that — comparing item (iii) of Theorem 3.4 to the corresponding item of Theorem 2.3 — we only asserted one pair of formulas for $u'(x)$ and $v'(y)$. The reason is that, in general, the formulae (86) do not hold true any more, the reason again being precisely that for the dual optimizer $\widehat{Y}_T(y)$ we may have $\mathbb{E}[\widehat{Y}_T(y)] < y$. Indeed, the validity of $u'(x) = \mathbb{E}[U'(\widehat{X}_T(x))]$ is tantamount to the validity of $y = \mathbb{E}[\widehat{Y}_T(y)]$.

Case 2: $\mathrm{dom}(U) = \mathbb{R}$

We now pass to the case of a utility function U satisfying Assumption 1.2 case 2 which is defined and finitely valued on all of $\mathbb{R}$. The reader should have in mind the exponential utility $U(x) = -e^{-\gamma x}$, for $\gamma > 0$, as the typical example.

We want to obtain a result analogous to Theorem 3.4 also in this setting. Roughly speaking, we get the same theorem, but the sets C and D considered above have to be chosen in a somewhat different way, as the optimal portfolio $\widehat{X}_T$ now may assume negative values too.

Firstly, we have to assume throughout the rest of this section that the semimartingale S is *locally bounded*. The case of non locally bounded processes is not yet understood and waiting for future research.

Next we turn to the question; what is the proper definition of the set $C(x)$ of terminal values X_T dominated by a random variable $x + (H \cdot S)_T$, where H is an "allowed" trading strategy? On the one hand we cannot be too liberal in the choice of "allowed" trading strategies as we have to exclude doubling strategies and similar schemes. We therefore maintain the definition of the value function $u(x)$ unchanged

$$u(x) = \sup_{H \in \mathcal{H}} \mathbb{E}\left[U\left(x + (H \cdot S)_T\right)\right], \quad x \in \mathbb{R}, \tag{130}$$

where we still confine H to run through the set $\mathcal{H}$ of admissible trading strategies, i.e., such that the process $((H \cdot S)_t)_{0 \le t \le T}$ is uniformly bounded from below. This notion makes good sense economically as it describes the strategies possible for an agent having a finite credit line.

On the other hand, in general, we have no chance to find the minimizer $\widehat{H}$ in (130) within the set of admissible strategies: already in the classical cases studied by Merton ([M 69] and [M 71]) the optimal solution $x + (\widehat{H} \cdot S)_T$ to (130) is *not* uniformly bounded from below; this random variable typically assumes low values with very small probability, but its essential infimum typically is minus infinity.

In [S 00] the following approach was used to cope with this difficulty: fix the utility function $U : \mathbb{R} \to \mathbb{R}$ and first define the set $C_U^b(x)$ to consist of all random variables G_T dominated by $x + (H \cdot S)_T$, for some *admissible* trading strategy H and such that $\mathbb{E}[U(G_T)]$ makes sense:

$$\begin{aligned} C_U^b(x) \quad = \quad & \big\{G_T \in L^0(\Omega, \mathcal{F}_T, \mathbf{P}) : \text{ there is } H \text{ admissible s.t.} && (131)\\ & G_T \le x + (H \cdot S)_T \text{ and } \mathbb{E}[|U(G_T)|] < \infty\big\}. && (132) \end{aligned}$$

Next we define $C_U(x)$ as the set of $\mathbb{R} \cup \{+\infty\}$-valued random variables X_T such that $U(X_T)$ can be approximated by $U(G_T)$ in the norm of $L^1(\mathbf{P})$, when G_T runs through $C_U^b(x)$:

$$\begin{aligned} C_U(x) \quad = \quad & \big\{X_T \in L^0(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbb{R} \cup \{+\infty\}) : U(X_T) \text{ is in} && (133)\\ & L^1(\mathbf{P})\text{-closure of } \{U(G_T) : G_T \in C_U^b(x)\}\big\}. && (134) \end{aligned}$$

The optimization problem (130) now reads

$$u(x) = \sup_{X_T \in C_U(x)} \mathbb{E}[U(X_T)], \quad x \in \mathbb{R}. \tag{135}$$

The set $C_U(x)$ was chosen in such a way that the value functions $u(x)$ defined in (130) and (135) coincide; but now we have much better chances to find the maximizer to (135) in the set $C_U(x)$.

Two features of the definition of $C_U(x)$ merit some comment: firstly, we have allowed $X_T \in C_U(x)$ to attain the value $+\infty$; indeed, in the case when $U(\infty) < \infty$ (e.g., the case of exponential utility), this is natural, as the set $\{U(X_T) : X_T \in C_U(x)\}$ should equal the $L^1(\mathbf{P})$-closure of the set $\{U(G_T) : G_T \in C_U^b(x)\}$. But we shall see that — under appropriate assumptions — the optimizer $\widehat{X}_T$, which we are going to find in $C_U(x)$, will almost surely be finite.

Secondly, the elements X_T of $C_U(x)$ are only *random variables* and, at this stage, they are not related to a *process* of the form $x + (H \cdot S)$. Of course, we finally want to find for each $X_T \in C_U(x)$, or at least for the optimizer $\widehat{X}_T$, a predictable, S-integrable process H having "allowable" properties (in order to exclude doubling strategies) and such that $X_T \leq x + (H \cdot S)_T$. We shall prove later that — under appropriate assumptions — this is possible and give a precise meaning to the word "allowable".

After having specified the proper domain $C_U(x)$ for the primal optimization problem (135), we now pass to the question of finding the proper domain for the dual optimization problem. Here we find a pleasant surprise: contrary to case 1 above, where we had to pass from the set $\mathcal{M}^a(S)$ to its closed, solid hull D, it turns out that, in the present case 2, the dual optimizer always lies in $\mathcal{M}^a(S)$. This fact was first proved by F. Bellini and M. Fritelli ([BF 00]).

We now can state the main result of [S 00]:

Theorem 3.5 *[S 00, Theorem 2.2] Let the locally bounded semi-martingale $S = (S_t)_{0\leq t\leq T}$ and the utility function U satisfy Assumptions 1.1, 1.2 case 2 and 1.3; suppose in addition that U has reasonable asymptotic elasticity. Define*

$$u(x) = \sup_{X_T \in C_U(x)} \mathbb{E}[U(X_T)], \qquad v(y) = \inf_{Q \in \mathcal{M}^a(S)} \mathbb{E}\left[V\left(y\tfrac{dQ}{d\mathbf{P}}\right)\right]. \tag{136}$$

Then we have:

(i) The value functions $u(x)$ and $v(y)$ are conjugate; they are continuously differentiable, strictly concave (resp. convex) on $\mathbb{R}$ (resp. on $]0,\infty[$) and satisfy

$$u'(-\infty) = -v'(0) = v'(\infty) = \infty, \qquad u'(\infty) = 0. \tag{137}$$

(ii) The optimizers $\widehat{X}_T(x)$ and $\widehat{Q}(y)$ in (136) exist, are unique and satisfy

$$\widehat{X}_T(x) = I\left(y\frac{d\widehat{Q}(y)}{d\mathbf{P}}\right), \qquad y\frac{d\widehat{Q}(y)}{d\mathbf{P}} = U'(\widehat{X}_T(x)), \tag{138}$$

where $x \in \mathbb{R}$ and $y > 0$ are related via $u'(x) = y$ or equivalently $x = -v'(y)$.

(iii) We have the following relations between u', v' and $\widehat{X}, \widehat{Q}$ respectively:

$$u'(x) = \mathbb{E}_{\mathbf{P}}[U'(\widehat{X}_T(x))], \qquad v'(y) = \mathbb{E}_{\widehat{Q}}\left[V'\left(y\frac{d\widehat{Q}(y)}{d\mathbf{P}}\right)\right] \tag{139}$$

$$xu'(x) = \mathbb{E}_{\mathbf{P}}[\widehat{X}_T(x)U'(\widehat{X}_T(x))], \qquad yv'(y) = \mathbb{E}_{\mathbf{P}}\left[y\frac{d\widehat{Q}(y)}{d\mathbf{P}}V'\left(y\frac{d\widehat{Q}(y)}{d\mathbf{P}}\right)\right] \tag{140}$$

(iv) If $\widehat{Q}(y) \in \mathcal{M}^e(S)$ and $x = -v'(y)$, then $\widehat{X}_T(x)$ equals the terminal value of a process of the form $\widehat{X}_t(x) = x + (H \cdot S)_t$, where H is predictable and S-integrable, and such that $\widehat{X}$ is a uniformly integrable martingale under $\widehat{Q}(y)$.

We refer to [S 00] for a proof of this theorem and further related results. We cannot go into the technicalities here, but a few comments on the proof of the above theorem are in order: the technique is to reduce case 2 to case 1 by approximating the utility function $U : \mathbb{R} \to \mathbb{R}$ by a sequence $(U^{(n)})_{n=1}^{\infty}$ of utility functions $U^{(n)} : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that $U^{(n)}$ coincides with U on $[-n, \infty[$ and equals $-\infty$ on $]-\infty, -(n+1)]$. For fixed initial endowment $x \in \mathbb{R}$, we then apply Theorem 3.4 to find for each $U^{(n)}$ the saddle-point $(\widehat{X}_T^{(n)}(x), \widehat{Y}_T^{(n)}(\widehat{y}_n)) \in C_U^b(x) \times D(\widehat{y}_n)$; finally we show that this sequence converges to some $(\widehat{X}_T(x), \widehat{y}\widehat{Q}_T) \in C_U(x) \times \widehat{y}\mathcal{M}^a(S)$, which then is shown to be the saddle-point for the present problem. The details of this construction are rather technical and lengthy (see [S 00]).

We have assumed in item (iv) that $\widehat{Q}(y)$ is equivalent to $\mathbf{P}$ and left open the case when $\widehat{Q}(y)$ is only absolutely continuous to $\mathbf{P}$. F. Bellini and M. Fritelli have observed ([BF 00]) that, in the case $U(\infty) = \infty$ (or, equivalently, $V(0) = \infty$), it follows from (136) that $\widehat{Q}(y)$ is equivalent to $\mathbf{P}$. But there are also other important cases where we can assert that $\widehat{Q}(y)$ is equivalent to $\mathbf{P}$: for example, for of the exponential utility $U(x) = -e^{-\gamma x}$, in which case the dual optimization becomes the problem of finding $\widehat{Q} \in \mathcal{M}^a(S)$ minimizing the relative entropy with respect $\mathbf{P}$, it follows from the work of Csiszar [C 75] (compare also [R 84], [F 00], [GR 00]) that the dual optimizer $\widehat{Q}(y)$ is equivalent to $\mathbf{P}$, provided only that there is at least one $Q \in \mathcal{M}^e(S)$ with finite relative entropy.

Under the condition $\widehat{Q}(y) \in \mathcal{M}^e(S)$, item (iv) tells us that the optimizer $\widehat{X}_T \in C_U(x)$ is almost surely finite and equals the terminal value of a process $x + (H \cdot S)$, which is a uniformly integrable martingale under $\widehat{Q}(y)$; this property qualifies H to be a "allowable", as it certainly excludes doubling strategies and related schemes. One may turn the point of view around and take this as the *definition* of the "allowable" trading strategies; this was done in [DGRSSS 00] for the case of exponential utility, where this approach is thoroughly studied and some other definitions of "allowable" trading strategies, over which the primal problem may be optimized, are also investigated. Further results on these lines were obtained in [KaS 00] for the case of exponential utility, and in [S 00b] for general utility functions.

We finish this survey with a brief account on the recent literature related to maximizing expected utility in financial markets. There are many aspects going beyond the basic problem surveyed above. We can only give a very brief indication on the many interesting papers and hope to have provided the reader with some introductory motivation to study this literature.

G. Zitkovic [Z 00] has analyzed the problem of optimizing expected utility of consumption during the time interval $[0, T]$. He obtained a similar result as Theorem 3.4 above, provided the utility functions $U_{t,\omega}$, which in this setting may depend on $t \in [0, T]$ and $\omega \in \Omega$ in an $\mathcal{F}_t$-measurable way, satisfy the reasonable elasticity condition in a uniform way.

Results related to the duality theory of utility maximization and notably to the dual optimizer $\widehat{Q} \in \mathcal{M}^e(S)$ were obtained in [F 00], [K 00], [XY 00], [GK 00], [GR 00] and [BF 00].

Utility maximization under transaction costs was investigated, e.g., in [HN 89], [CK 96], [CW 00] and [DPT 00]; in the latter two papers the phenomenon arising in Theorem 3.4 is of crucial importance: for the dual optimizer one has to perform a similar enlargement as the passage from $\mathcal{M}^a(S)$ to D encountered in Theorem 3.4 above.

The theme of random endowment, which is intimately related to the concept of utility based hedging of contingent claims is treated in [KJ 98], [KR 00], [CSW 00], [D 00], [JS 00], [CH 00], and in the context of minimizing expected shortfall, which leads to non-smooth utility functions, in [C 00] and [FL 00]. Non smooth utility functions also come up in a natural way in [DPT 00] and in [L 00].

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