Introduction to the Mathematics of Financial Markets

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Abstract

In this introductory course we review some of the basic concepts of Mathematical Finance. We start with an account on the thesis of L. Bachelier, which was defended as "Théorie de la Spéculation" in Paris in 1900. We hope that this historic approach gives a good motivation for a critical appreciation of the modern theory.

In section 2 we then present the basic framework of the modern no-arbitrage theory in the simple setting of finite probability spaces Ω.

The celebrated Black-Scholes model, based on geometric Brownian motion, is presented in section 3. It is compared to Bachelier's model, which is based on (arithmetic) Brownian motion.

The first three sections are kept on a relatively low level of technical sophistication. In section 4 we pass to a higher level of technicality and review the general theory of semi-martingale models of financial markets. We discuss in some detail the "fundamental theorem of asset pricing", which establishes the relation between the no-arbitrage theory on the one hand, and martingale theory on the other.

Finally, in section 5 we briefly discuss some of the applications of the fundamental theorem.

Contents

1 Introduction: Bachelier's Thesis from 1900

The fact that this course is given in the year 2000 at the école d'été in Saint Flour makes it particularly appealing to start this course with a review of the seminal thesis of Louis Bachelier: "Théorie de la Spéculation" [B 00]. This historical account will provide a good motivation for the general theory. We note, however, that readers only interested in a presentation of the theory in modern terms, can immediately pass to section 2.

Bachelier's thesis was defended in Paris on March 29, 1900, and H. Poincaré was a member of the thesis committee. He wrote a very positive and insight-full report on this thesis (this opinion as well as many other value judgements below only reflect my personal point of view). One may consult this report in Courtault et al. [CK 00], where one can also find a copy of the handwritten manuscript of Poincaré's report. We also refer to the interview of M. Taqqu with B. Bru [T 00] for an account on the personal life of Bachelier, who — in spite of his brilliant and original work, and the fame and support of his thesis adviser — remained an outsider to the French mathematical establishment during all of his life.

L. Bachelier was born in 1870 and became an orphan at the age of 19. In order to make a living, he had to work at the Bourse de Paris where he was exposed on a daily basis with the erratic movement of prices of financial securities.

In these days there was massive trade at the Bourse de Paris in the so-called "rentes", which were perpetual bonds paying an annual interest rate, typically 3 % (paid in 4 quarterly coupons of 75 centimes per 100 francs par value). The reason why these instruments had such importance in France goes back to the French revolution, when many wealthy aristocrats left the country and lost their property. When they returned after the restauration, they wanted their property back, but this turned out to be impossible after 25 years. The solution adopted by the government in order to recompensate them, was to issue "rentes", and to distribute an appropriate amount of them among the expropriated noble-men. While the quaterly coupons would provide them with an adequate income, the capital was never paid. These rentes were passed on in the families and they were also traded massively at the Bourse de Paris (for more information see [T 00]). Of course, they were not necessarily traded at par value but rather at changing prices similarly as in today's bond markets.

We spoke in some detail about the "rentes", because their special properties are important to understand the choice of Bachelier's model for the stock price process.

(i) There was a (very) liquid market, and price fluctuations happened "in continuous time", similarly as in the major stock and bond markets of today.

(ii) The price of a "rente" would typically not deviate too much from its par value, e.g. 100 francs; hence the absolute price changes (expressed in francs) and the relative price changes (expressed in percent) would roughly be the same.

(iii) The price fluctuations were relatively mild, if compared, e.g., with today's price fluctuations of stocks: one may deduce from the data provided by L. Bachelier that the standard deviation of the price change of a "rente" with par value of 100 francs over a year was about 2.4 francs (which roughly corresponds to a yearly volatility of 2.4 percent in the Black-Scholes model analyzed in section 3 below).

L. Bachelier was interested in designing a rational theory for the prices of term contracts. The two forms which were traded at the Bourse at that time also play a basic role today as forward contracts and options.

Definition 1.1 *A forward contract on an underlying security S consists of the* right and the obligation *to buy a fixed quantity (which we normalize to be one) of the underlying security, at a fixed price K and a fixed time T in the future.*

The underlying was in Bachelier's case typically a "rente", but it may just as well be any risky security such as a share, a foreign currency, a commodity etc.

Depending on the value of the "strike price" K, the present day value (i.e., at time $t = 0$) of such a forward contract could be positive or negative. The price $K = F$ at which a forward is contracted today at price zero is called the *forward price* of the underlying S (see, e.g., [H 99] or any introductory text on Mathematical Finance for more explanation).

We shall show now — in a similar way as Bachelier did in 1900 — that the forward price F is determinated by some very elementary *no-arbitrage arguments.* For the sake of clarity, we provide an example in a slightly different economic context than the one considered by Bachelier.

Example 1.2 Let $X = (X_t)_{0 \le t \le T}$ model the exchange rate of the US $ vs. the €, i.e., the price of 1 US $ in terms of €. The present day rate X_0 (the "spot price") can be looked up in the newspaper, hence this is just a positive number, say $X_0 = 1.1$. On the other hand, for $t > 0$, we do not know the exchange rate X_t. Later on we shall model X as a stochastic process, but presently it is not even necessary to speak about probability at all. X_t simply is some quantity which will be known at time t.

To compare cash-flows at different times we assume that there are "cash-accounts" B_t^d and B_t^f for € and US $ respectively, which are given by

$$B_t^d = e^{r_d t}, \quad B_t^f = e^{r_f t}, \tag{1}$$

where d stands for "domestic", i.e. €, while f stands for "foreign", i.e. US $. The idea behind the notion of "cash account" is that an investor has the possibility of investing in a "riskless" way, which means that the value of her investment in the "cash-account" will develop deterministically in a way which is known in advance. The reader should think of a bank account (in € or US $ resp.) yielding an interest rate of r_d or r_f respectively. We in advance shall also assume that these investments may be either positive or negative ("long" or "short" in the financial lingo), in other words we may invest or borrow at the same conditions. Of course, this is an unrealistic assumption for small investors, but we should think of large investors (banks, investment funds, broker houses etc.) for which this assumption is a close approximation to reality.

Claim 1.3 *Given the time horizon T, there is a unique forward price F which does not allow arbitrage opportunites, namely*

$$F = X_0 e^{(r_d - r_f)T}. \tag{2}$$

We have not yet defined the notion of arbitrage — and we shall give a formal mathematical definition only much later. But the best way to grasp this — very primitive and economically convincing — concept is to consider the subsequent argument.

Consider two portfolios which can be established on the market today.

Portfolio A: Invest $e^{-r_f T}$ US \$ into a US \$ cash account. This investment will be worth one US \$ at time T, and we can buy it today at a price of $e^{-r_f T} X_0$ €.

Portfolio B: Invest $e^{-r_d T} F$ € into a € cash account and buy one forward contract with maturity T and strike-price $K = F$. A moment's reflection reveals that this investment will also be worth one US \$ at time T and that we can buy it today at a price of $e^{-r_d T} F$ € (recall that F is defined in such a way that we can "buy" (i.e., enter into) a forward contract today at cost zero).

Hence the portfolios A and B are worth the same at time T (independent of how the exchange rate $(X_t)_{0 \leq t \leq T}$ develops!). We therefore claim that they also must have the same value today which results in equation (2).

Indeed, suppose for example that $F > X_0 e^{(r_d - r_f)T}$. In this case an "arbitrageur" would profit of the situation by buying portfolio A and selling portfolio B, thus obtaining the strictly positive difference $e^{-r_d T} F - e^{r_f T} X_0$, as a *riskless profit*: at time T the two positions will cancel out surely.

If the inequality is of the form $F < X_0 e^{(r_d - r_f)T}$ just reverse the roles of portfolio A and B. Also note that we have given our example in terms of the rather symbolic quantity of one US \$. But of course there is no normalizing factor in front of the above argument and — if the market circumstances permit — you are free to multiply it with your favourite power of 10. Hence it is economically quite obvious that a market, where equation (2) is violated, cannot be in equilibrium as such an arbitrage opportunity would quickly be exploited by economic agents; a moment's reflection reveals that the market forces triggered by an arbitrageur behaving according to the above recipe will act towards making a possible violation of the identity (2) become smaller, and that people would continue to exploit such a violation of the "no-arbitrage principle" up to the point where (2) is satisfied to a sufficient degree as to make this arbitrage opportunity unattractive, even for a large investor.

The reader also should note that it is not necessary that all market participants behave rationally (and that they are aware of the identity (2)). It suffices that some of them (in theory even one would suffice!), who are ready and able to act with large sums on the market, are aware of (2) and eager to exploit arbitrage opportunities, whenever they come up.

Let us recapitulate the assumptions on the financial market which we have made above (more or less tacitly) in order for the no-arbitrage argument — and therefore the formula (2) — to be valid: we assumed that we can go long and short in the cash accounts (1) as well as in the forward contract at prices, which do not depend on the sign of the investment, without any transaction costs and with arbitrarily large quantities. As mentioned above, these assumptions are not fully satisfied in practice, but the economic situation of the "big players" in the market is quite close to these assumptions.

The attentive reader has noticed that we did not fully rely on the assumption (1) that there exist "riskless" cash accounts behaving according to (1), for all $0 \leq t \leq T$;

all we needed was, that the relation holds true for $t = T$. In other words and using the financial lingo, we had to assume the existence of "riskless" (in practice this means that the government guarantees for the payment) "zero coupon bonds" maturing at time T, i.e., a contract, which pays 1 € (or 1 US $) at time T. Such contracts — or close approximations to them — are indeed traded in massive volume in financial markets.

At this point the reader is advised to convince herself — by consulting the financial section of a standard newspaper — that the above arguments are not merely theoretical but confirm very well to reality: the forward price of a currency depends on the difference of the interest rates in the corresponding currencies, pertaining to the maturity T, via (2) — and it only depends on this difference. Also observe that, in the case $r_d = r_f$, (2) reduces to $F = X_0$, i.e. the forward price then simply equals the spot price.

Let us turn back to L. Bachelier and the "rentes" traded at the Bourse de Paris. There was a liquid market in forward contracts on these "rentes" and Bachelier noticed the above relation between the spot price and the forward price. To link to our US $/€ example, the role of the accumulated interest of a "rente" plays a similar role as the interest rate r_f for the foreign currency, at least for periods $[0, T]$ which contain no coupon payment (in the case of coupon payments one has to make some rather straightforward adaptations). On the other hand, there was a complicated system of partial recompensation of the buyer of a forward contract with respect to this accumulated interest, called — "contangoes" (in french: "reports") — which — roughly speaking — plays a similar role as r_d above. The details are quite complicated, only of historical interest, and not relevant for our purposes. We shall therefore assume that the system of contangoes would fully recompensate the accumulated interest of the "rentes"; while this was not the case in reality, it was explicitly mentioned as a theoretical case by Bachelier. This corresponds to the case $r_d = r_f$ for the case of foreign exchange considered above, and implies — by similar no-arbitrage arguments — that the forward price (called the "true price" by Bachelier) coincides with the spot price.

Assumption 1.4 *We assume for the rest of this section that, for every maturity T, the spot price S_0 of the underlying security, and the forward price F with respect to T, coincide.*

We shall see later that this convenient assumption does not restrict the generality of the argument. What it does in practice: it dispenses us of making boring calculations of upcounting and discounting as reflected by the identity (2).

One final comment on whether Bachelier used the same no-arbitrage arguments as we did above: Bachelier does not argue by no-arbitrage but simply states that bond prices must "logically increase" by the accumulated interest which is tantamount to (2). He would simply appeal to common sense without explicitly mentioning the rather obvious no-arbitrage arguments. He saves this for more complicated securities where the argument becomes less obvious, as we shall presently see.

After this elementary treatment of forward contracts and forward prices in the first pages of his thesis, Bachelier passes to the case of options, which — in today's terminology — were European options.

Definition 1.5 *A European call (resp. put) option on an underlying security S consists in the* right (but not the obligation) *to buy (resp. to sell) a fixed quantity (which we normalize to be one) of the underlying security, at a fixed price K and a fixed time T in the future.*

In fact, there is a slight — but for the mathematical modelling rather crucial — difference between the way options are traded today and the way they were traded in Bachelier's days, at least in France. Nowadays the option premium C, i.e., the price, the buyer of an option has to pay, in order to enter into the contract, (the letter C standing for call option) is paid up front, i.e., at $t = 0$. In 1900 it was paid at the exercise time $t = T$ of the option. We denote the latter premium by $\widehat{C}$ to indicate that it corresponds to the upcounted premium C (from $t = 0$ to $t = T$) with respect to the risk free rate of interest (more precisely and in modern terminology: with respect to the zero coupon rate with maturity T). Fixing the letter K for the strike price of the option one arrives — after a moment's reflection — at the usual "hockey-stick" shape for the pay-off function of the option at time T. We draw the value of the option as a function of the price S_T of the underlying at time T:

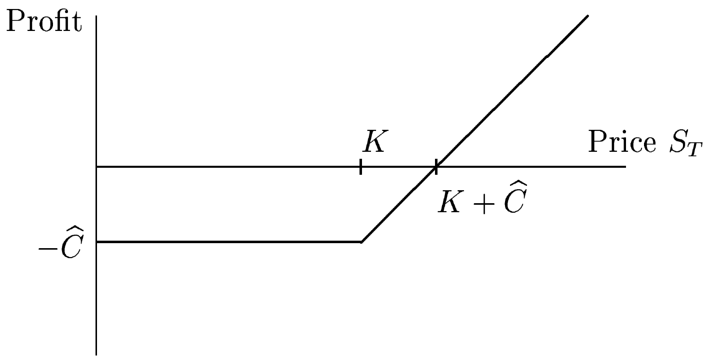

Figure 1: Pay-off function of a call option at time T.

This famous picture appears explicitly (with different letters for notation) in Bachelier's thesis. In fact, Bachelier compares the pay-off function of an option to the pay-off function of a forward contract with forward price F:

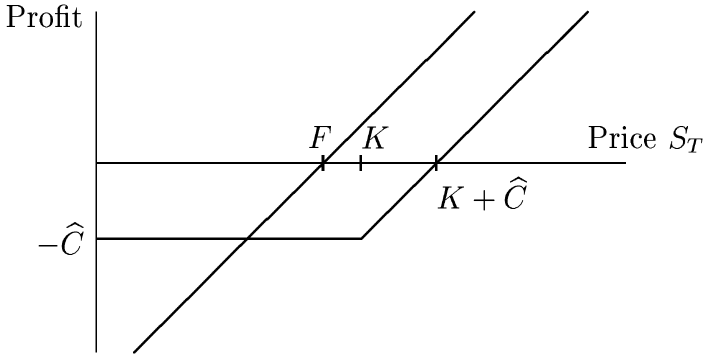

Figure 2: Pay-off function of a call option and a forward contract at time T.

Today the quotation system in option markets usually fixes the strike price K while the premium C is variable and fluctuates (the reader might look up the financial section of any standard newspaper); in Bachelier's times it was done the other way round (at least for the "rentes"): the (upcounted) premium $\widehat{C}$ was fixed, typically $\widehat{C} = 50$, 25, or 10 centimes, and the strike price K would fluctuate according to demand and supply. In fact, the way people were quoting options was in terms of the "spread" $K + \widehat{C} - F$, which is very natural as we now shall see.

Bachelier gives the following numerical example: Suppose that the forward price F (for fixed horizon T which at these times was in the order of one or two months) for a "rente" equals 104 francs. He then continues: *"If we buy a forward contract on 3000 francs par value, we expose ourselves to a potential loss which may become considerable if a fall in the market occurs. To avoid this risk, we could buy an option at 50 centimes paying no longer 104 francs but 104.15 francs, for example."* In our notation this amounts to $F = 104$, $K = 103.65$, $\widehat{C} = 0.50$, $K + \widehat{C} = 104.15$, and the spread $K + \widehat{C} - F = 0.15$. The idea is that one agrees to pay $K + \widehat{C} - F$ (the "spread") more than in the case of a forward contract when exercising the contract at time T; en revanche, one has limited the maximal loss to $\widehat{C} = 0.50$.

He then remarks the "obvious fact" that the spread is a decreasing function of the premium $\widehat{C}$ and again he does not bother to give the — rather trivial — corresponding no-arbitrage argument (which we leave to the attentive reader). But then he also observes the concavity of this function by a less trivial combination of investments: this combination of options is known today under the name of "butterfly" in finance. We don't give the details here; the interested reader may look it up in Bachelier's thesis [B 00, p. 24] and compare it to the "butterfly" argument as explained, e.g., in [H 99]. Bachelier does not use the word arbitrage, which is today's terminology, but refers to *"operations in which one of the traders would profit regardless of eventual prices"*, which amounts to the same, and is in fact a very pretty description of the notion of arbitrage. Working at the bourse he was very aware of the no-arbitrage principle [B 00, p. 24]: *"We will see that such spreads are never found in practice"*.

After these preparations, L. Bachelier passes to the central topic, *Probabilities in Operations on the Exchange.* He had already addressed the basic difficulty of introducing probability in the context of the stock exchange in the introduction to the thesis in a very sceptical way: *"The calculus of probabilities, doubtless, could never be applied to fluctuations in security quotations, and the dynamics of the Exchange will never be an exact science."*

Nevertheless he now proceeds to model the price change of securities by a probability distribution distinguishing

> *"two kinds of probabilities:*
>
> *(i) The probability which might be called "mathematical", which can be determined a priori and which is studied in games of chance.*
>
> *(ii) The probability dependent on future events and, consequently impossible to predict in a mathematical manner.*
>
> *This last is the probability that the speculator tries to predict."*

My personal interpretation of this — somewhat confusing — definition is the following: sitting daily at the Bourse and watching the movement of prices, Bachelier got the same impression that we get today when observing price movements in financial markets, e.g., on the internet. The development of the charts of prices of stocks, indices etc. on the screen or blackboard resembles a "game of chance". On the other hand, the second kind of probability seems to refer to the expectations of a speculator who has a personal opinion on the development of prices. Bachelier continues: *"His (the speculator's) inductions are absolutely personal, since his counterpart in a transaction necessarily has the opposite opinion."*

Here he is led to the remarkable conclusion, which in today's terminology is called the "efficient market hypothesis":

> *"It seems that the market, the aggregate of speculators, at a given instant can believe in neither a market rise nor a market fall since, for each quoted price, there are as many buyers as sellers."*

He then makes clear that this principle should be understood in terms of "true prices", i.e., forward prices (compare the up- and discounting arguments as well as assumption 1.4 above). Finally he ends up with his famous dictum:

> *"In sum, the consideration of true prices permits the statement* of this fundamental *principle:*
> ***The mathematical expectation of the speculator is zero.***"

This is a truly fundamental principle and the reader's admiration for Bachelier's pathbreaking work will increase even more when continuing to the subsequent paragraph of Bachelier's thesis:

> *It is necessary to evaluate the generality of this principle carefully: It means that the market, at a given instant, considers not only currently negotiable transactions, but even those which will be based on a subsequent fluctuation in prices as having a zero expectation.*
> *For example, I buy a bond with the intention of selling it when it will have appreciated by 50 centimes. The expectation of this complex transaction is zero exactly as if I intended to sell my bond on the liquiditation date, or at any time whatever.*

In my opinion, in these two paragraphs, the basic ideas underlying the concepts of martingales, stopping times, trading strategies, and Doob's stopping theorem already appear in a very intuitive way. It also sets the basic theme of the modern approach to option pricing which is based on the notion of a martingale.

In the remainder of this introductory review of Bachelier's thesis we shall discuss the implications of this **fundamental principle** and we shall address the following natural basic question:

Is the fundamental principle of L. Bachelier true?

There are, at least, two aspects to this question:

(i) Is it true, from a practical point of view, i.e., does it agree with data from financial markets?

(ii) Is it true, from a mathematical point of view, i.e., are there theorems that support his claim?

But let us first look at the implications of the *fundamental principle*: In order to draw conclusions from it, Bachelier had to determine the probability distribution of the random variable S_T (the price of the underlying security at expiration time T), or, more generally, on the entire stochastic process $(S_t)_{0\leq t\leq T}$. It is important to note that Bachelier had the approach of considering this object as a *process*, i.e., by thinking of the *pathwise behaviour* of the trajectories $(S_t(\omega))_{0\leq t\leq T}$; this was very natural for him, as he was constantly exposed to observing the behaviour of the prices, as t "varies in continuous time".

To fix the process S, Bachelier fixes the maturity time T and chooses the coordinates such that the forward price F which — according to our assumption 1.4 — coincides with the current price S_0 of the underlying security, is at the origin. Then Bachelier assumes — more or less tacitly — that, for $0 \leq t \leq T$, the probability $p_{x,t}dx$, that the price S_t of the underlying security, starting at time $t_0 = 0$ from the point $x = 0$, lies at time t in the infinitesimal interval $(x, x + dx)$, is *symmetric around the origin and homogeneous in time t as well as in space x*.

Of course, Bachelier notices that this creates a problem, as it gives positive probabilities to negative values of the underlying security, which is absurd. But one should keep in mind the proportions mentioned above: a typical yearly standard deviation σ of the prices of the bonds considered by L. Bachelier was of the order of 2.4 %. Hence the region where the bond price after a year becomes negative is roughly 40 standard deviations away from the mean; anticipating that Bachelier uses the normal distribution this is — in his words — *"considered completely negligible"*, as the time horizons for the options were just fractions of a year. On the other hand, we should be warned when considering Bachelier's results asymptotically for $t \to \infty$ (or $\sigma \to \infty$ which roughly amounts to the same), as in these circumstances the effect of assigning positive probabilities to negative values of S_t is not *"completely negligible"* any more.

After these specifications, Bachelier argues that *"by the principle of joint probabilities"* (apparently he means the independence of the increments), we obtain

$$p_{z,t_1+t_2} = \int_{-\infty}^{+\infty} p_{x,t_1} p_{z-x,t_2} dx. \tag{3}$$

In other words, he obtains what we call today the Chapman-Kolmogoroff equation. Then he observes that *"this equation is confirmed by the function"*

$$p_{x,z} = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2\sigma^2 t}\right), \tag{4}$$

concluding that *"evidently the probability is governed by the Gaussian law already famous in the calculus of probabilities"*.

Some remarks seem in order here: firstly, for the convenience of the reader who looks up Bachelier's original text, we mention that Bachelier did not use the quantity

σ for the parametrisation but rather the quantity $H = \frac{\sigma}{\sqrt{2\pi}}$. Secondly, he obviously did not bother about the uniqueness of the solution to (3). Thirdly, he was well aware — and explicitly mentions — that he models the price movements in absolute terms and not in relative terms (w.r.t. the stock prices). As already mentioned, this distinction is not very important in the case of the "rentes", where the current price is typically close to the par value of 100 francs.

Summing up, Bachelier derived from some basic principles the transition law of Brownian Motion and it's relation to the Chapman-Kolmogoroff equation.

Bachelier then gives an *"Alternative Determination of the Law of Probability"*. He approximates the continuous time model $(S_t)_{t\geq 0}$ by a random walk, i.e., a process which during the time interval Δt moves up or down with probability $\frac{1}{2}$ by Δx. He clearly works out that Δx must be of the order $(\Delta t)^{\frac{1}{2}}$ and — using only Stirling's formula — he obtains the convergence of the one-dimensional marginal distributions of the random walk to those of Brownian motion.

Now a section follows, which is not directly needed for the subsequent applications in finance, but which — retrospectively — is of utmost mathematical importance: *"Radiation of probability"*. Consider the random walk model and suppose that the grid in space is given by $\ldots, x_{n-2}, x_{n-1}, x_n, x_{n+1}, x_{n+2}, \ldots$ having the same distance $\Delta x = x_n - x_{n-1}$, for all n, and such that at time t these points have probabilities $\ldots, p^t_{n-2}, p^t_{n-1}, p^t_n, p^t_{n+1}, p^t_{n+2}, \ldots$ under the random walk under consideration. What are the probabilities $\ldots, p^{t+\Delta t}_{n-2}, p^{t+\Delta t}_{n-1}, p^{t+\Delta t}_n, p^{t+\Delta t}_{n+1}, p^{t+\Delta t}_{n+2}, \ldots$ of these points at time $t + \Delta t$? A moment's reflection reveals the rule which is so nicely described by Bachelier in the subsequent phrases:

> *"Each price x during an element of time radiates towards its neighboring price an amount of probability proportional to the difference of their probabilities.*
>
> *I say proportional because it is necessary to account for the relation of Δx to Δt.*
>
> *The above law can, by analogy with certain physical theories, be called the law of radiation or diffusion of probability."*

Passing formally to the continuous limit and denoting by $P_{x,t}$ the distribution function associated to the density function (4)

$$P_{x,t} = \int_{-\infty}^{x} p_{z,t} dz \tag{5}$$

Bachelier deduces in an intuitive and purely formal way the relation

$$\frac{\partial P}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial x} = \frac{1}{c^2}\frac{\partial^2 P}{\partial x^2} \tag{6}$$

where $c > 0$ is a constant. Of course, the heat equation was known to Bachelier: he notes that *"this is a Fourier equation"*.

Hence Bachelier in 1900 very explicitly discovered the fundamental relation between Brownian motion and the heat equation; this fact was rediscovered five years later by A. Einstein and resulted in a goldmine of mathematical investigation through the work of Kolmogoroff, Kakutani, Feynman, Kac, and many others up to recent research. It is worth noting that H. Poincaré in his report on Bachelier's thesis apparently saw the seminal importance of this idea when he wrote *"On peut regretter que M. Bachelier n'ait pas developpé d'avantage cette partie de sa thèse"* (One may regret that M. Bachelier did not develop further this part of his thesis).

With all these considerations L. Bachelier has fixed the model for the price changes of the underlying security — namely the normal distribution — up to the crucial parameter σ, which he calls the *"coefficient of instability or of nervousness of a security"* (strictly speaking he considers $\frac{\sigma}{\sqrt{2\pi}}$ rather than σ, which is just a matter of normalization). Fixing the parameter σ and applying the "fundamental principle" to the pay-off function in figure 2 one obtains — using the identity $F = S_0$ from assumption 1.4 — the equation

$$-\widehat{C} + \int_{K-S_0}^{\infty} (x - (K - S_0)) f(x)dx = 0, \tag{7}$$

where

$$f(x) = \frac{1}{\sigma\sqrt{2\pi T}} e^{-\frac{x^2}{2\sigma^2 T}}, \tag{8}$$

which clearly determines the relation between the premium $\widehat{C}$ of the option and $K - S_0$ and therefore also the relation between $\widehat{C}$ and the "spread" $K + \widehat{C} - S_0$. In other words, equation (7) determines the price for the option and therefore solves the basic problem considered by Bachelier.

It is straightforward to derive from (7) an "option pricing formula" by calculating the integral in (7) (compare, e.g., [Sh 99]): denoting by $\varphi(x)$ the standard normal density function, i.e., $\varphi(x)$ equals (8) for $\sigma^2 T = 1$, by $\Phi(x)$ the corresponding distribution function, and using the relation $\varphi'(x) = -x\varphi(x)$, an elementary calculation reveals that

$$\begin{aligned} \widehat{C} &= \int_{\frac{K-F}{\sigma\sqrt{T}}}^{\infty} \left(x\sigma\sqrt{T} - (K - F) \right) \varphi(x)dx \\ &= (F - K)\Phi\left(\frac{F-K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\varphi\left(\frac{F-K}{\sigma\sqrt{T}}\right). \end{aligned} \tag{9}$$

Interestingly, Bachelier does not bother to write up this easy formula which gives $\widehat{C}$ as a function of K (the way which is useful in determining option prices today). As mentioned above, he was rather interested in expressing inversely the "spread" $K + \widehat{C} - F$ as a function of $\widehat{C}$, and apparently there is no explict way of writing down this relation.

Instead, he does something much more interesting: he first specializes to the case of *simple options* (this is terminology from 1900), when $K = F$, which at his time were the usual options on commodities; in modern terms they correspond to so called at-the-money options where the strike price K equals the forward price F (which in our setting is equal to the spot price S_0 by assumption 1.4). In this case the solution to (7) obviously results in

$$\widehat{C} = \frac{\sigma}{\sqrt{2\pi}} \sqrt{T}, \tag{10}$$

which is a remarkably simple formula. Bachelier also uses this formula to turn the point of view upside down, or — in modern terms — to determine the "implied volatility", thus discovering yet one more basic idea of modern mathematical finance: if we can observe the (upcounted) premium $\widehat{C}$ of an at-the-money option on the market, formula (10) determines very directly the "nervousness" parameter $\frac{\sigma}{\sqrt{2\pi}}$ and therefore specifies the probability distribution $p_{x,t}$.

Still, formula (10) depends on the parameter σ and Bachelier — following the reflexes of a true mathematician — wanted to find quantities invariant under variations of the parameter σ and the expiration date T: For example, he determines the probability that the buyer of an at-the-money option (i.e., $K = F$) makes a profit. Glancing at figure 2 this probability p equals

$$p = \int_{\widehat{C}}^{\infty} f(x)dx = \int_{\frac{\sigma\sqrt{T}}{\sqrt{2\pi}}}^{\infty} f(x)dx, \tag{11}$$

where $f(x)$ is given by (8). Calculating this expression, the term $\sigma\sqrt{T}$ cancels out, and we obtain

$$p = \int_{\frac{1}{\sqrt{2\pi}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1 - \Phi\left(\frac{1}{\sqrt{2\pi}}\right) \approx 0.345. \tag{12}$$

In other words, according to Bachelier's model, the buyer of an at-the-money option makes a profit in 34,5 % of the cases, and a loss in 65,5 % of the cases. Isn't it a remarkable and surprising result that this number does not depend on any parameter? Bachelier also derives explicit numbers (not depending on any parameter) for the probability of success in a number of similar situations.

Then he treats the case of options where the strike price K is not necessarily equal to the forward price F, i.e., options which are not necessarily at the money. He uses a quadratic approximation of the behaviour of the relation between K and $\widehat{C}$ determined by (7) in a neighbourhood of $K = F$ which again yields very explicit and practical formulae, displaying a good fit for the values appearing in practice, i.e.,when $|K - F|$ is small as compared to F.

After all these derivations Bachelier compares his theoretical results with the financial data observed for the "rentes" in the period of 1894-1898.

He just considers averages over these five years and in particular the "nervousness parameter" $\frac{\sigma}{\sqrt{2\pi}}$ is an average estimate, while it becomes clear from the remarks of Bachelier, that the "nervousness" $\frac{\sigma}{\sqrt{2\pi}}$ of the market was varying in time (just as it does today).

He estimates the yearly standard deviation of the price movement of a rente to be approximately equal to 240 centimes, which corresponds to the above mentioned 2.4 % of the par value of 100 francs.

Then he compares the quantities derived from his model (using this parameter) to the empirical financial data (taking again averages over these five years).

This comparison of calculated figures with observed data does not live up to the standards of a modern statistical analysis; also the match is not overwhelmingly good — the difference sometimes being in the range of 10 or 20 percent — but it shows that the qualitative features are well captured.

To sum up the issue of the match of his theory with empirical financial data Bachelier makes the remarkable comment:

> *"If, with respect to several questions treated in this study, I have compared the results of observation with those of theory, it was not to verify formulae established by mathematical methods, but only to show that the market, unwittingly, obeys a law which governs it, the law of probability."*

It is interesting to have a look into Poincaré's report on Bachelier's thesis where he gives an argument in favor of Bachelier's *fundamental principle* (which, of course, is the basis of the above methodology) relying on the law of large numbers; Poincaré also clearly stresses the relative weakness of this argument (the reader should compare the argument below involving the law of large numbers to the much more convincing no-arbitrage arguments encountered above):

> *"One should not expect a very exact verification. The principle of the mathematical expectation holds in the sense that, if it were violated, there would always be people who would act so as to re-establish it and they would eventually notice this. But they would only notice it, if the deviations were considerable. The verification, then, can only be gross. The author of the thesis gives statistics where this happens in a very satisfactory manner."*

In the final part of his thesis L. Bachelier makes another seminal discovery: the law of the maximum of a Brownian path. Here we again see clearly that Bachelier had a *pathwise* approach to stochastic processes. The fact that the density function of the maximum of the Brownian path equals twice the density of the corresponding Gaussian density function on the positive axis (while it is of course zero on the negative axis) is today the standard example for the use of the "reflection principle", which reduces this fact almost to a triviality.

Interestingly, Bachelier does not derive it in this way, but rather by approximation with a discrete random walk and using a combinatorial result obtained in 1888 by D. André, called the solution to the ballot problem ("problème du scrutin").

Using this theorem and passing in an appropriate way to the limit, Bachelier obtains the result on the distribution of the maximum of Brownian motion. It is only then that he uses the reflection principle to *interpret* this result:

> *"The interpretation of this result is very simple: The price cannot be exceeded at the moment t without having been attained previously. The probability $\mathcal{P}$ is therefore equal to the probability P multiplied by the probability that, given that the price was quoted previously, it will be exceeded at the moment t, i.e., multiplied by $\frac{1}{2}$. Thus*
>
> $$\mathcal{P} = \frac{P}{2} \text{ ."} \tag{13}$$

To explain the notation: letting $c > 0$ denote "the price" referred to above, and $(W_t)_{t \geq 0}$ Brownian motion, the letter $\mathcal{P}$ denotes the probability $\mathbb{P}\{W_t \geq c\}$ while

P denotes $\mathbb{P}\{\sup_{0\le u\le t} W_u \ge c\}$, so that (13) describes the well known law of the maximum of a Brownian path.

Bachelier was led to consider this problem by a very interesting idea from the financial point of view, which may be considered as a precursor of the idea of dynamical hedging, which in turn is the central idea of modern mathematical finance.

Consider again the buyer of an at-the-money option with $K = F = S_0$ and a premium $\widehat{C}$. We have seen above that the probability of success of the buyer of such an option is

$$\mathbb{P}[S_T \ge S_0 + \widehat{C}] = 1 - \Phi\left(\frac{1}{\sqrt{2\pi}}\right) \approx 0.345. \tag{14}$$

Now suppose that the buyer of this option follows the subsequent strategy: at the first moment when S_t reaches the level $S_0 + \widehat{C}$ (if this happens before T), she "locks in" her profit by going short (i.e., selling) one unit of the underlying security. Of course, the "first moment when ..." is a stopping time τ in modern terminology. A moment's reflection reveals that in the case $\tau \le T$ the speculator cannot end up with a negative result and will have a strictly positive gain, when, in addition to $\tau \le T$, the price S_T at expiration time happens to be less than K. But, of course, this operation of "locking in" the profit only happens if S_t attains the level $S_0 + \widehat{C}$ for some $0 \le t \le T$, while in the other case the speculator will end up with a loss.

What is the probability of success (i.e., a non-negative result) of a speculator pursuing this strategy? Clearly it equals

$$\mathbb{P}\left[\max_{0\le t\le T} S_t \ge S_0 + \widehat{C}\right] \tag{15}$$

for which we obtain, using the law of the maximum of Brownian motion,

$$\mathbb{P}\left[\max_{0\le t\le T} S_t \ge S_0 + \widehat{C}\right] = 2\left(1 - \Phi\left(\frac{1}{\sqrt{2\pi}}\right)\right) \approx 0.69. \tag{16}$$

In other words, the probability of a non-negative result of this strategy is about 69 %. Again we find it remarkable that this result does not depend on any parameter.

Let us try to give a résumé of this review of Bachelier's remarkable thesis and to compare it with the modern theory, in particular with the Black-Scholes model considered below.

The usual argument against Bachelier and in favor of Black-Scholes is the fact that Bachelier's model of Brownian motion assigns positive probability to negative prices of the underlying stock, while the Black-Scholes model (using geometric Brownian motion) does not. (For the remainder of this section we assume that the reader is already sufficiently familiar with the basic features of the Black-Scholes model as discussed in section 3 below.)

In my opinion this argument is to a large extent a myth: in basic applications of statistics (say, quality control) there are good reasons to model the quantities under consideration (say, the length of a screw) by a normal distribution. Apparently nobody worries that this model also assigns positive probability to a negative length of the screw, although this is at least as absurd as a negative stock price. The reason is, that — if expressed numerically — these probabilities are *"completely negligible"*, as was so nicely phrased by Bachelier.

One might compare the relation of modelling price processes with Brownian motion as opposed to geometric Brownian motion, to that of using linear interest as opposed to continuously compounded interest for cash accounts. Of course, the latter one is logically more appealing, but we all know, that the difference between these two procedures is very minor for short periods (say, less than a year, in the case of reasonable values of the interest rate). On the other hand, in the long run the difference is spectacular.

Similarly, the differences between the Bachelier and the Black-Scholes option pricing formulae are very minor, as long as σ and T remain in a reasonable range, which certainly was the case for the options on the rentes considered by Bachelier (compare the more quantitative discussion at the end of section 3). On the other hand it is worth noting that, for $T \to \infty$, Bachelier's formula $\widehat{C} = \frac{\sigma}{\sqrt{2\pi}}\sqrt{T}$ (see (10) above) for the option price assigns arbitrarily large values to the premium of an option, while an obvious no-arbitrage argument (using assumption 1.4 and the non-negativity of the underlying security) reveals that $\widehat{C}$ is certainly less than S_0.

In my opinion, L. Bachelier has obtained an option pricing model which, for practical purposes, is just as satisfactory as the model obtained by Black and Scholes some 70 years later, with the shortcomings of these models being very similar (e.g., underestimation of extreme movements of the underlying by using normal or lognormal distributions). But there is one crucial idea which L. Bachelier has missed and which is of central importance for the modern theory: the concept of *dynamic hedging* which allows to reduce Bachelier's "fundamental principle" to no-arbitrage considerations. The use of this idea to determine option prices is due to R. Merton (in a footnote of the original Black-Scholes paper [BS 73] this is explicitly acknowledged) and plays a truly fundamental role.

On the other hand, we have seen above that L. Bachelier was already close to this idea when considering trading strategies where the selling of a security would happen at a *stopping time*. But for a full-fledged theory of dynamic hedging, Bachelier would have had to make quite a number of additional pioneering steps in his lonely endeavour of investigating Brownian motion. His situation was in sharp contrast to the situation encountered by the researchers in Mathematical Finance in the last third of the 20^{th} century, who could build on a well-established theory of stochastic integration, as notably developed by K. Itô and by the school of P.A. Meyer in Strasbourg.

In any case, let us stop here with the review of Bachelier's seminal achievements and turn to a systematic development of the modern theory of option pricing, which is based on the notion of *no-arbitrage*.

2 Models of Financial Markets on Finite Probability Spaces

In order to reduce the technical difficulties of the theory of option pricing to a minimum, we assume throughout this section that the probability space Ω underlying our model will be finite, say, $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$. This assumption implies that all functional-analytic delicacies pertaining to different topologies on $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $L^0(\Omega, \mathcal{F}, \mathbb{P})$ evaporate, as all these spaces are simply $\mathbb{R}^N$ (we assume w.l.o.g. that the sigma-algebra $\mathcal{F}$ is the power set of Ω and that $\mathbb{P}$ assigns strictly positive value to each $\omega \in \Omega$). Hence all the functional analysis, which we shall need in section 4 for the case of more general processes, reduces to simple linear algebra in the setting of the present chapter.

Nevertheless we shall write $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, $L^1(\Omega, \mathcal{F}, \mathbb{P})$ etc. below (knowing very well that these spaces are isomorphic in the present setting) to indicate, what we shall encounter in the setting of the general theory.

Definition 2.1 *A model of a* financial market *is an $\mathbb{R}^{d+1}$-valued stochastic process $S = (S)_{t=0}^T = (S_t^0, S_t^1, \ldots, S_t^d)_{t=0}^T$, based on and adapted to the filtered stochastic base $(\Omega, \mathcal{F}, (\mathcal{F})_{t=0}^T, \mathbb{P})$. We shall assume that the zero coordinate S^0, which we call the cash account, satisfies $S_t^0 \equiv 1$, for $t = 0, 1, \ldots, T$. The letter ΔS_t denotes the increment $S_t - S_{t-1}$.*

Definition 2.2 *$\mathcal{H}$ denotes the set of* trading strategies *for the financial market S. An element $H \in \mathcal{H}$ is an $\mathbb{R}^d$-valued process $(H_t)_{t=1}^T = (H_t^1, H_t^2, \ldots, H_t^d)_{t=1}^T$ which is predictable, i.e. each H_t is $\mathcal{F}_{t-1}$-measurable.*

We then define the stochastic integral $(H \cdot S)$ as the $\mathbb{R}$-valued process $((H \cdot S)_t)_{t=0}^T$ given by

$$(H \cdot S)_t = \sum_{j=1}^{t} (H_j, \Delta S_j), \quad t = 0, \ldots, T, \tag{17}$$

where $(.,.)$ denotes the inner product in $\mathbb{R}^d$.

The reader might be puzzeled why we chose S to be $\mathbb{R}^{d+1}$-valued, while we chose H to be $\mathbb{R}^d$-valued. The reason is that we defined the zero coordinate S^0 of S to be identically equal to 1 so that $\Delta S_t^0 \equiv 0$ and this coordinate can not contribute to the stochastic integral (17). We note that this assumption does not restrict the generality of the model as we always may choose the cash account as numéraire. This means, that the stock prices are expressed in units of the cash account, or — in more practical terms — we have expressed stock prices in discounted terms.

On the other hand we want to stress for later use (the change of numéraire theorem 2.13 below) the role of the cash account — which we choose as numéraire — in the definition of a financial market, although the coordinate S^0 presently only serves as a dummy.

Definition 2.3 *We call the subspace K of $L^0(\Omega, \mathcal{F}, \mathbb{P})$ defined by*

$$K = \{(H \cdot S)_T : H \in \mathcal{H}\} \tag{18}$$

the set of contingent claims attainable at price 0.

The economic interpretation is the following: the random variables $f = (H \cdot S)_T$, for some $H \in \mathcal{H}$, are precisely those contingent claims, i.e., the pay-off functions at time T depending on $\omega \in \Omega$, that an economic agent may replicate with zero initial investment, by pursuing some predictable trading strategy H.

For $a \in \mathbb{R}$, we call the *set of contingent claims attainable at price a* the affine space K_a obtained by shifting K by the constant function $a\mathbf{1}$, in other words the random variables of the form $a + (H \cdot S)_T$, for some trading strategy H. Again the economic interpretation is that these are precisely the contingent claims that an economic agent may replicate with an initial investment of a by pursuing some predictable trading strategy H.

Definition 2.4 *We call the convex cone C in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ defined by*

$$C = \{g \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \text{ s.t. there is } f \in K, f \geq g\} \tag{19}$$

the set of contingent claims super-replicable at price 0.

Economically speaking, a contingent claim $g \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is *super-replicable at price* 0, if we can achieve it with zero net investment, subsequently pursuing some predictable trading strategy H — thus arriving at some contingent claim f — and then, possibly, "throwing away money" to arrive at g. This operation of "throwing away money" may seem awkward at this stage, but we shall see later that the set C plays an important role in the development of the theory. Observe that C is a convex cone containing the negative orthant $L^\infty_-(\Omega, \mathcal{F}, \mathbb{P})$. Again we may define C_a as the *contingent claims super-replicable at price a* if we shift C by the constant function $a\mathbf{1}$.

Definition 2.5 *A financial market S satifies the* no-arbitrage condition *(NA) if*

$$K \cap L^0_+(\Omega, \mathcal{F}, \mathbb{P}) = \{0\} \tag{20}$$

or, equivalently,

$$C \cap L^\infty_+(\Omega, \mathcal{F}, \mathbb{P}) = \{0\}, \tag{21}$$

where 0 *denotes the function identically equal to zero.*

In other words we now have formalized the concept of an arbitrage possibility: it consists of the existence of a trading strategy H such that — starting from an initial investment zero — the resulting contingent claim $f = (H \cdot S)_T$ is non-negative and not identically equal to zero. If a financial market does not allow for arbitrage we say it satisfies the *no-arbitrage condition (NA)*.

Definition 2.6 *A probability measure Q on $(\Omega, \mathcal{F})$ is called an* equivalent martingale measure *for S, if $Q \sim \mathbb{P}$ and S is a martingale under Q.*

We denote by $\mathcal{M}^e(S)$ the set of equivalent martingale probability measures and by $\mathcal{M}^a(S)$ the set of all (not necessarily equivalent) martingale probability measures. The letter a stands for "absolutely continuous with respect to $\mathbb{P}$" which in the present setting (finite Ω and $\mathbb{P}$ having full support) automatically holds true, but which will be of relevance for general probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ later. We shall often identify a measure Q on $(\Omega, \mathcal{F})$ with its Radon-Nikodym derivative $\frac{dQ}{d\mathbb{P}} \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.

Lemma 2.7 *For a probability measure Q on $(\Omega, \mathcal{F})$ the following are equivalent:*

(i) $Q \in \mathcal{M}^a(S)$,

(ii) $\mathbf{E}_Q[f] = 0$, *for all* $f \in K$,

(iii) $\mathbf{E}_Q[g] \leq 0$, *for all* $g \in C$.

Proof The equivalences are rather trivial, as (ii) is tantamount to the very definition of S being a martingale under Q, and the equivalence of (ii) and (iii) is straightforward. ■

After having fixed these formalities we may formulate and prove the central result of the theory of pricing and hedging by no-arbitrage, sometimes called the "fundamental theorem of asset pricing", which in its present form (i.e., finite Ω) is due to Harrison and Pliska [HP 81].

Theorem 2.8 (Fundamental Theorem of Asset Pricing) *For a financial market S modeled on a finite stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ the following are equivalent:*

(i) S satisfies (NA).

(ii) $\mathcal{M}^e(S) \neq \emptyset$.

Proof (ii) $\Rightarrow$ (i): This is the obvious implication. If there is some $Q \in \mathcal{M}^e(S)$ then by lemma 2.7 we have that

$$\mathbf{E}_Q[g] \leq 0, \quad \text{for } g \in C. \tag{22}$$

On the other hand, if there were $g \in C \cap L^\infty_+$, $g \neq 0$, then, using the assumption that Q is equivalent to $\mathbb{P}$, we would have

$$\mathbf{E}_Q[g] > 0, \tag{23}$$

a contradiction.

(i) $\Rightarrow$ (ii) This implication is the important message of the theorem which will allow us to link the no-arbitrage arguments with martingale theory. We give a functional analytic existence proof, which will be generalizable — in spirit — to more general situations.

By assumption the space K intersects L^∞_+ only at 0. We want to separate the disjoint convex sets $L^\infty_+ \backslash \{0\}$ and K by a hyperplane induced by a linear functional $Q \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Unfortunately this is a situation, where the usual versions of the separation theorem (i.e., the Hahn-Banach Theorem) do not apply (even in finite dimensions!).

One way to overcome this difficulty (in finite dimension) is to consider the convex hull of the unit vectors $(\mathbf{1}_{\{\omega_n\}})_{n=1}^N$ in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ i.e.

$$P := \left\{ \sum_{n=1}^N \mu_n \mathbf{1}_{\{\omega_n\}} : \mu_n \geq 0, \sum_{n=1}^N \mu_n = 1 \right\}. \tag{24}$$

This is a convex, compact subset of $L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$ and, by the *(NA)* assumption, disjoint from K. Hence we may strictly separate the sets P and K by a linear functional $Q \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* = L^1(\Omega, \mathcal{F}, \mathbb{P})$, i.e., find $\alpha < \beta$ such that

$$\begin{aligned} \langle Q, f\rangle &\leq \alpha \quad \text{for} \quad f \in K, \\ \langle Q, h\rangle &\geq \beta \quad \text{for} \quad h \in P. \end{aligned} \tag{25}$$

As K is a linear space, we have $\alpha \geq 0$ and may, in fact, replace α by 0. Hence $\beta > 0$. Therefore $\langle Q, \mathbf{1}\rangle > 0$, and we may normalize Q such that $\langle Q, \mathbf{1}\rangle = 1$. As Q is strictly positive on each $\mathbf{1}_{\{\omega_n\}}$, we therefore have found a probability measure Q on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ such that condition (ii) of lemma 2.7 holds true. In other words, we found an equivalent martingale measure Q for the process S. ■

Corollary 2.9 *Let S satisfy (NA) and $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ be an attainable contingent claim so that*

$$f = a + (H \cdot S)_T, \tag{26}$$

for some $a \in \mathbb{R}$ and some trading strategy H.

Then the constant a and the process $(H \cdot S)$ are uniquely determined by (26) *and satisfy, for every $Q \in \mathcal{M}^e(S)$,*

$$a = \mathbf{E}_Q[f], \qquad \textit{and} \qquad a + (H \cdot S)_t = \mathbf{E}_Q[f|\mathcal{F}_t] \textit{ for } 0 \leq t \leq T. \tag{27}$$

Proof As regards the uniqueness of the constant $a \in \mathbb{R}$, suppose that there are two representations $f = a^1 + (H^1 \cdot S)_T$ and $f = a^2 + (H^2 \cdot S)_T$ with $a^1 \neq a^2$. Assuming w.l.o.g. that $a^1 > a^2$ we find an obvious arbitrage possibility: we have $a^1 - a^2 = ((H^1 - H^2) \cdot S)_T$, i.e. the trading strategy $H^1 - H^2$ produces a strictly positive result at time T, a contradiction to *(NA)*.

As regards the uniqueness or the process $H \cdot S$ we simply apply a conditional version of the previous argument: assume that $f = a + (H^1 \cdot S)_T$ and $f = a + (H^2 \cdot S)_T$ such that the processes $H^1 \cdot S$ and $H^2 \cdot S$ are not identical. Then there is $0 < t < T$ such that $(H^1 \cdot S)_t \neq (H^2 \cdot S)_t$; w.l.o.g. $A := \{(H^1 \cdot S)_t > (H^2 \cdot S)_t\}$ is a non-empty event, which clearly is in $\mathcal{F}_t$. Hence, using the fact that $(H^1 \cdot S)_T = (H^2 \cdot S)_T$, the trading strategy $H := (H^2 - H^1)\chi_A \cdot \chi_{]t,T]}$ is a predictable process producing an arbitrage, as $(H \cdot S)_T = 0$ outside A, while $(H \cdot S)_T = (H^1 \cdot S)_t - (H^2 \cdot S)_t > 0$ on A, which again contradicts *(NA)*.

Finally, the equations in (27) result from the fact that, for every predictable process H and every $Q \in \mathcal{M}^a(S)$, the process $H \cdot S$ is a Q-martingale. ■

Denote by $\text{cone}(\mathcal{M}^e(S))$ and $\text{cone}(\mathcal{M}^a(S))$ the cones generated by the convex sets $\mathcal{M}^e(S)$ and $\mathcal{M}^a(S)$, respectively. The subsequent result clarifies the polar relation between these cones and the cone C. Recall (see, e.g., [Sch 66]) that, for a pair (E, E') of vector spaces in separating duality via the scalar product $\langle ., .\rangle$, the polar C^0 of a set C in E is defined as

$$C^0 = \{g \in E' : \langle f, g\rangle \leq 1, \text{ for all } f \in C\}. \tag{28}$$

In the case when C is closed under multiplication with positive scalars (e.g., if C is a convex cone) the polar C^0 may equivalently be defined by

$$C^0 = \{g \in E' : \langle f, g\rangle \leq 0, \text{ for all } f \in C\}. \tag{29}$$

The *bipolar theorem* (see, e.g., [Sch 66]) states that the bipolar $C^{00} := (C^0)^0$ of a set C in E is the $\sigma(E, E')$-closed convex hull of C.

After these general considerations we pass to the concrete setting of the cone $C \subseteq L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ of contingent claims super-replicable at price 0. Note that in our finite-dimensional setting this convex cone is closed as it is the algebraic sum of the closed linear space K (a linear space in $\mathbb{R}^N$ is always closed) and the closed polyhedral cone $L^\infty_-(\Omega, \mathcal{F}, \mathbb{P})$ (the verification, that the algebraic sum of a space and a polyhedral cone in $\mathbb{R}^N$ is closed, is an easy, but not completely trivial exercise). Hence we deduce from the bipolar theorem, that C equals its bipolar C^{00}.

Proposition 2.10 *Suppose that S satisfies (NA). Then the polar of C is equal to cone($\mathcal{M}^a(S)$) and $\mathcal{M}^e(S)$ is dense in $\mathcal{M}^a(S)$. Hence the following assertions are equivalent for an element $g \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$*

(i) $g \in C$,

(ii) $\mathbf{E}_Q[g] \leq 0$, *for all* $Q \in \mathcal{M}^a(S)$,

(iii) $\mathbf{E}_Q[g] \leq 0$, *for all* $Q \in \mathcal{M}^e(S)$,

Proof The fact that the polar C^0 and cone($\mathcal{M}^a(S)$) coincide, follows from lemma 2.7 and the observation that $C \supseteq L^\infty_-(\Omega, \mathcal{F}, \mathbb{P})$ implies $C^0 \subseteq L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$. Hence the equivalence of (i) and (ii) follows from the bipolar theorem.

As regards the density of $\mathcal{M}^e(S)$ in $\mathcal{M}^a(S)$ we first deduce from theorem 2.8 that there is at least one $Q^* \in \mathcal{M}^e(S)$. For any $Q \in \mathcal{M}^a(S)$ and $0 < \mu \leq 1$ we have that $\mu Q^* + (1-\mu)Q \in \mathcal{M}^e(S)$, which clearly implies the density of $\mathcal{M}^e(S)$ in $\mathcal{M}^a(S)$. The equivalence of (ii) and (iii) now is obvious. ∎

The subsequent theorem tells us precisely what the principle of no arbitrage can tell us about the possible prices for a contingent claim f. It goes back to the work of D. Kreps [K 81] and was subsequently extended by several authors.

For given $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, we call $a \in \mathbb{R}$ an *arbitrage-free price*, if in addition to the financial market S, the introduction of the contingent claim f at price a does not create an arbitrage possibility. Mathematically speaking, this can be formalized as follows. Let $C^{f,a}$ denote the cone spanned by C and the linear space spanned by $f - a$; then a is an arbitrage-free price for f if $C^{f,a} \cap L^\infty_+(\Omega, \mathcal{F}, \mathbb{P}) = \{0\}$.

Theorem 2.11 (Pricing by No-Arbitrage) *Assume that S satisfies (NA) and let $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Define*

$$\overline{\pi}(f) = \sup \{\mathbf{E}_Q[f] : Q \in \mathcal{M}^e(S)\}, \tag{30}$$

$$\underline{\pi}(f) = \inf \{\mathbf{E}_Q[f] : Q \in \mathcal{M}^e(S)\}, \tag{31}$$

Either $\underline{\pi}(f) = \overline{\pi}(f)$, in which case f is attainable at price $\pi(f) := \underline{\pi}(f) = \overline{\pi}(f)$, i.e. $f = \pi(f) + (H \cdot S)_T$ for some $H \in \mathcal{H}$; therefore $\pi(f)$ is the unique arbitrage-free price for f.

Or $\underline{\pi}(f) < \overline{\pi}(f)$, in which case $\{\mathbf{E}_Q[f] : Q \in \mathcal{M}^e(S)\}$ equals the open interval $]\underline{\pi}(f), \overline{\pi}(f)[$, which in turn equals the set of arbitrage-free prices for the contingent claim f.

Proof First observe that the set $\{\mathbf{E}_Q[f] : Q \in \mathcal{M}^e(S)\}$ forms a bounded non-empty interval in $\mathbb{R}$, which we denote by I.

We claim that a number a is in I, iff a is an arbitrage-free price for f. Indeed, supposing that $a \in I$ we may find $Q \in \mathcal{M}^e(S)$ s.t. $\mathbf{E}_Q[f - a] = 0$ and therefore $C^{f,a} \cap L^\infty_+(\Omega, \mathcal{F}, \mathbb{P}) = \{0\}$.

Conversely suppose that $C^{f,a} \cap L^\infty_+(\Omega, \mathcal{F}, \mathbb{P}) = \{0\}$. Note that $C^{f,a}$ is a closed convex cone (it is the algebraic sum of the linear space $\text{span}(K, f - a)$ and the closed, polyhedral cone $L^\infty_-(\Omega, \mathcal{F}, \mathbb{P})$). Hence by the same argument as in the proof of theorem 2.8 there exists a probability measure $Q \sim \mathbb{P}$ such that $Q|_{C^{f,a}} \leq 0$. This implies that $\mathbf{E}_Q[f - a] = 0$, i.e., $a \in I$.

Now we deal with the boundary case: suppose that a equals the right boundary of I, i.e., $a = \overline{\pi}(f) \in I$, and consider the contingent claim $f - \overline{\pi}(f)$; by definition we have $\mathbf{E}_Q[f - \overline{\pi}(f)] \leq 0$, for all $Q \in \mathcal{M}^e(S)$, and therefore by proposition 2.10, that $f - \overline{\pi}(f) \in C$. We may find $g \in K$ such that $g \geq f - \overline{\pi}(f)$. If the sup in (30) is attained, i.e., if there is $Q^* \in \mathcal{M}^e(S)$ such that $\mathbf{E}_{Q^*}[f] = \overline{\pi}(f)$, then we have $0 = \mathbf{E}_{Q^*}[g] \geq \mathbf{E}_{Q^*}[f - \overline{\pi}(f)] = 0$ which in view of $Q^* \sim \mathbb{P}$ implies that $f - \overline{\pi}(f) \equiv g$; in other words f is attainable at price $\overline{\pi}(f)$. This in turn implies that $\mathbf{E}_Q[f] = \overline{\pi}(f)$, for all $Q \in \mathcal{M}^e(S)$, and therefore I is reduced to the singleton $\{\overline{\pi}(f)\}$.

Hence, if $\underline{\pi}(f) < \overline{\pi}(f)$, $\overline{\pi}(f)$ cannot belong to the interval I, which is therefore open on the right hand side. Passing from f to $-f$, we obtain the analogous result for the left hand side of I, which therefore equals $I =]\underline{\pi}(f), \overline{\pi}(f)[$. ■

Corollary 2.12 (complete financial markets) *For a financial market S satisfying the no-arbitrage condition (NA) the following are equivalent:*

(i) $\mathcal{M}^e(S)$ consists of a single element Q.

(ii) Each $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ may be represented as

$$f = a + (H \cdot S)_T, \quad \text{for some } a \in \mathbb{R}, \text{ and } H \in \mathcal{H}. \tag{32}$$

In this case $a = \mathbf{E}_Q[f]$, the stochastic integral $(H \cdot S)$ is unique, and we have that

$$\mathbf{E}_Q[f|\mathcal{F}_t] = \mathbf{E}_Q[f] + (H \cdot S)_t, \quad t = 0, \ldots, T. \tag{33}$$

Proof The implication (i) $\Rightarrow$ (ii) immediately follows from the preceding theorem; for the implication (ii) $\Rightarrow$ (i), note that, (32) implies that, for elements Q_1, $Q_2 \in \mathcal{M}^a(S)$, we have $\mathbf{E}_{Q_1}[f] = a = \mathbf{E}_{Q_2}[f]$; hence it suffices to note that if $\mathcal{M}^e(S)$ contains two different elements Q_1, Q_2 we may find $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ s.t. $\mathbf{E}_{Q_1}[f] \neq \mathbf{E}_{Q_2}[f]$. ■

Let us pause here for a moment and recapitulate the above results from an economic point of view. In particular we address the question: how does this theory relate to Bachelier's *fundamental principle*?

We consider a model S of a financial market satisfying the assumptions of corollary 2.12. The reason why these (models of) financial markets are called *complete* in the Mathematical Finance literature is related to assertion (ii) above: in such a market any contingent claim f is already replicable by an initial investment a and a properly chosen trading strategy H. We shall see in the next section that the arch-example of a complete financial market in discrete time is the random walk, also called the binomial model. We have seen that this model was already considered by Bachelier (over a grid in arithmetic progression); some 70 years later, Cox, Ross and Rubinstein [CRR 79] studied this model over a grid in geometric progression.

The basic problem of Bachelier, as well as of modern Mathematical Finance in general, is that of assigning a price a to a contingent claim f; corollary 2.12 tells us that, in the case of a complete market, we simply have to take the expectation $\mathbf{E}_Q[f]$, similarly as Bachelier proposed in his "fundamental principle". But now the argument in favor of this methodology is based on a no-arbitrage argument, which is more robust from an economic point of view than the equilibrium argument used by Bachelier.

Also, the message of corollary 2.12 is not quite identical to Bachelier's "fundamental principle". The subtle difference is that in modern Mathematical Finance one takes the expectation with respect to a *risk neutral* probability measure Q, i.e., a measure under which S is a martingale and which does not necessarily coincide with the *physical* measure P. This distinction between P and Q does not show up in Bachelier's work (although he also is speaking about "two kinds of probability", but apparently he has something different in mind in this passage of his thesis). Bachelier argues somehow in the opposite direction as compared to the modern approach: he postulates that the process S has to be a martingale already under the "physical" measure P (this is what his "fundamental principle" amounts to in modern terminology).

The distinction between the measure Q and P is one of the crucial features of the modern approach to Mathematical Finance. It is implicit in the early work of Black and Scholes [BS 73] and Merton [M 73], and has clearly been crystallized in the later work of Harrison, Kreps and Pliska ([HK 79], [HP 81], [K 81]).

In this respect Bachelier's approach really misses something crucial: for example, there is massive empirical evidence that — in the long run — stocks perform better than bonds. At least, this happened in the previous hundred or two hundred years. Many people believe that this will also be the case in the future (but, of course, we don't know that). In any case, Bachelier has no way of modelling such a phenomenon without violating the "fundamental principle".

One might try to argue in favor of Bachelier that such a long term effect is not of crucial importance for short term option prices and may therefore be ignored.

But there are also other obstructions to the somewhat naive application of the "fundamental principle", which involve logical inconsistencies (which is, of course, particularly annoying from a mathematical part of view): let's take up again the foreign exchange example 1.2 and assume, mainly for notational convenience, that the domestic and foreign interest rates r_d and r_f equal zero. The stochastic process $(X_t)_{0\le t\le T}$ models the price of one US\$ in terms of €. By applying Bachelier's "fundamental principle" to the situation of a €-investor "speculating" in US\$, we must have

$$X_0 = \mathbf{E}[X_T]. \tag{34}$$

On the other hand, the same principle applied to the situation of a US\$-investor "speculating" in € implies

$$X_0^{-1} = \mathbf{E}[X_T^{-1}]. \tag{35}$$

But Jensen's inequality tells us that (34) and (35) cannot hold simultaneously (except for the trivial case when X_T is constant). Hence we find a logical obstruction to the "fundamental principle" of Bachelier.

At this stage a distiction between the measure P and Q is unavoidable and we also see from the above argument that the "risk-neutral" measure Q apparently depends on the choice of a numéraire.

We therefore pass to a thorough analysis of the role of the numéraire S^0 in our modelling of a financial market. In particular, we investigate what happens under a "change of numéraire", i.e., by passing from one unit of denomination, say €, to another one, say US $.

Let us consider once more the basic example 1.2 of a financial market consisting of a €-bond and a $-bond (which we now consider in discrete time, to confirm with the setting of this section). We now drop the assumption $r_d = r_f = 0$ and assume that these bonds develop (expressed in terms of € and $ respectively) by

$$B_t^{€} = e^{r_d t}, \quad B_t^{\$} = e^{r_f t}, \quad t = 0, 1, \ldots, T. \tag{36}$$

Denoting again by $(X_t)_{0 \leq t \leq T}$ the stochastic process modelling the exchange rate, the value (in terms of €) of an investment into the $-bond is given by the stochastic process $(e^{r_f t} X_t)_{t=0}^T$. But note that this refers to the € as numéraire, which *is not a traded asset*, unless we have $r_d = 0$. This may seem odd a first glance; but remember our standing assumption that we can go long and short in traded assets at the same conditions. If the Euro were a traded asset, this would imply that we could borrow Euros at nominal value (i.e., without paying interest); combining this operation with an investment into a €-bond paying positive interest, clearly creates an arbitrage.

We have agreed to choose a traded asset as numéraire: from the point of view of a €-investor, the natural choice in our example is the €-bond. Hence from her point of view the financial market is modeled by

$$S_t = (S_t^0, S_t^1) = \left(1, e^{(r_f - r_d)t} X_t\right), \quad t = 0, 1, \ldots, T, \tag{37}$$

where S now is expressed in terms of units of the €-bond.

But adopting the point of view of a $-investor it is natural to express everything in terms of the $-bond, i.e.

$$\check{S}_t = \left(\frac{S_t^0}{S_t^1}, \frac{S_t^1}{S_t^1}\right) = \left(e^{(r_d - r_f)t} X_t^{-1}, 1\right), \quad t = 0, 1, \ldots, T. \tag{38}$$

The previous theorem 2.11 and corollary 2.12 tell us, how to relate the arbitrage free prices of derivative securities $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ to the expectations under the "risk-neutral" probabilities $Q \in \mathcal{M}^e(S)$.

How do these things change, when we pass to a new numéraire? Of course, the arbitrage free prices should remain unchanged (after denominating things in the new numéraire), as the notion of arbitrage should not depend on whether we do the book-keeping in € or in $. On the other hand, we shall presently see that the risk-neutral measures Q do depend on the choice of numéraire.

Let us analyze the situation in the proper degree of generality: the model of a financial market $S = (S_t^0, S_t^1, \ldots, S_t^d)_{t=0}^T$ is defined as above. Recall that we assumed that the traded asset S^0 serves as numéraire, i.e., the value S_t^j of the j'th asset at time t is expressed in units of S^0. In particular, we have $S_t^0 = 1$, for all $0 \leq t \leq T$.

We also assume that the process $(S_t^1)_{0\leq t\leq T}$ is strictly positive; choosing this asset as the new numéraire we find the process $\check{S}$ denoting the prices of the assets $S^0, S^1, \ldots, S^d$ in terms of S^1:

$$\check{S} = \left(\frac{S_t^0}{S_t^1}, 1, \frac{S_t^2}{S_t^1}, \ldots, \frac{S_t^d}{S_t^1} \right)_{t=0}^T . \tag{39}$$

To link with the previous example, we might have that S^0 is a cash account in €, S^1 a cash account in US \$, while $S^2, \ldots, S^d$ model some other stocks, commodities etc.

We now have the proper setting to formulate the theorem clarifying the situation:

Theorem 2.13 (change of numéraire) *Assume that the financial market* $S = (S_t^0, S_t^1, \ldots, S_t^d)$ *satisfies (NA) and recall that we have assumed* $S_t^0 \equiv 1$*, i.e., we have chosen the zero coordinate as numéraire.*

We also assume that the first coordinate $(S_t^1)_{t=0}^T$ *is a strictly positive process, so that we may define the* "process S in terms of the numéraire S^1" *by passing to*

$$\check{S} = \left(\frac{S_t^0}{S_t^1}, 1, \frac{S_t^2}{S_t^1}, \ldots, \frac{S_t^d}{S_t^1} \right)_{t=0}^T . \tag{40}$$

Then the set $\mathcal{M}^e(\check{S})$ *of equivalent martingale measures for* $\check{S}$ *equals*

$$\mathcal{M}^e(\check{S}) = \left\{ \check{Q} : \frac{d\check{Q}}{d\mathbb{P}} = \frac{S_T^1}{S_0^1} \frac{dQ}{d\mathbb{P}},\ Q \in \mathcal{M}^e(S) \right\} . \tag{41}$$

For a contingent claim $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ *the interval of arbitrage-free prices therefore does not depend on the chosen numéraire, as we have*

$$\{\mathbf{E}_Q[f] : Q \in \mathcal{M}^e(S)\} = \left\{ S_0^1 \mathbf{E}_{\check{Q}} \left[\frac{f}{S_T^1} \right] : \check{Q} \in \mathcal{M}^e(\check{S}) \right\} . \tag{42}$$

Proof Note that the fact that S is a Q-martingale implies that $\mathbf{E}_Q[\frac{S_T^1}{S_0^1}] = 1$, for all $Q \in \mathcal{M}^e(S)$, so that the set defined by the right hand side of (41) consists of probability measures. Also note that, by our assumption on the strict positivity of S^1, these measures are equivalent to $\mathbb{P}$.

We now calculate the space $\check{K} \subseteq L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ of claims attainable at price 0 with respect to $\check{S}$,

$$\check{K} = \left\{ (L \cdot \check{S})_T : L \in \mathcal{H} \right\} . \tag{43}$$

We claim that

$$\check{K} = \left\{ \frac{1}{S_T^1} f : f \in K \right\} . \tag{44}$$

In fact, we claim more generally, that the class of processes of the form $S_t^1 (L \cdot \check{S})_t$ coincides with the class of processes $(H \cdot S)_t$ where L and H run through the predictable processes.

Economically speaking this means that the possible gains processes $H \cdot S$, obtained by trading with respect to some trading strategy H in terms of the numéraire S^0, coincide with the possible gains processes $S^1(L \cdot \check{S})$, where $L \cdot \check{S}$ run through the possible gains processes in terms of the numéraire S^1, which subsequently are transformed into units of the numéraire S^0 by multiplication with S^1.

To verify this — economically rather obvious — identity in a formal way we use a little stochastic calculus, namely the stochastic version of the product formula (similarly as in [DS 95, theorem 11]). We have

$$S_t = S_t^1 \check{S}_t, \tag{45}$$

the right hand side refering to multiplication of the positive scalar $S_t^1(\omega)$ with the $(d+1)$-vector $\check{S}_t(\omega) = (\frac{S_t^0(\omega)}{S_t^1(\omega)}, 1, \frac{S_t^2(\omega)}{S_t^1(\omega)}, \dots, \frac{S_t^d(\omega)}{S_t^1(\omega)})$. Hence by elementary algebra we obtain

$$\Delta S_t = S_{t-1}^1 \Delta \check{S}_t + \check{S}_{t-1} \Delta S_t^1 + \Delta S_t^1 \Delta \check{S}_t. \tag{46}$$

Now we fix any predictable process L and calculate the increment of the process $S_t^1(L \cdot \check{S})_t$ in a similar way:

$$\begin{aligned} \Delta(S_t^1(L \cdot \check{S})_t) &= (L \cdot \check{S})_{t-1} \Delta S_t^1 + S_{t-1}^1 \Delta((L \cdot \check{S})_t) + \Delta S_t^1 \Delta((L \cdot \check{S})_t) \\ &= (L \cdot \check{S})_{t-1} \Delta S_t^1 + L_t(S_{t-1}^1 \Delta \check{S}_t + \Delta S_t^1 \Delta \check{S}_t) \\ &= (L \cdot \check{S})_{t-1} \Delta S_t^1 + L_t(S_{t-1}^1 \Delta \check{S}_t + \check{S}_{t-1} \Delta S_t^1 + \Delta S_t^1 \Delta \check{S}_t) \\ &\quad - L_t \check{S}_{t-1} \Delta S_t^1 \\ &= ((L \cdot \check{S})_{t-1} - L_t \check{S}_{t-1}) \Delta S_t^1 + L_t \Delta S_t, \end{aligned} \tag{47}$$

where in the last equality we have used (46). In other words, the increment $\Delta(S_t^1(L \cdot \check{S})_t)$ of the process $S^1(L \cdot \check{S})$ is the product of some $\mathcal{F}_{t-1}$-measurable functions with the increments ΔS_t^1 and ΔS_t respectively. Noting that ΔS_t^1 is just one of the coordinates of ΔS_t, we conclude that the process $S_t^1(L \cdot \check{S}_t)$ may be represented as a stochastic integral of the form $(H \cdot S)$ for some predictable process H. Reversing the roles of S and $\check{S}$ and using the strict positivity of the process S^1, we also conclude that each process of the form $(H \cdot S)$ may be presented in the form $S^1(L \cdot \check{S})$, for some predictable process L, which shows in particular (44).

Hence the linear map $M : L^\infty \to L^\infty$ of multiplication by the function $\frac{S_0^1}{S_T^1}$

$$Mf = \frac{S_0^1}{S_T^1} f \tag{48}$$

maps K bijectively onto $\check{K}$. By basic linear algebra the adjoint M^* of M, which is equal to M, maps the polar of $\check{K}$ onto the polar of K and therefore $cone(\mathcal{M}^a(\check{S}))$ and $cone(\mathcal{M}^e(\check{S}))$ onto $cone(\mathcal{M}^a(S))$ and $cone(\mathcal{M}^e(S))$ respectively. Hence we obtain the identity (41). Finally observe that equality (42) is an immediate consequence of equality (41), noting that, when Q runs through $\mathcal{M}^e(S)$, then $M^{-1}(Q)$ runs through $\mathcal{M}^e(\check{S})$. ■

Corollary 2.14 (change of numéraire in a complete market) *Assume in addition to the assumptions of theorem 2.13 that $\mathcal{M}^e(S)$ consists of a singleton $\{Q\}$. Then $\mathcal{M}^e(\check{S}) = \{\check{Q}\}$ where $\frac{d\check{Q}}{d\mathbb{P}} = \frac{S_T^1}{S_0^1}\frac{dQ}{d\mathbb{P}}$.*

For $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, we obtain the unique arbitrage free price as

$$\mathbf{E}_Q[f] = S_0^1 \mathbf{E}_{\check{Q}}\left[\frac{f}{S_T^1}\right]. \blacksquare \tag{49}$$

We finish this section by a dynamic version of theorem 2.11 on pricing by no arbitrage, due to D. Kramkov (in a much more general setting; see [K 96] and section 5 below).

Theorem 2.15 (Optional Decomposition) *Assume that S satisfies (NA) and let $V = (V_t)_{t\geq 0}$ be an adapted process.*

The following assertions are equivalent:

(i) V is a super-martingale for each $Q \in \mathcal{M}^e(S)$.

(i') V is a super-martingale for each $Q \in \mathcal{M}^a(S)$

(ii) V may be decomposed into $V = V_0 + H \cdot S - C$, where $H \in \mathcal{H}$ and $C = (C_t)_{t\geq 0}$ is an increasing adapted process starting at $C_0 = 0$.

To explain the terminology *"optional decomposition"* let us compare this theorem with Doob's celebrated decomposition theorem for non-negative super-martingales $(V_t)_{t\geq 0}$ (see, e.g., [P 90]): this theorem asserts that, for a non-negative (adapted, càdlàg) process V, we have equivalence between the following two statements:

(i) V is a super-martingale (with respect to the fixed measure $\mathbb{P}$),

(ii) V may be decomposed in a unique way into $V = V_0 + M - C$, where $M = (M_t)_{t\geq 0}$ is a local martingale (with respect to $\mathbb{P}$) and C an increasing predictable process s.t. $M_0 = C_0 = 0$.

We see the similarity in spirit, but, of course, there are differences. As regards condition (i) the difference is that, in the setting of the optional decomposition theorem, the super-martingale properly pertains to *all* martingale measures Q for the process S. As regards condition (ii) the role of the local martingale M in Doob's theorem is taken by the stochastic integral $H \cdot S$. A decisive difference between the two theorems is that, in theorem 2.15, the decomposition is not unique any more and one cannot choose, in general, C to be predictable. The process C can only be chosen to be adapted and therefore optional (for finite Ω, a process is adapted iff it is optional).

The economic interpretation of the optional decomposition theorem goes as follows: a process of the form $V = V_0 + H \cdot S - C$ describes the wealth process of an economic agent, starting at an initial wealth V_0, subsequently investing in the financial market according to the trading strategy H, and consuming as described by the process C: the random variable C_t models the accumulated consumption during the time interval $\{1, \ldots, t\}$. The message of the optional decomposition theorem is that these wealth processes are characterised by condition (i) (or, equivalently, (i')).

Proof of theorem 2.15 First assume that $T = 1$, i.e., we have a one-period model $S = (S_0, S_1)$. In this case the present theorem is an immediate consequence of theorem 2.11: if V is a super-martingale under each $Q \in \mathcal{M}^e(S)$, then

$$\mathbf{E}_Q[V_1] \le V_0, \qquad \text{for all } Q \in \mathcal{M}^e(S). \tag{50}$$

Hence there is a predictable trading strategy H such that $V_0 + (H \cdot S)_1 \ge V_1$. Letting $C_0 = 0$ and writing $\Delta C_1 = C_1 = V_1 - (V_0 + (H \cdot S)_1)$ we have obtained the desired decomposition.

Recall our general assumption that $\mathcal{F}_0$ is trivial; it implies that the trading strategy $H = H_1$ simply is a vector in $\mathbb{R}^d$, as an $\mathcal{F}_0$-measurable function is constant. But this assumption is not at all essential for the above argument: if $\mathcal{F}_0$ is not trivial, we simply apply the above argument to each of the atoms of the sigma-algebra $\mathcal{F}_0$ to obtain an $\mathcal{F}_0$-measurable function H_1.

Hence we may apply, for each fixed $t \in \{1, \ldots, T\}$, the same argument as above to the one-period financial market (S_{t-1}, S_t) based on $(\Omega, \mathcal{F}, \mathbb{P})$ and adapted to the filtration $(\mathcal{F}_{t-1}, \mathcal{F}_t)$. We thus obtain an $\mathcal{F}_{t-1}$-measurable $\mathbb{R}^d$-valued function H_t and a non-negative $\mathcal{F}_t$-measurable function ΔC_t such that

$$\Delta V_t = (H_t, \Delta S_t) - \Delta C_t, \tag{51}$$

where $(.,.)$ denotes the inner product in $\mathbb{R}^d$.

This finishes the construction of the optional decomposition: define the predictable process H as $(H_t)_{t=1}^T$, and the adapted increasing process C by $C_t = \Sigma_{j=1}^t \Delta C_j$.

This shows the implication (i) $\Rightarrow$ (ii); the implications (ii) $\Rightarrow$ (i') $\Rightarrow$ (i) are trivial. ■

3 The Binomial Model, Bachelier's Model and the Black-Scholes Model

The canonical example of a finite probability space Ω, to which the no-arbitrage theory applies very nicely, is the *binomial* model. Let $\Omega = \{-1, +1\}^T$ be equipped with the filtration $(\mathcal{F}_t)_{t=0}^T$, where $\mathcal{F}_t$ is generated by the first t coordinate maps on Ω. As probability measure $\mathbb{P}$ we chose the uniform measure on $\mathcal{F} = \mathcal{F}_T$, but we remark that the subsequent results do not depend on this special choice of $\mathbb{P}$; the only property of $\mathbb{P}$ which is needed is, that $\mathbb{P}$ assigns positive mass to each point of Ω.

Consider a financial market based on $(\Omega, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ consisting of a cash account $B_t := (S_t^0)_{t=0}^T \equiv 1$ and a risky asset (stock) $(S_t^1)_{t=0}^T$ which is an $\mathbb{R}$-valued adapted process defined on $(\Omega, (\mathcal{F}_t)_{t=0}^T, \mathcal{F})$. By abuse of notation we also shall write S for the one-dimensional process S^1.

To avoid trivialities we assume that

$$\mathbb{P}[S_t \neq S_{t-1} | \mathcal{F}_{t-1}] > 0 \quad \text{everywhere, for } t = 1, \ldots, T. \tag{52}$$

It ist rather obvious and very intuitive that S does not allow arbitrage iff

$$\mathbb{P}[S_t > S_{t-1} | \mathcal{F}_{t-1}] > 0 \text{ and } \mathbb{P}[S_t < S_{t-1} | \mathcal{F}_{t-1}] > 0, \quad \text{for } t = 1, \ldots, T. \tag{53}$$

It is just as obvious — using, e.g., backward induction (compare [LL 96]) — that in this case there exists a unique equivalent martingale measure Q. Hence we know that, for any contingent claim $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, we can find a trading strategy H such that

$$f = \mathbf{E}_Q[f] + (H \cdot S)_T \tag{54}$$

and that we have, for every $t = 0, \ldots, T$,

$$\mathbf{E}_Q[f|\mathcal{F}_t] = \mathbf{E}_Q[f] + (H \cdot S)_t. \tag{55}$$

We now specialize to two concrete cases for the financial market model S: the first example is the simple random walk; this was considered by Bachelier as a discrete approximation to Brownian motion. The second one is the multiplicative version of the random walk — i.e., $(\ln(\frac{S_t}{S_0}))_{t=0}^T$ is a random walk, possibly with drift. In finance the latter model is called the Cox, Ross, and Rubinstein model [CRR 79]. These authors analyzed this model as a discrete analogue to geometric Brownian motion.

In the former case, i.e., the simple random walk, where $(S_t - S_{t-1})_{t=1}^T$ are i.i.d. random variables taking values $\pm\sigma\Delta x$ with probability $\frac{1}{2}$, the original measure $\mathbb{P}$ is already the unique martingale measure for the process $(S_t)_{t=0}^T$. Hence we deduce from corollary 2.14 that the unique arbitrage-free price of a contingent claim $f \in L^\infty(P)$ is given by $\mathbf{E}_P[f]$, which justifies Bachelier's "fundamental principle" on the basis of no-arbitrage arguments for the model of a simple random walk.

In the Cox-Ross-Rubinstein case the original measure $\mathbb{P}$ — in general — is not a martingale measure, but it is easy to explicitly calculate the density $\frac{dQ}{d\mathbb{P}}$ (which amounts to a discrete version of Girsanov's theorem).

In both cases the pricing formulae for an option reduce to the calculation of the expected value of the hockey-stick function $f(x) = (x - K)_+$ with respect to a binomial distribution, placed on a sequence of points in arithmetic progression in the former and on a sequence in geometric progression in the latter case.

We leave the elementary but somewhat cumbersome calculations of the resulting formulae in the first case to the energetic reader (who may also find the calcuations essentially in Bachelier's thesis) and in the second case we refer to the beautiful book by Lamberton and Lapeyre [LL 96], where these calculations are presented in a clean and transparent way.

We now pass on to the continuous limits of these models (if properly normalised), where — as usual in mathematics — the results and formulae become more elegant and more transparent.

To do so, we recall the *martingale representation theorem* for Brownian motion, which is the continuous analogue to the elementary considerations on the binomial model above.

Theorem 3.1 *(see, e.g., [RY 91]) Let $(W_t)_{0\le t\le T}$ be a standard Brownian motion modeled on $(\Omega, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$, where $(\mathcal{F}_t)_{0\le t\le T}$ is the natural (saturated) filtration generated by W.*

Then $\mathbb{P}$ is the unique measure on $\mathcal{F}_T$ which is absolutely continuous with respect to $\mathbb{P}$, and under which W is a martingale.

Correspondingly, for every function $f \in L^1(\Omega, \mathcal{F}_T, \mathbb{P})$ there is a unique predictable process $H = (H_t)_{0\le t\le T}$ such that

$$f = \mathbf{E}[f] + (H \cdot W)_T, \tag{56}$$

and

$$\mathbf{E}[f|\mathcal{F}_t] = \mathbf{E}[f] + (H \cdot W)_t, \quad 0 \le t \le T, \tag{57}$$

which implies in particular that $(H \cdot W)$ is a uniformly integrable martingale.

Bachelier's model revisited:

Let us restate Bachelier's model in the framework of the formalism developed above: let $B_t \equiv 1$ and $S_t = S_0 + \sigma W_t$, $0 \le t \le T$, where S_0 is the current stockprice, $\sigma > 0$ is a fixed constant, and W is standard Brownian motion on its natural base $(\Omega, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$.

Fixing the strike price K, we want to price and hedge the contingent claim

$$f(\omega) = (S_T(\omega) - K)_+ \in L^1(\Omega, \mathcal{F}_T, \mathbb{P}). \tag{58}$$

Using the martingale representation theorem we may find a trading strategy $\overline{H}$ s.t.

$$\begin{aligned} f &= \mathbf{E}[f] + (\overline{H} \cdot W)_T \\ &= \mathbf{E}[f] + (H \cdot S)_T, \end{aligned} \tag{59}$$

where $H = \frac{\overline{H}}{\sigma}$. Noting that $B_t \equiv 1$ implies that assumption 1.4 is satisfied, we deduce from (9) above that

$$C(S_0, T) := \mathbf{E}[f] = (S_0 - K)\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\varphi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right). \tag{60}$$

By the same token we obtain, for every $0 \le t \le T$, and conditionally on the stock price having the value S_t at time t,

$$C(S_t, T-t) := \mathbf{E}[f|S_t] = (S_t - K)\Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}\varphi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right). \quad (61)$$

Hence this solves the pricing problem, which now is based on the no-arbitrage considerations rather than on accepting Bachelier's *fundamental principle*, as we now have the "replication formula" (59).

But what is the trading strategy H, in other words, the recipe to replicate the option by trading dynamically? Economic intuition suggests that we have

$$H(S,t) = \frac{\partial}{\partial S} C(S, T-t). \quad (62)$$

Indeed, consider the following heuristic reasoning using infinitesimals: suppose at time t the stock price equals S_t so that the value of the option equals $C(S_t, t)$. During the infinitesimal interval $(t, t+dt)$ the Brownian motion W_t will move by $dW_t = W_{t+dt} - W_t = \epsilon_t \sqrt{dt}$, where $\mathbb{P}[\epsilon_t = 1] = \mathbb{P}[\epsilon_t = -1] = \frac{1}{2}$, so that S_t will move by $dS_t = S_{t+dt} - S_t = \epsilon_t \sigma \sqrt{dt}$. Hence the value of the option $C(S_t, t)$ will move by $dC_t = C(S_{t+dt}, T-(t+dt)) - C(S_t, T-t) \approx \epsilon_t \frac{\partial C}{\partial S}(S_t, T-t)\sigma\sqrt{dt}$, where we neglect terms of smaller order than $\sqrt{dt}$. In other words, the ratio between the up or down movement of the underlying stock S and the option is

$$\begin{aligned} dC_t : dS_t &= \epsilon_t \frac{\partial C}{\partial S}(S_t, T-t)\sigma\sqrt{dt} : \epsilon_t \sigma \sqrt{dt} \quad (63) \\ &= \frac{\partial C}{\partial S}(S_t, T-t). \end{aligned}$$

If we want to replicate the option by investing the proper quantity H of the underlying stock, formula (63) suggests that this quantity should equal $\frac{\partial C}{\partial S}(S_t, T-t)$.

After these motivating remarks, let us deduce the equation

$$H(S_t, t) = \frac{\partial C}{\partial S}(S_t, T-t) \quad (64)$$

more formally. Consider the stochastic process

$$C(S_t, T-t) = C(S_0 + \sigma W_t, T-t), \quad 0 \le t \le T, \quad (65)$$

of the value of the option. By Itô's formula

$$dC(S_t, t) = \frac{\partial C}{\partial S} dS_t + \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2\right) dt, \quad (66)$$

where we have used $dS_t = \sigma dW_t$. One readily deduces from formula (61) that C verifies the heat equation with parameter $\frac{\sigma^2}{2}$ displayed in (68) below (time is running into the negative direction in the present setting). In particular, for the function C defined in (61), the drift term in (66) vanishes as it must be the case according to the general theory (the option price process is a martingale by (57)). Hence (66) reduces to the formula

$$C(S_t, T-t) = C(S_0, T) + (H \cdot S)_t, \quad (67)$$

where H is given by (64). Rephrasing this result once more we have shown that the trading strategy H, whose existence was guaranteed by the martingale representation, is of the form (64).

One more word on the fact that $C(S, T-t)$ satisfies the heat equation, which may be verified by simply calculating the partial derivatives in (61). Admitting this calculation, we concluded above that the drift term in (66) vanishes. One may also turn the argument around to conclude from (57) that the drift term in (66) must vanish, which then *implies* that $C(S, T-t)$ must satisfy the heat equation (time running inversely)

$$\frac{\partial C}{\partial t}(S, T-t) = -\frac{\sigma^2}{2}\frac{\partial^2 C}{\partial S^2}(S, T-t). \tag{68}$$

Imposing the boundary condition $C(S, T-T) = C(S, 0) = (S-K)_+$ one may *derive* from this p.d.e. by standard methods the solution (61). This is, in fact, how F. Black and M. Scholes originally proceeded (in the framework of their model, which we shall analyse in a moment). Let us also give the heuristic argument to deduce the p.d.e. (C10) from Bachelier's "fundamental principle" and Itô's formula.

Suppose there is a *"formula"* $C(S_t, T-t)$ which gives the value of an option for every $0 \leq t \leq T$ and $S_t \in \mathbb{R}$. By assumption, at the terminal date $t = T$ we have the boundary condition $C(S_T, T-T) = C(S_T, 0) = (S_T - K)_+$.

Applying Bachelier's fundamental principle (remember this wonderful passage following the formulation of his "fundamental principle", which describes the idea of a martingale!) the stochastic process $(C(S_t, T-t))_{0\leq t\leq T}$ should be a martingale. Therefore the drift term in (66) should vanish, which amounts to formula (68).

The Black-Scholes model:

This model of a stock market was proposed by the famous economist P. Samuelson in 1965 ([S 65]), who at this time was aware of Bachelier's work. In fact, triggered by a question of J. Savage, it was P. Samuelson who had rediscovered Bachelier's work for the economic literature some years before 1965.

The model is usually called the Black-Scholes model today and became the standard reference model in the context of option pricing:

$$\begin{aligned} B_t &= e^{rt}, \\ S_t &= S_0 e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}, \quad 0 \leq t \leq T. \end{aligned} \tag{69}$$

Again W is a standard Brownian motion with natural base $(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$.

The paramenter r models the "riskless rate of interest", while the parameter μ models the average increase of the stock price. Indeed using Itô's formula one may describe the model equivalently by the differential equations:

$$\frac{dB_t}{B_t} = rdt, \tag{70}$$

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t. \tag{71}$$

The numéraire in this model is just the relevant currency (say €); in order to remain consistent with the above theory, we shall rather follow our usual procedure of taking a traded asset as numéraire, namely the bond. We then have

$$\begin{aligned}\widetilde{B}_t &\equiv 1, \\ \widetilde{S}_t &= S_0 e^{\sigma W_t + (\mu - r - \frac{\sigma^2}{2})t}.\end{aligned} \tag{72}$$

The tilde indicates that we now have denominated B_t and S_t in terms of the bond B_t, i.e., we have discounted them. We shall write ν for $\mu - r$ which is called the "excess return". The only thing we have to keep in mind by passing to the bond as numéraire, is that now quantities have to be expressed in terms of the bond: in particular, if K denotes the strike price of an option at time T (expressed in € at time T), we have to express it as Ke^{-rT} units of the bond.

Contrary to Bachelier's setting, the process

$$\widetilde{S}_t = S_0 e^{\sigma W_t + (\nu - \frac{\sigma^2}{2})t}, \quad 0 \le t \le T, \tag{73}$$

is *not* a martingale under $\mathbb{P}$ (unless $\nu = 0$, which typically is not the case).

The unique martingale measure Q for $\widetilde{S}$ (which is absolutely $\mathbb{P}$-continuous) is given by Girsanov's theorem (see [RY 91] or any introductory text to Mathematical Finance)

$$\frac{dQ}{d\mathbb{P}} = \exp\left(-\frac{\nu}{\sigma} W_T - \frac{\nu^2}{2\sigma^2} T\right). \tag{74}$$

Let us price and hedge the contingent claim $f(\omega) = (\widetilde{S}_T(\omega) - Ke^{-rT})_+$, which is the pay-off function of the European call option with exercise time T and strike price K (expressed in terms of €).

Noting that $(W_t + \nu t)_{t \ge 0}$ is a standard Brownian motion under Q and applying theorem 3.1 to the Q-martingale $\widetilde{S}$, we may calculate

$$\begin{aligned}C(S_0, T) = \mathbf{E}_Q[f] &= \mathbf{E}_Q\left[\left(S_0 e^{\sigma(W_T + \nu T) - \frac{\sigma^2}{2}T} - Ke^{-rT}\right)_+\right] \\ &= S_0 \mathbf{E}_Q\left[e^{\sigma\sqrt{T}Z - \frac{\sigma^2 T}{2}} \chi_{\{S_T \ge K\}}\right] - Ke^{-rT} Q[S_T \ge K],\end{aligned} \tag{75}$$

where Z denotes a $N(0,1)$-distributed random variable under Q.

After an elementary but tedious calculation (see, e.g., [LL 96]) this yields the famous *Black-Scholes formula*

$$\begin{aligned}C(S_0, T) &= S_0 \Phi\left(\frac{\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \\ &\quad - Ke^{-rT} \Phi\left(\frac{\ln(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)\end{aligned} \tag{76}$$

and, by the same token, for $0 \le t \le T$, and $S_t > 0$,

$$\begin{aligned}C(S_t, T - t) &= S_0 \Phi\left(\frac{\ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}\right) \\ &\quad - Ke^{-rT} \Phi\left(\frac{\ln(\frac{S_t}{K}) + (r - \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}\right).\end{aligned} \tag{77}$$

Let us take some time to contemplate on this truly remarkable formula (for which R. Merton and M. Scholes received the Nobel prize in economics in 1997; F. Black unfortunately had passed away in 1995).

1.) As a warm-up consider the limits as $\sigma \to \infty$ (which yields $C(S_0, T) = S_0$) and $\sigma \to 0$ (which yields $C(S_0, T) = (S_0 - Ke^{-rT})_+$). The reader should convince herself that this does make sense economically. For an extremely risky underlying S, an option on one unit of S is almost as valuable as one unit of S itself (think, for example, of a call option on a lottery ticket with $K = 100$ and exercise time T, such that T is later than the drawing where it is decided, whether the ticket wins a million or not). On the other hand, if the underlying S is almost riskless a similar consideration reveals that the value of an option is almost equal to its "inner value" $(S_0 - Ke^{-rT})_+$.

This behavior of the Black-Scholes formula should be contrasted to Bachelier's formula (specializing to the case $S_0 = K$ and $r = 0$)

$$C^{\text{Bachelier}}(S_0, T) = \frac{\sigma}{\sqrt{2\pi}} \sqrt{T} \tag{78}$$

obtained in (10) above, which tends to infinity as $\sigma \to \infty$; this limiting behaviour is economically absurd and contradicts an obvious no-arbitrage argument which — using the fact that S_T is non-negative — shows that the value of a call option always must be less than the value of the underlying stock.

The reason for this difference in the behaviour of the Black-Scholes formula and Bachelier's one, for large values of σ, is that geometric Brownian motion always remains positive, while Brownian motion may also attain negative values, a fact which has strong effects for very large σ or — what amounts to the same, at least in the case $r = 0$ — for very large T. Nevertheless we shall presently see that — for reasonable values of σ and T — the Black-Scholes formula and Bachelier's formula (78) are very close. This seems to be the essential fact, keeping in mind Keynes' dictum telling us, *not* to look at the limit $T \to \infty$: in the long run we all are dead.

2.) Let us compare the Black-Scholes formula (76) and Bachelier's formula (78) more systematically. To do so we specialize in the Black-Scholes formula to $r = 0$ and $S_0 = K$, and we have to let the σ in the Black-Scholes formula, which we now denote by σ^{BS}, correspond to the σ appearing in Bachelier's formula, denoted by σ^{B}. As the former pertains to the relative standard deviation of stock prices and the latter to the absolute standard deviation, we roughly find the correspondence — at least for small values of T —

$$\sigma^{\text{B}} \approx \sigma^{\text{BS}} S_0 \tag{79}$$

Hence, in this special case, the Black-Scholes and Bachelier option prices to be compared are

$$C^{\text{BS}} = S_0 \left[\Phi\left(\frac{\sigma^{\text{BS}}\sqrt{T}}{2} \right) - \Phi\left(-\frac{\sigma^{\text{BS}}\sqrt{T}}{2} \right) \right], \tag{80}$$

while

$$C^{\text{B}} = \frac{\sigma^{\text{B}}}{\sqrt{2\pi}}\sqrt{T} \approx S_0 \frac{\sigma^{\text{BS}}}{\sqrt{2\pi}}\sqrt{T}. \tag{81}$$

The difference of the two quantitaties is best understood by looking at the shaded area in the subsequent graph involving the density $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ of the standard normal distribution, and noting that $\varphi(0) = \frac{1}{\sqrt{2\pi}}$.

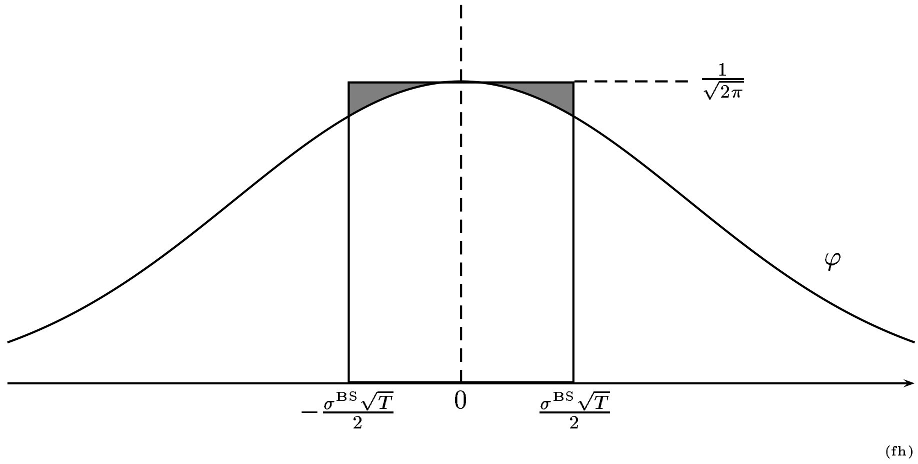

Figure 3: Comparison of the Bachelier with the Black-Scholes formula.

Developping $\varphi(x)$ into a Taylor series around zero and using $\varphi''(0) = -\frac{1}{\sqrt{2\pi}}$ we get the asymptotic expression

$$C^{\mathrm{B}} - C^{\mathrm{BS}} = S_0 \left[\frac{1}{24\sqrt{2\pi}} \left(\sigma^{\mathrm{BS}}\sqrt{T} \right)^3 \right] + o\left(\left(\sigma^{\mathrm{BS}}\sqrt{T} \right)^3 \right), \tag{82}$$

which indicates a very good fit, if $\sigma^{\mathrm{BS}}\sqrt{T}$ is small. Evaluating this expression for the empirical data reported by Bachelier, i.e., $\sigma^{\mathrm{BS}} \approx 2.4\,\%$ on a yearly basis, and $T \approx 2\,\text{months} = \frac{1}{6}\,\text{year}$ (this is a generous upper bound for the periods considered by Bachelier which were ranging between 10 and 45 days) we find

$$C^B - C^S \approx S_0 \frac{1}{24\sqrt{2\pi}} \left(0.024\sqrt{\tfrac{1}{6}} \right)^3 \approx 1.56 * 10^{-8} S_0. \tag{83}$$

Hence for this data the difference of the option value obtained from Bachelier's or the Black-Scholes model is of the order 10^{-8} times the value S_0 of the underlying; keeping in mind, that for Bachelier's data, the price of an option was of the order of $S_0/100$, we find that the difference is of the order 10^{-6} of the price of the option.

In view of all the uncertainties involved in option pricing, in particular as regards the estimation of σ, one might be tempted to call this quantity "completely negligible, a priori" (this expression was used by Bachelier when discussing the drawbacks of the normal distribution giving positive probability to negative stock prices).

3.) Let us now comment on the role of the riskless rate of interest r, appearing in the Black-Scholes formula and the reason why this variable does not show up in Bachelier's formula: noting the obvious fact that

$$\ln\left(\frac{S_0}{K}\right) + rT = \ln\left(\frac{S_0}{Ke^{-rT}}\right), \tag{84}$$

one readily observes that this quantity only enters the Black-Scholes formula (76) via the discounting of the strike price, i.e., transforming K units of $€_{t=T}$ into Ke^{-rT} units of $€_{t=0}$. When comparing the setting of Black-Scholes to that of Bachelier one should recall that the option premium in Bachelier's days pertained to a payment at time T or, in modern terms, was expressed in terms of a zero coupon bond maturing at time T. Under the assumption of a constant riskless interest — as is the case in the Black-Scholes model — this amounts to considering the present day quantities upcounted by e^{rT}. This was perfectly taken into account by Bachelier, who stressed that the quantities appearing in his formulae have to be understood in terms of "true prices", i.e., forward prices in modern terminology, which amounts to upcounting by e^{rT} in the present setting. In fact, we have seen in section 1 that Bachelier did even more, as he in addition was considering the "contangoes", which — in modern terminology — correspond to a continuous yield on the stock.

The bottom line of these considerations on the role of r is: when we assumed that $r = 0$ in the above comparison of the Bachelier and Black-Scholes option pricing methodology, this assumption did not restrict the generality of the argument: it also applies to $r \neq 0$ as Bachelier denoted the relevant quantities in terms of "true prices".

4.) What is the partial differential equation satisfied by the solution (77) of the Black-Scholes formula? Again we specialize to the case $r = 0$ in order to focus the attention of the reader to the crucial aspect, but we note that now we *do restrict the generality* and refer to any introductory text to Mathematical Finance (e.g., [LL 96]) for the Black-Scholes partial differential equation in the case of a riskless rate of interest $r \neq 0$.

From the Martingale Representation Theorem 3.1 above we know that the Black-Scholes option price *process*

$$C(S_t, T-t)_{0 \leq t \leq T} \tag{85}$$

is a martingale under the measure Q defined in (74). Hence, denoting by $(\widetilde{W}_t)_{0 \leq t \leq T}$ a standard Brownian motion under Q, using $dS_t = \sigma S_t d\widetilde{W}_t$, and working under the measure Q, we obtain from Itô's formula

$$dC_t = dC(S_t, T-t) = \frac{\partial C}{\partial S} \sigma S_t d\widetilde{W}_t + \left(\frac{\sigma^2}{2} S_t^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} \right) dt. \tag{86}$$

We first observe, using again $\sigma S_t d\widetilde{W}_t = dS_t$, that — similarly as in the context of Bachelier — the replicating trading strategy $H_t(\omega)$ is given by $\frac{\partial C}{\partial S}(S_t(\omega), T-t)$. In the lingo of finance this quantity is called the "Delta" of the option (which depends on S_t and t), and the trading strategy H is called "delta-hedging".

Next we pass to the drift term: as it must vanish, we arrive at the "Black-Scholes partial differential equation"

$$\frac{\partial C}{\partial t}(S, T-t) = -\frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2}(S, T-t), \text{ for } S \geq 0, t \geq 0. \tag{87}$$

This is the multiplicative analogue of the heat equation (68) and may, in fact, easily be reduced to a heat equation (with drift) by passing to logarithmic coordinates $x = \ln(S)$.

Exactly as in Bachelier's case we may proceed by solving the partial differential equation (87) for the boundary condition $C(S, T-T) = C(S, 0) = (S-K)_+$ and $C(0,t) = 0$ to obtain the Black-Scholes formula.

In the lingo of finance, the quantity $-\frac{\partial C}{\partial t}$ is called the "Theta" and the quantity $\frac{\partial^2 C}{\partial S^2}$ the "Gamma" of the option. Hence the p.d.e. (87) allows for the following economic interpretation: the loss of value of the option, when time to maturity $T-t$ decreases (and S remains fixed), is equal to the "convexity" or "gamma" of the option price (as a function of S) at time t, normalized by $\frac{\sigma^2}{2}S^2$ (in the case of the Bachelier model the normalisation was simply $\frac{\sigma^2}{2}$). This has a good economic interpretation and today's option traders think in these terms. They speak about "selling or buying convexity" or rather "going gamma-short or gamma-long" which amounts to the same thing. The interpretation of (87) is that, for the buyer of an option, the convexity of the function $C(S, T-t)$ in the variable S corresponds to a kind of insurance with respect to price movements of S. As there is no such thing as a free lunch, this insurance costs (proportional to the second derivative) and a positive $\frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2}$ is reflected by a negative partial derivative $\frac{\partial C}{\partial t}$ of $C(S, T-t)$ with respect to time t.

Let us illustrate this fact by reasoning once more heuristically with infinitesimal movements of Brownian motion: we want to explain the infinitesimal change of the option price when "time increases by an infinitesimal while the stock price S remains constant". To do so we apply the heuristic analogue of the Brownian bridge: consider the infinitesimal interval $[t, t+2dt]$ and assume that the driving Q-Brownian motion $\widetilde{W}$ moves in the first half $[t, t+dt]$ from $\widetilde{W}_t$ to $\widetilde{W}_t + \epsilon_t\sqrt{dt}$, where ϵ_t is a random variable with $Q[\epsilon_t = 1] = Q[\epsilon_t = -1] = \frac{1}{2}$, while in the second half $[t+dt, t+2dt]$ it moves back to $\widetilde{W}_t$. What happens during this time interval to a "hedger" who proceeds according to the Black-Scholes trading strategy H described above, which replicates the option? At time t she holds $\frac{\partial C}{\partial S}(S_t, T-t)$ units of the stock. Following first the scenario $\epsilon_t = +1$, the stock has a price of $S_t + \sigma S_t\sqrt{dt}$ at time $t+dt$. Appart from being happy about this up movement, the hedger now (i.e., at time $t+dt$) adjusts the portfolio to hold $\frac{\partial C}{\partial S}(S_t + \sigma S_t\sqrt{dt}, T-(t+dt))$ units of stock, which results in a net buy of $\frac{\partial^2 C}{\partial S^2}(S_t, T-t)\sigma S_t\sqrt{dt}$ units of stock, where we neglect terms of smaller order than $\sqrt{dt}$. In the next half $[t+dt, t+2dt]$ of the interval the stock price S drops again to the value $S_{t+2dt} = S_t$ and the hedger readjusts the portfolio by selling again the $\frac{\partial^2 C}{\partial S^2}(S_t, T-t)\sigma S_t\sqrt{dt}$ units of stock (neglecting again terms of smaller order than $\sqrt{dt}$). It seems at first glance that the gains made in the first half are precisely compensated by the losses in the second half, but a closer inspection shows that the hedger did "buy high" and "sell low": the quantitiy $\frac{\partial^2 C}{\partial S^2}(S_t, T-t)\sigma S_t\sqrt{dt}$ was bought at price $S_t + \sigma S_t\sqrt{dt}$ at time $t+dt$, and sold at price S_t at time $t+2dt$, resulting in a total loss of

$$\left(\frac{\partial^2 C}{\partial S^2}(S_t, T-t)\sigma S_t\sqrt{dt}\right)\left(\sigma S_t\sqrt{dt}\right) = \sigma^2 S_t^2\frac{\partial^2 C}{\partial S^2}(S_t, T-t)dt. \qquad (88)$$

Going through the scenario $\epsilon_t = -1$, one finds that the hedger did first "sell low" and then "buy high" resulting in the same loss (where again we neglect infinitesimals resulting in effects (with respect to the final result) of smaller order than dt).

Keeping in mind that this was achieved during an interval of total length $2dt$ we have found a heuristic explanation for the Black-Scholes equation (87) (we also note that the same argument, applied to Bachelier's model, yields a heuristic explanation of the heat equation (68)). The general phenomenon behind this fact is that, in the case of convexity, the "wobbling" of Brownian motion, which is of order $\sqrt{dt}$ in an interval of length dt, causes the hedger to have systematic losses, which are proportional to $\frac{\partial^2 C}{\partial S^2}$ as well as to the increment $d\langle S\rangle_t$ of the quadratic variation process $\langle S\rangle_t = \int_0^t \sigma^2 S_u^2 du$ of the stock price process S.

5.) When deriving the Black-Scholes formula (76) we did not go through the (elementary but tedious) trouble of explicitly calculating (75). We shall now furnish an explicit derivation of the formula which has the merit of yielding an interpretation of the two probabilities appearing in (76). It also allows for a better understanding of the formula (for example, for the remarkable fact, that the parameter μ has disappeared) and which also dispenses us of some troubles in the calculation.

As observed in (75) above, the contingent claim $f(\omega) = (\widetilde{S}_T(\omega) - Ke^{-rT})_+$ (expressed in terms of the numeraire B_t) splits into

$$\begin{aligned}(\widetilde{S}_T - Ke^{-rT})_+ &= \widetilde{S}_T\chi_{\{\widetilde{S}_T\geq Ke^{-rT}\}} - Ke^{-rT}\chi_{\{\widetilde{S}_T\geq Ke^{-rT}\}} \\ &= \widetilde{S}_T\chi_{\{S_T\geq K\}} - Ke^{-rT}\chi_{\{S_T\geq K\}} \\ &= f_1 - f_2.\end{aligned} \tag{89}$$

We have to calculate $\mathbf{E}_Q[f_1]$ and $\mathbf{E}_Q[f_2]$ under the risk-neutral measure Q defined in (74). This is easy for f_2 and we do not have to use the explicit form of the density (74) provided by Girsanov's theorem. It suffices to observe that $\widetilde{S}_t = S_0\exp(\sigma\widetilde{W}_t - \frac{\sigma^2}{2}t)$ where $\widetilde{W}$ is a Brownian motion under Q. So

$$X := \frac{\ln\left(\frac{\widetilde{S}_T}{S_0}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}} \sim N(0,1) \qquad \text{under } Q, \tag{90}$$

whence

$$\begin{aligned}\mathbf{E}_Q[f_2] &= e^{-rT}K\,Q[\widetilde{S}_T \geq e^{-rT}K] \\ &= e^{-rT}K\,Q\left\{\frac{\ln\left(\frac{\widetilde{S}_T}{S_0}\right)+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}} \geq \frac{\ln\left(\frac{e^{-rT}K}{S_0}\right)+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right\} \\ &= e^{-rT}K\,Q\left\{X \geq \frac{\ln\left(\frac{e^{-rT}K}{S_0}\right)+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right\} \\ &= e^{-rT}K\,\Phi\left(\frac{\ln\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right),\end{aligned} \tag{91}$$

which yields the second term of the Black-Scholes formula.

Why was the calculation of $\mathbf{E}_Q[f_2]$ so easy? Because the amount Ke^{-rT} is just a constant (expressed in terms of the present numéraire); hence the calculation of the expectation reduced to the calculation of the probability of an event, namely the prohability that the option will be exercised, with respect to Q.

To proceed similarly with the calculation of $\mathbf{E}_Q[f_1]$ we make a change of numéraire, now choosing the risky asset S in the Black-Scholes model (69) as numéraire. Under this numéraire the model reads

$$\begin{aligned} \frac{B_t}{S_t} &= S_0^{-1} e^{-\sigma W_t + (r-\mu+\frac{\sigma^2}{2})t} \\ \frac{S_t}{S_t} &\equiv 1 \end{aligned} \tag{92}$$

where W is a standard Brownian motion under $\mathbb{P}$. The reader certainly has noticed the symmetry with (72). But what ist the probability measure $\check{Q}$ under which the process $\frac{B_t}{S_t}$ becomes a martingale? Using Girsanov we can explicitly calculate the density $\frac{d\check{Q}}{d\mathbb{P}}$; but, in fact, we don't really need this full information. All we need is to observe that we may write

$$\frac{B_t}{S_t} = S_0^{-1} e^{-\sigma \check{W}_t - \frac{\sigma^2}{2}t}, \tag{93}$$

where $\check{W}$ is a standard Brownian motion under $\check{Q}$ (the reader worried by the minus sign in front of $\sigma\check{W}_t$ may note that $-\check{W}$ also is a standard Brownian motion under $\check{Q}$). We now apply the change of numéraire theorem (in the form of corollary 2.14) to calculate $\mathbf{E}_Q[f_1]$. In fact, we have only proved this theorem for the case of finite Ω, but we trust in the reader's faith that it also applies to the present case (for a thorough investigation for the validity of this theorem for general locally bounded semi-martingale models we refer to [DS 95]). Applying this theorem we obtain

$$\begin{aligned} \mathbf{E}_Q[f_1] &= \mathbf{E}_Q\left[\widetilde{S}_T \chi_{\{\frac{S_T}{B_T} \geq e^{-rT}K\}}\right] \\ &= S_0\, \mathbf{E}_{\check{Q}}\left[\frac{\widetilde{S}_T}{\widetilde{S}_T} \chi_{\{\frac{B_T}{S_T} \leq e^{rT}K^{-1}\}}\right] \\ &= S_0\, \mathbf{E}_{\check{Q}}\left[\chi_{\{S_0^{-1} e^{-\sigma\check{W}_T - \frac{\sigma^2}{2}T} \leq e^{rT}K^{-1}\}}\right] \\ &= S_0\, \check{Q}\left[S_0 e^{\sigma\check{W}_T + \frac{\sigma^2}{2}T} \geq e^{-rT}K\right]. \end{aligned} \tag{94}$$

Noting that $\check{W}_T/\sqrt{T}$ is $N(0,1)$-distributed under $\check{Q}$, this expression is completely analogous to that appearing in (91), with the exception that now there is a plus in front of the term $\frac{\sigma^2}{2}T$. Hence we get

$$\mathbf{E}_Q[f_1] = S_0 \Phi\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right), \tag{95}$$

which is the first term appearing in the Black-Scholes formula. We now may interpret $\Phi\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$ as the probability, that the option will be exercised, with respect to $\check{Q}$.

4 The No-Arbitrage Theory for General Processes

We now again take up the theme of the no-arbitrage theory as developed in section 2: what can we deduce from applying the no-arbitrage principle with respect to pricing and hedging of derivative securities?

While we obtained satisfactory and mathematically rigorous answers to these questions in the case of a finite underlying probability space Ω in section 2, we saw in section 3, that the basic examples for this theory, the Bachelier and the Black-Scholes model, do not fit into this easy setting, as they involve Brownian motion.

In section 3 we coped with the difficulty either by using well-known results from stochastic analysis (e.g., the martingale representation theorem 3.1 for the Brownian filtration), or by appealing to the faith of the reader, that the results obtained in the finite case also carry over — mutatis mutandis — to more general situations, as we did when applying the change of numéraire theorem to the calculation of the Black-Scholes model.

In the present chapter we want to develop a "théorie génerale of no-arbitrage" applying to a general framework of stochastic processes. The development of Mathematical Finance since the work of Black, Merton and Scholes made it clear, that the relatively poor fit of the Black-Scholes model (as well as Bachelier's model) to empirical data (especially with respect to extremal behaviour, i.e., large changes in prices) makes it necessary for many applications, to pass to more general models; in some cases these models still have continuous paths, but also processes (in continuous time) with jumps are increasingly gaining importance.

We adopt the following general framework: let $S = (S_t)_{t\geq 0}$ be an $\mathbb{R}^{d+1}$-valued stochastic process based on and adapted to the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$. Again we assume that the zero coordinate S^0, called the bond, is normalised to $S_t^0 \equiv 1$.

We first will make a technical assumption, namely that the process S is *bounded*, i.e., that there exists a sequence $(\tau_n)_{n=1}^{\infty}$ of stopping times, increasing a.s. to $+\infty$, such that the stopped processes $S_t^{\tau_n} = S_{t\wedge\tau_n}$ are uniformly bounded, for each $n \in \mathbb{N}$. Note that continuous processes — or, more generally, càdlàg processes with uniformly bounded jumps — are locally bounded. This assumption will be very convenient for technical reasons, and only at the end of this section we shall indicate, how to extend to the general case of processes, which are not necessarily locally bounded.

We have chosen $[0, \infty[$ for the time index set in order to allow for maximal generality; of course this also covers the case of a compact interval $[0, T]$, which is relevant in most applications, by assuming that S_t is constant, for $t \geq T$. We shall always assume that the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual assumptions of right continuity and saturatedness, and that S has a.s. càdlàg trajectories.

How to define the trading strategies H, which played a crucial role in the preceding sections? A very elementary approach, corresponding to the role of step functions in integration theory, is formalized by the subsequent concept.

Definition 4.1 *(compare, e.g., [P 90]) For a locally bounded stochastic process S we call an $\mathbb{R}^d$-valued process $H = (H_t)_{t\geq 0}$ a* simple trading strategy *(or, speaking* *more mathematically, a* simple integrand*), if H is of the form*

$$H = \sum_{i=1}^{n} h_i \chi_{]\!]\tau_{i-1},\tau_i]\!]}, \tag{96}$$

where $0 = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_n$ are finite stopping times and h_i are $\mathcal{F}_{\tau_{i-1}}$-measureable, $\mathbb{R}^d$-valued functions.

We then may define, similarly as in definition 2.2, the stochastic integral $(H \cdot S)$ as the stochastic process

$$\begin{aligned} (H \cdot S)_t &= \sum_{i=1}^{n} \left(h_i, S_{\tau_i \wedge t} - S_{\tau_{i-1} \wedge t}\right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{d} h_i^j \left(S^j_{\tau_i \wedge t} - S^j_{\tau_{i-1} \wedge t}\right), \quad 0 \leq t < \infty, \end{aligned} \tag{97}$$

and its terminal value as the random variable

$$(H \cdot S)_\infty = \sum_{i=1}^{n} (h_i, S_{\tau_i} - S_{\tau_{i-1}}). \tag{98}$$

We call H admissible *if, in addition, the stopped process S^{τ_n} and the functions $h_1, \ldots, h_n$ are uniformly bounded.*

This definition is a well known building block for developing a stochastic integration theory (see, e.g., [P 90]). It has a clear economic interpretation in the present context: at time τ_{i-1} an investor decides to adjust her portfolio in the assets $S^1, \ldots, S^j, \ldots, S^d$ by fixing her investment in asset S^j to be $h_i^j(\omega)$ units; we allow h_i^j to have arbitrary sign (holding a negative quantity means borrowing or "going short"), and to depend on the random element ω in an $\mathcal{F}_{\tau_{i-1}}$-measurable way, i.e., using the information available at time τ_{i-1}. The funds for adjusting the portfolio in this way simply are financed by taking the appropriate amount from (or putting into) the "cash box", modeled by the numéraire $S^0 \equiv 1$. The investor holds this portfolio fixed up to time τ_i. During this period the value of the risky stocks S^j, $j = 1, \ldots, d$, changed from $S^j_{\tau_{i-1}}(\omega)$ to $S^j_{\tau_i}(\omega)$ resulting in a total gain (or loss) given by the random variable $(h_i, S_{\tau_i} - S_{\tau_{i-1}})$. At time τ_i, for $i < n$, the investor readjusts the portfolio again and at time τ_n she liquidates the portfolio, i.e., converts all her positions into the numéraire. Hence the random variable $(H \cdot S)_{\tau_n} = (H \cdot S)_\infty$ models the total gain (in units of the numéraire S_0) which she finally, i.e., at time τ_n, obtained by adhering to the strategy H; the process $(H \cdot S)_t$ models the gains accumulated up to time t.

The concept of a simple trading strategy is designed in a purely algebraic way, avoiding limiting procedures, in order to be on safe grounds.

The next crucial ingredient in developing the theory is the proper generalisation of the notion of an equivalent martingale measure.

Definition 4.2 *A probability measure Q on $\mathcal{F}$ which is equivalent (resp. absolutely continuous with respect) to $\mathbb{P}$ is called an* equivalent *(resp.* absolutely continuous*)* local martingale measure*, if S is a local martingale under Q.*

We denote by $\mathcal{M}^e(S)$ (resp. $\mathcal{M}^a(S)$) the family of all such measures, and say that S satisfies the condition of the existence of an equivalent local martingale measure (EMM), if $\mathcal{M}^e(S) \neq \emptyset$.

Note that, by our assumption of local boundedness of S, we have that S is a local Q-martingale, iff S^τ is a Q-martingale for each stopping time τ such that S^τ is uniformly bounded.

Why did we use the notion of a local martingale instead of the more familiar notion of a martingale? The reason is, that it is the natural degree of generality. The subsequent easy lemma (whose proof is an obvious consequence of the chosen concepts and left to the reader) shows that this notion serves just as well as the notion of a martingale for the present purpose of a no-arbitrage theory. On the other hand, the restriction to the notion of martingale measures would make it impossible to formulate the general version of the fundamental theorem of asset pricing (theorem 2.8 below), as may bee seen from easy examples (see, e.g., [DS 94a]).

Lemma 4.3 *A locally bounded semi-martingale S is a local martingale under Q iff*

$$\mathbf{E}_Q\left[(H \cdot S)_\infty\right] = 0, \tag{99}$$

for each admissible simple trading strategy H.

For later use we note that the "$=$" in (99) may equivalently be replaced by "$\leq$" (or "$\geq$").

We define the subspace K^{simple} of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ of contingent claims available at price zero via an admissible simple trading strategy by

$$K^{\text{simple}} = \{(H \cdot S)_\infty : H \text{ simple, admissible}\} \tag{100}$$

and by C^{simple} the convex cone in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ of contingent claims dominated by some $f \in K$

$$C^{\text{simple}} = K^{\text{simple}} - L^\infty_+ = \left\{f - k : f \in K^{\text{simple}}, k \geq 0\right\}. \tag{101}$$

Definition 4.4 *S satisfies the no-arbitrage condition (NA) with respect to simple integrands, if $K^{\text{simple}} \cap L^\infty_+(\Omega, \mathcal{F}, \mathbb{P}) = \{0\}$ (or, equivalently, $C^{\text{simple}} \cap L^\infty_+(\Omega, \mathcal{F}, \mathbb{P}) = \{0\}$).*

We want to prove a fundamental theorem of asset pricing analogous to theorem 2.8 above. But now things are more delicate and the notion of *(NA)* defined above is not sufficiently strong to imply this result:

Proposition 4.5 *The condition (EMM) of the existence of an equivalent local martingale measure implies the condition (NA) of no-arbitrage with respect to simple integrands, but not vice versa.*

Proof *(EMM) $\Rightarrow$ (NA)*: this is an immediate consequence of lemma 4.3, noting that for $Q \sim P$, and a non-negative function $f \geq 0$, which does not vanish almost surely, we have $\mathbf{E}_Q[f] > 0$.

(NA) $\not\Rightarrow$ (EMM): we give an easy counterexample which is just an infinite random walk.

Let $t_n = 1 - \frac{1}{n+1}$ and define the $\mathbb{R}$-valued process S to start at $S_0 = 1$, and to be constant except for jumps at the points t_n which are defined as

$$\Delta S_{t_n} = 2^{-n} \epsilon_n \tag{102}$$

such that $(\epsilon_n)_{n=1}^{\infty}$ are independent random variables taking the values $+1$ or -1 with probabilities

$$P[\epsilon_n = 1] = \frac{1 + \alpha_n}{2}, \quad P[\epsilon_n = -1] = \frac{1 - \alpha_n}{2}, \tag{103}$$

where $(\alpha_n)_{n=1}^{\infty}$ is a sequence in $]-1, +1[$ to be specified below.

Clearly this well-defines a bounded process S, for which there is a unique measure Q on $(\Omega, \mathcal{F}) = (\{-1,1\}^{\mathbb{N}}, \text{Borel}\,(\{-1,1\}^{\mathbb{N}}))$, under which S is a martingale; this measure is given by

$$Q[\epsilon_n = 1] = Q[\epsilon_n = -1] = \frac{1}{2}, \tag{104}$$

and $(\epsilon_n)_{n=1}^{\infty}$ are independent under Q.

By a result of Kakutani (see, e.g. [W 91]) we know that Q is either equivalent to P, or P and Q are mutually singular, depending on whether $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ or not.

Taking, for example, $\alpha_n = \frac{1}{2}$, for all $n \in \mathbb{N}$, we have constructed a process S on $(\Omega, \mathcal{F}, \mathbb{P})$, for which there is no equivalent (local) martingale measure Q. On the other hand, it is an easy and instructive exercise to show that, *for simple trading strategies*, there are no-arbitrage opportunities for the process S. ■

The example in the above proof shows, why the no-arbitrage condition defined in 4.4 is too narrow: it is intuitively rather obvious that by a sequence of properly scaled bets on a (sufficiently) biased coin one can "produce something like an arbitrage", while a finite number of bets (as formalized by definition 4.1) does not suffice to do so.

But here we are starting to move on thin ice, and it will be the crucial issue to find a mathematically precise framework, in which the above intuitive insight can be properly formalized.

A decisive step in this direction was done in the work of D. Kreps [K 81], who realized that the purely algebraic notion of no-arbitrage with respect to simple integrands has to be complemented with a topological notion:

Definition 4.6 *(compare [K 81]) S satisfies the condition of* no free lunch *(NFL), if the closure $\overline{C}$ of C^{simple}, taken with respect to the weak-star topology of $L^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$, satisfies*

$$\overline{C} \cap L^{\infty}(\Omega, \mathcal{F}, \mathbb{P}) = \{0\}. \tag{105}$$

This strengthening of the condition of no-arbitrage is taylor-made so that the subsequent version of the fundamental theorem of asset pricing holds true.

Theorem 4.7 (Kreps - Yan) *A locally bounded process S satisfies the condition of no free lunch (NFL), iff condition (EMM) of the existence of an equivalent local martingale measure is satisfied:*

$$\text{(NFL)} \iff \text{(EMM)}. \tag{106}$$

Proof *(EMM) ⇒ (NFL)*: This is still the easy part. By lemma 4.3 we have $\mathbf{E}_Q[f] \leq 0$, for each $Q \in \mathcal{M}^e(S)$ and $f \in C^{\text{simple}}$, and this inequality also extends to the weak-star closure $\overline{C}$. On the other hand, if *(EMM)* would hold true and *(NFL)* were violated, there would exist a $Q \in \mathcal{M}^e(S)$ and $f \in \overline{C}$, $f \geq 0$ not vanishing almost surely, whence $\mathbf{E}_Q[f] > 0$, a contradiction.

(NFL) ⇒ (EMM): We follow the strategy of the proof for the case of finite Ω, but have to refine the argument:

Step 1 (Hahn-Banach argument): We claim that, for fixed $f \in L^\infty_+$, $f \not\equiv 0$, there is $g \in L^1_+$ which, viewed as a linear functional on L^∞, is less than or equal to zero on $\overline{C}$, and such that $\langle f, g\rangle > 0$. To see this, apply the separation theorem (e.g., [Sch 66, th. II, 9.2]) to the σ^*-closed convex set $\overline{C}$ and the compact set $\{f\}$ to find $g \in L^1$ and $\alpha < \beta$ such that $g|_{\overline{C}} \leq \alpha$ and $\langle f, g\rangle > \beta$. Since $0 \in C$ we have $\alpha \geq 0$. As $\overline{C}$ is a cone, we have that g is zero or negative on $\overline{C}$ and, in particular, nonnegative on L^∞_+, i.e. $g \in L^1_+$. Noting that $\beta > 0$ we have proved step 1.

Step 2 (Exhaustion Argument): Denote by $\mathcal{G}$ the set of all $g \in L^1_+$, $g \leq 0$ on C. Since $0 \in \mathcal{G}$ (or by step 1), $\mathcal{G}$ is nonempty.

Let $\mathcal{S}$ be the family of (equivalence classes of) subsets of Ω formed by the supports of the elements $g \in \mathcal{G}$. Note that $\mathcal{S}$ is closed under countable unions, as for a sequence $(g_n)_{n=1}^\infty \in \mathcal{G}$, we may find strictly positive scalars $(\alpha_n)_{n=1}^\infty$, such that $\sum_{n=1}^\infty \alpha_n g_n \in \mathcal{G}$. Hence there is $g_0 \in \mathcal{G}$ such that, for $S_0 = \{g_0 > 0\}$, we have

$$P(S_0) = \sup\{P(S) : S \in \mathcal{G}\}. \tag{107}$$

We now claim that $P(S_0) = 1$, which readily shows, that g_0 is strictly positive almost surely. Indeed, if $P(S_0) < 1$, then we could apply step 1 to $f = \chi_{(\Omega\setminus S_0)}$ to find $g_1 \in \mathcal{G}$ with

$$\langle f, g_1\rangle = \int_{\Omega\setminus S_0} g_1(\omega)dP(\omega) > 0 \tag{108}$$

Hence, g_0+g_1 would be an element of $\mathcal{G}$ whose support has P-measure strictly bigger than $P(S_0)$, a contradiction.

Normalize g_0 so that $||g_0||_1 = 1$ and let Q be the measure on $\mathcal{F}$ with Radon-Nikodym derivative $dQ/dP = g_0$. We conclude from lemma 4.3 that Q is a local martingale measure for S, so that $\mathcal{M}^e(S) \neq 0$. ■

Some comments on the Kreps-Yan theorem seem in order: this theorem was obtained by D. Kreps [K 81] in a more general setting and under a — rather mild — additional separability assumption; the reason for the need of this assumption is that D. Kreps did not use the above exhaustion argument, but rather some sequential procedure relying on the separability of $L^1(\Omega, \mathcal{F}, \mathbb{P})$. Independently, and at about the same time, Ji-An Yan [Y 80] proved in a different context, namely the characterisation of semi-martingales as good integrators, and without a direct relation to finance, a general theorem. C. Stricker [S 90] observed, that Yan's theorem may be applied, to quickly yield the above theorem without any separability assumption. We therefore took the liberty to name it after these two authors.

The message of the theorem is, that the assertion of the "fundamental theorem of asset pricing" 2.8 is valid for general processes, if one is willing to interpret the notion of "no arbitrage" in a somewhat liberal way, crystallized in the notion of "no free lunch" above.

What is the economic interpretation of a "free lunch"? By definition S violates the assumption *(NFL)*, if there is a function $g_0 \in L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$, $g_0 \neq 0$, and nets $(g_\alpha)_{\alpha\in I}, (f_\alpha)_{\alpha\in I}$ in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, such that $f_\alpha = (H^\alpha \cdot S)_\infty$ for some admissible, simple integrand $H^\alpha, g_\alpha \leq f_\alpha$, and $\lim_{\alpha\in I} g_\alpha = g_0$, the limit converging with respect to the weak-star topology of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Speaking economically: an arbitrage opportunity would be the existence of a trading strategy H such that $(H \cdot S)_\infty \geq 0$, almost surely, and $P[(H \cdot S)_\infty > 0] > 0$. Of course, this is the dream of each arbitrageur, but we have seen, that — for the purpose of the fundamental theorem to hold true — this is asking for too much (at least, if we only allow for simple admissible trading strategies). Instead, a free lunch is the existence of a contingent claim $g_0 \geq 0$, $g_0 \neq 0$, which may, in general, not be written as (or dominated by) a stochastic integral $(H \cdot S)_\infty$ with respect to a simple admissible integrand; but there are contingent claims g_α "close to g_0", which can be obtained via the trading strategy H^α, and subsequently "throwing away" the amount of money $f_\alpha - g_\alpha$.

This triggers the question whether we can do somewhat better than the above — admittedly complicated — procedure. Can we find a requirement sharpening the notion of "no free lunch", i.e., being closer to the original notion of "no arbitrage" and such that a — properly formulated — version of the "fundamental theorem" still holds true?

Here are some mathematically precise questions related to our attempt to make the process of taking the weak-star closure more understandable:
(i) is it possible, in general, to replace the net $(g_\alpha)_{\alpha\in I}$ above by a sequence $(g_n)_{n=0}^\infty$?
(ii) can we choose the net $(g_\alpha)_{\alpha\in I}$ (or, hopefully, the sequence $(g_n)_{n=0}^\infty$) such that $(g_\alpha)_{\alpha\in I}$ remains bounded in $L^\infty(P)$ (or at least such that the negative parts $((g_\alpha)_-)_{\alpha\in I}$ remain bounded)? This latter issue is crucial from an economic point of view, as it pertains to the question whether the approximation of f by $(g_\alpha)_{\alpha\in I}$ can be done *respecting a finite credit line.*
(iii) is it really necessary to allow for the "throwing away of money"?

It turns out that questions (i) and (ii) are intimately related and, in general, the answer to these questions is no. In fact, the study of the pathologies of the operation of taking the weak-star closure is an old theme of functional analysis. On the very last pages of S. Banach's original book ([B 32]) the following example is given: there is a separable Banach space X such that, for every given fixed number $n \geq 1$ (say $n = 35$), there is a convex cone C in the dual space X^*, such that $C \subsetneq C^{(1)} \subsetneq C^{(2)} \subsetneq \ldots \subsetneq C^{(n)} = C^{(n+1)} = \overline{C}$, where $C^{(k)}$ denotes the sequential weak-star closure of $C^{(k-1)}$, i.e., the limits of weak-star convergent *sequences* $(x_i)_{i=0}^\infty$, with $x_i \in C^{(k-1)}$, and $\overline{C}$ denotes the weak-star closure of C. In other words, by taking the limits of weak-star convergent *sequences* in C we do not obtain the weak-star closure of C immediately, but we have to repeat this operation precisely n times, when finally this process stabilizes to arrive at the weak-star closure $\overline{C}$.

In Banach's book this construction is done for $X = c_0$ and $X^* = l^1$ while our present context is $X = L^1(P)$ and $X^* = L^\infty(P)$. Adapting the ideas from Banach's book, it is possible to construct a semi-martingale S such that the corresponding convex cone C^{simple} has the following property: taking the weak-star sequential closure $(C^{\text{simple}})^{(1)}$, the resulting set intersects $L^\infty_+(P)$ only in $\{0\}$; but doing the operation twice, we obtain the weak-star closure $C^{(2)} = \overline{C}$, and $\overline{C}$ intersects $L^\infty_+(P)$ in a non-trivial way (see [DS 94, example 7.8]). Hence we cannot reduce to sequences

$(g_n)_{n=0}^{\infty}$ in the definition of *(NFL)*. The construction of this example uses a process with jumps; for continuous processes the situation is, in fact, nicer, and in this case it is possible to give positive answers to questions (i) and (ii) above (see [S 90], [D 92] and [DS 94]).

As regards question (iii), the dividing line again is the continuity of the process S (see [S 90] and [D 92] for positive results for continuous processes, and [S 94] for a counterexample S, where S is a process with jumps).

Summing up the above discussion: the theorem of Kreps and Yan is a beautiful and mathematically precise extension of the fundamental theorem of asset pricing 2.8 to a general framework of stochastic processes in continuous time. However, in general, the concept of passing to the weak-star closure does not allow for a clear-cut economic interpretation. It is therefore desirable to prove versions of the above theorem, where the closure with respect to the weak-star topology is replaced by the closure with respect to some finer topology (ideally the topology of uniform concergence, which allows for an obvious and convincing economic interpretation).

To do so, let us contemplate once more, where the above encountered difficulties related to the weak-star topology originated from: they are essentially caused by our restriction to consider only *simple, admissible trading strategies.* These nice and simple objects can be defined without any limiting procedure, but we should not forget, that they are only auxiliary gimmicks, playing the same role as step functions in integration theory. The concrete examples of trading strategies encountered in section 3 for the case of the Bachelier and the Black-Scholes model led us already out of this class: of course, they are not simple trading strategies.

Hence we have to pass to a suitable class of more general trading strategies than just the simple, admissible ones. Among other pleasant and important features, this will have the following effect on the corresponding sets C and K: these sets will turn out to be "closer to their closures" (ideally they will already be closed in the relevant topology), than the above considered sets C^{simple} and K^{simple}; the reason is that the passage from simple to more general intergrands *involves already a limiting procedure.*

Let us do in a more systematic way our search for an appropriate class of trading strategies:

First of all, one has to restrict the choice of the integrands H to make sure that the process $H \cdot S$ exists. Besides the qualitative restrictions coming from the theory of stochastic integration, one has to avoid problems coming from so-called doubling strategies. This was already noted in the paper by Harrison and Pliska (1979). To explain this remark, let us consider the classical doubling strategy. We toss a coin, and when heads comes up, the player is paid 2 times his bet. If tails comes up, the player loses his bet. The strategy is well known: the player doubles his bet until the first time he wins. If he starts with 1 €, his final gain (= last pay out - total sum of the preceding bets) is 1 € almost surely. He has an almost sure win. The probability that heads will eventually show up, is indeed one (even if the coin is not fair). However, his accumulated losses are not bounded from below. Everybody, especially the casino boss, knows that this is a very risky way of winning 1 €. This type of strategy has to be ruled out: there should be a lower bound on the player's loss. The described doubling strategy is known for centuries and in French it is still referred to as "la martingale".

Here is the definition of the class of intergrands which turns out to be appropriate for our purposes.

Definition 4.8 *Fix an $\mathbb{R}^{d+1}$-valued stochastic process $S = (S_t)_{t \geq 0}$ as defined in the beginning of this section, which we now also* assume to be a semi-martingale. *An $\mathbb{R}^d$-valued predictable process $H = (H_t)_{t \geq 0}$ is called an* admissible integrand *for the semi-martingale S, if*

(i) H is S-integrable, i.e., the stochastic integral $H \cdot S = ((H \cdot S)_t)_{t \geq 0}$ is well-defined in the sense of stochastic integration theory for semi-martingales,

(ii) there is a constant M such that

$$(H \cdot S)_t \geq -M, \qquad a.s., \text{ for all } t \geq 0. \tag{109}$$

Let us comment on this definition: we place ourselves into the "théorie générale" of integration with respect to semi-martingales: here we are on safe grounds as the theory developed, in particular by P.-A. Meyer and his school, tells us precisely what it means that a predictable process H is S-integrable (see, e.g., [P 90]). But in order to do so we have to make sure that S is a semi-martingale: this is precisely the class of processes allowing for a satisfactory integration theory, as we know from the theorem of Bichteler and Dellacherie.

How natural is the assumption, that S is a semi-martingale, from an economic point of view? In fact, it fits very naturally into the present no-arbitrage framework: it is shown in ([DS 94, theorem 7.2]) that, for a locally bounded, càdlàg process S, the assumption, that the closure of C^{simple} *with respect to the norm topology of $L^\infty(P)$* intersects $L^\infty(P)_+$ only in $\{0\}$, implies already that S is a semi-martingale. This assumption therefore is implied by a very mild strengthening of the no-arbitrage condition for simple, admissible integrands. Loosely speaking, the message of this theorem is that a no-arbitrage theory only makes sense, if we start with a semi-martingale model for the financial market S.

As regards condition (ii) in the above definition, this is a strong and economically convincing requirement to rule out the above discussed doubling strategy, as well as similar schemes, which try to make a final gain at the cost of possibly going very deep into the red. Condition (ii) goes back to the original work of Harrison and Pliska [HP 81]: there is a finite credit line M obliging the investor to finance her trading in such a way, that this credit line is respected at all times $t \geq 0$.

Definition 4.9 *Let*

$$K = \{(H \cdot S)_\infty : H \text{ admissible and } (H \cdot S)_\infty = \lim_{t \to \infty} (H \cdot S)_t \text{ exists a.s.}\}, \tag{110}$$

which forms a convex cone of functions in $L^0(\Omega, \mathcal{F}, \mathbb{P})$, and

$$C = \{g \in L^\infty(P) : g \leq f \text{ for some } f \in K\}. \tag{111}$$

We say that S satisfies the condition of no free lunch with vanishing risk *(NFLVR), if*

$$\overline{C} \cap L^\infty_+(P) = \{0\}, \tag{112}$$

where $\overline{C}$ now denotes the closure of C with respect to the norm topology of $L^\infty(P)$.

Comparing the present definition to the notion of "no free lunch" *(NFL)*, the weak-star topology has been replaced by the topology of uniform convergence. Taking up again the discussion following the Kreps-Yan theorem 4.7, we now find a better economic interpretation: S allows for a *free lunch with vanishing risk*, if there is $f \in L^\infty_+(P)\backslash\{0\}$ and sequences $(f_n)_{n=0}^\infty = ((H^n \cdot S)_\infty)_{n=0}^\infty \in K$, for a sequence $(H^n)_{n=0}^\infty$ of admissible integrands, and $(g_n)_{n=0}^\infty$ satisfying $g_n \leq f_n$, such that

$$\lim_{n\to\infty} \|f - g_n\|_\infty = 0. \tag{113}$$

In particular the negative parts $((g_n)_-)_{n=0}^\infty$ tend to zero uniformly, which explains the term *"vanishing risk"*.

We now have all the ingredients to formulate a *general version of the fundamental theorem of asset pricing.*

Theorem 4.10 ([DS 94, corr.1.2]) *The following assertions are equivalent for an $\mathbb{R}^{d+1}$-valued locally bounded semi-martingale model $S = (S_t)_{t\geq 0}$ of a financial market:*

(i) (EMM), i.e., there is a probability measure Q, equivalent to P, such that S is a local martingale under Q.

(ii) (NFLVR), i.e., S satisfies the condition of no free lunch with vanishing risk.

The present theorem is a sharpening of the Kreps-Yan theorem, as it replaces the weak-star convergence in the definition of "no free lunch" by the economically more convincing notion of uniform convergence. The price to be paid for this improvement is, that now we have to place ourselves into the context of *general admissible*, instead of *simple admissible* integrands.

The proof of theorem 4.10 as given in [DS 94] is surprisingly long and technical; despite of several attempts, no essential simplification of this proof has been achieved so far. We are not able to go in detail through this proof, but we shall try to give a "guided tour" through it, which should motivate and help the interested reader to find her way through the arguments in [DS 94].

We start by observing that the implication (i) $\Rightarrow$ (ii) still is the easy one: supposing that S is a local martingale under Q and H is an admissible trading strategy, we may deduce from a result of Ansel-Stricker ([AS 94], see also [E 80]) and the fact that $H \cdot S$ is bounded from below, that $H \cdot S$ is a local martingale under Q, too. Using the boundedness from below of $H \cdot S$, we also conclude that $H \cdot S$ is a Q-super-martingale, so that

$$\mathbf{E}_Q[(H \cdot S)_\infty] \leq 0. \tag{114}$$

Hence $\mathbf{E}_Q[g] \leq 0$, for all $g \in C$, and this equality extends to the norm closure $\overline{C}$ of C (in fact, it also extends to the weak-star closure of C, but we don't need this stronger result for the proof of the present theorem).

Summing up, we have proved that *(EMM)* implies *(NFLVR)*.

Before passing to the reverse implication let us still have a closer look at the crucial inequality (114): its message is that the notion of equivalent local martingale measures Q and admissible integrands H has been designed in such a way, that the basic intuition behind the notion of a martingale holds true: *you cannot win in average by betting on a martingale.* Note, however, that the notion of admissible integrands does not rule out the possibility *to lose in average by betting on S.* An example, already noted in [HP 81], is the so-called "suicide stategy H". Consider a simplified roulette, where red and black both have probability $\frac{1}{2}$, and as usual, when winning, your bet is doubled. The strategy consists in placing one € on red and then walking to the bar and regarding the roulette from a distance: if it happens that consecutively only red turns up in the next couple of games, you may watch a huddle of chips piling up with exponential growth. But, inevitably, i.e., with probability one, black will eventually turn up, which will cause the huddle — including your original € — to disappear. Translating this story into the language of stochastic integration, we have a martingale S (in fact, a random walk) and an admissible trading strategy H such that we have a strict inequality in (114). Of course, the present process $H \cdot S$ corresponding to the "suicide strategy", is just the process corresponding to the "doubling strategy" with opposite sign.

We now discuss the hard implication *(NFLVR) ⇒ (EMM)* of theorem 4.10. It is reduced to the subsequent theorem wich may be viewed as the "abstract" version of theorem 4.10:

Theorem 4.11 ([DS 94, theorem 4.2]) *In the setting of theorem 4.10 assume that (ii) holds true, i.e., that S satisfies (NFLVR).*
Then the cone $C \subseteq L^\infty(P)$ is weak-star closed.

The fact that theorem 4.11 implies theorem 4.10 now follows immediately from the Kreps-Yan theorem: theorem 4.11 tells us that we don't have to bother about passing to the weak-star closure of C any more, as assumption (ii) of theorem 4.10 implies that C *already is weak-star closed.* In other words, our program of choosing the *"right"* class of admissible integrands was successful: the "passage to the limit" which was necessary in the context of the Kreps-Yan theorem, i.e., the passage from C^{simple} to its weak-star closure, is already taken care of by the "passages to the limit" in the stochastic integration theory of general admissible integrands for the semi-martingale S.

In fact, theorem 4.11 tells us that — under the assumption of *(NFLVR)* — C equals precisely the weak-star closure of C^{simple} (the fact that C^{simple} is weak-star dense in C follows from the general theory of stochastic integration, which is based on the idea of approximating a general integrand by simple integrands).

By rephrasing theorem 4.10 in the form of theorem 4.11, we did not come closer to a proof yet. But we see more clearly, what the heart of the matter is: for a net $(H^\alpha)_{\alpha\in I}$ of admissible integrands, $f_\alpha = (H^\alpha \cdot S)_\infty$ and $g_\alpha \le f_\alpha$ such that $(g_\alpha)_{\alpha\in I}$ weak-star converges to $f \in L^\infty(P)$, we have to show that we can find an admissible integrand H such that $f \le (H \cdot S)_\infty$. This will prove theorem 4.11 and therefore 4.10. Loosely speaking, we have to be able to pass from a net $(H^\alpha)_{\alpha\in I}$ of admissible trading strategies to a limiting admissible trading stategy H.

The first good news on our way to prove this result is that in the present context we may reduce from the case of a general net $(H^\alpha)_{\alpha\in I}$ to the case of a sequence $(H^n)_{n=0}^\infty$. This follows from a good old friend from functional analysis, the theorem of Krein-Smulian (see, e.g., [Sch 66]): this theorem implies that a convex set C in a dual Banach space X^* is weak-star closed, iff it is relatively weak-star closed in each bounded subset of X^*. Using some easy additional facts from general functional analysis (see [DS 94, theorem 2.1]) we may conclude that the convex cone C in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is weak-star closed iff it is weak-star sequentially closed. The reader should note the subtle difference to the example from Banach's book discussed after the Kreps-Yan theorem 4.7 above: to pass from a convex set $C \subseteq L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ to its weak-star closure, *it does*, in general, *not suffice* to add all the weak-star sequential limits. But to check, whether a convex set C is already weak-star closed, *it does suffice* to check, whether the weak-star sequential limits remain within C.

Once we have reduced to the case of sequences $(H^n)_{n=0}^\infty$ we may exploit another good friend from functional analysis, the theorem of Banach-Steinhaus (also called principle of uniform boundedness): if *a sequence* $(g_n)_{n=0}^\infty$ in X^* is weak-star convergent, the norms $(\|g_n\|)_{n=0}^\infty$ remain bounded. This result implies that we may reduce to the case that the sequence $(H^n)_{n=0}^\infty$ admits a uniform bound M such that $H^n \cdot S \geq -M$, for all $n \in \mathbb{N}$.

Putting together these reductions from general functional analysis, it will suffice for the proof of theorem 4.11 to prove the following result:

Proposition 4.12 *Under the hypotheses of theorem 4.11, let $(H^n)_{n=0}^\infty$ be a sequence of admissible integrands such that*

$$(H^n \cdot S)_t \geq -1, \qquad \textit{a.s., for } t \geq 0 \textit{ and } n \in \mathbb{N}.. \tag{115}$$

Also assume that $f_n = (H^n \cdot S)_\infty$ converges almost surely to f. Then there is an admissible integrand H such that

$$(H \cdot S)_\infty \geq f. \tag{116}$$

To convince ourselves that proposition 4.12 indeed implies theorem 4.11, we still have to justify one more reduction step which is contained in the statement of proposition 4.12: we may reduce to the case, when $(f_n)_{n=0}^\infty$ converges almost surely. This is done by an elementary lemma in the spirit of Komlos' theorem ([DS 94, lemma A 1.1]). In its simplest form it states the follwing: Let $(g_n)_{n=0}^\infty$ be an arbitrary sequence of random variables uniformly bounded from below. Then we may find convex combinations $h_n \in \text{conv}(f_n, f_{n+1,\ldots})$ converging almost surely to an $\mathbb{R} \cup \{+\infty\}$-valued random variable f. For more refined variations on this theme see [DS 99].

Note that the passage to convex combinations does not cost anything in the present context, where our aim is to find a limit to a given sequence in a locally convex vector space; hence the above lemma allows us to reduce to the case where we may assume, in addition to (115), that $(f_n)_{n=0}^\infty = ((H^n \cdot S)_\infty)_{n=0}^\infty$ converges almost surely to a function $f : \Omega \mapsto \mathbb{R} \cup \{+\infty\}$. Using the assumtion *(NFLVR)* we can show in the present context that f is a.s. finitely valued.

Summing up, proposition 4.12 is a statement about the possibility of passing to a (kind of) limit H, for a given sequence $(H^n)_{n=0}^\infty$ of admissible integrands. The crucial hypothesis is the uniform one-sided boundedness (115); apart from this strong assumption, we only have an information on the a.s. convergence of *the terminal values* $((H^n \cdot S)_\infty)_{n=0}^\infty$, but we do not have any a priori information on the convergence of *the processes* $((H^n \cdot S)_{t \geq 0})_{n=0}^\infty$.

Let us compare proposition 4.12 with the literature. An important theorem of J. Memin [M 80] states the following: if a sequence of stochastic integrals $((H^n \cdot S)_{t\geq 0})_{n=0}^{\infty}$ on a given semi-martingale S converges with respect to the semi-martingale topology, then the limit exists (as a semi-martingale) and is of the form $H \cdot S$ for some S-integrable predictable process H.

This theorem finally will play an important role in proving proposition 4.12; but we still have a long way to go, before we can apply it, as the assumptions of proposition 4.12 a priori do not tell us anything about the convergence of the sequence of processes $((H^n \cdot S)_{t\geq 0})_{n=0}^{\infty}$.

Another line of results in the spirit of proposition 4.12 assumes that the process S is a (local) martingale. The arch-example is the theorem of Kunita-Watanabe (see, e.g. [P 90] or [Y 78]): suppose that S is a locally L^2-bounded martingale, that each $(H^n \cdot S)_{t\geq 0}$ is an L^2-bounded martingal, and that the sequence $((H^n \cdot S)_{t\geq 0})_{n=0}^{\infty}$ is Cauchy in the Hilbert space of square-integrable martingales (equivalently: that the sequence of terminal values $((H^n \cdot S)_{\infty})_{n=0}^{\infty}$ is Cauchy in the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$). Then the limit exists (as a square-integrable martingale) and it is of the form $(H \cdot S)_{t\geq 0}$.

As the proof of this theorem is very simple and allows for some insight into the present theme, we sketch it (assuming, for simplicity, that S is $\mathbb{R}$-valued): denote by $\langle S\rangle_t$ the predictable, quadratic variation process of the L^2-bounded martingale S, which defines a finite measure $d\langle S\rangle_t$ on the sigma-algebra $\mathcal{P}$ of predictable subsets of $\Omega \times \mathbb{R}_+$. Denoting by $L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, d\langle S\rangle_t)$ the corresponding Hilbert space, the stochastic integration theory is designed in such a way that we have the isometric identity

$$\|H\|_{L^2(\Omega\times\mathbb{R}_+,\mathcal{P},d\langle S\rangle_t)} = \|(H \cdot S)_\infty\|_{L^2(\Omega,\mathcal{F},\mathbb{P})}, \tag{117}$$

for each predictable process H, for which the left hand side of (117) is finite.

Hence the assumption that $((H^n \cdot S)_{t\geq 0})_{n=0}^{\infty}$ is Cauchy in the Hilbert space of square-integrable martingales is tantamount to the assumption that $(H^n)_{n=0}^{\infty}$ is Cauchy in $L^2(\Omega\times\mathbb{R}_+, \mathcal{P}, d\langle S\rangle_t)$. Now, once more, the stochastic integration theory is designed in that way that $L^2(\Omega\times\mathbb{R}_+, \mathcal{P}, d\langle S\rangle_t)$ consists precisely of the S-integrable, predictable processes H such that $H\cdot S$ is an L^2-bounded martingale. Hence by the completeness of the Hilbert space $L^2(\Omega\times\mathbb{R}_+, \mathcal{P}, d\langle S\rangle_t)$ we can pass from the Cauchy-sequence $(H^n)_{n=0}^{\infty}$ to its limit $H \in L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, d\langle S\rangle_t)$, thus finishing the sketch of the proof of the Kunita-Watanabe theorem.

The above argument shows in a nice and transparent way how to deduce from a completeness property of the *space of predictable integrands* H a completeness property of the corresponding *space of stochastic integrals* $H \cdot S$. In the context of the theorem of Kunita-Watanabe, the functional analytic background for this argument is reduced to the — almost trivial — isometric identification of the two corresponding Hilbert spaces in (117).

Using substantially more refined arguments, M. Yor [Y 78] was able to extend this result to the case of Cauchy sequences $(H^n \cdot S)_{n=0}^{\infty}$ of martingales bounded in L^p, for arbitrary $1 \leq p \leq \infty$, the most delicate and interesting case being $p = 1$.

After this review of some of the previous literature on the topic of completeness of the space of stochastic integrals, let us turn back to propostion 4.12.

Unfortunately the theorems of Kunita-Watanabe and Yor do not apply to its proof, as we don't assume that S is a local martingale. It is precisely the point, that we finally want to *prove* that S is a local martingale with respect to some measure Q equivalent to P.

But in our attempt to build up some motivation for the proof of proposition 4.12, let us cheat for a moment and suppose that we know already that S is a local martingale under some equivalent measure Q and let $(H^n)_{n=0}^{\infty}$ be a sequence of S-integrable predictable processes satisfying (115). Using again the theorem of Ansel-Stricker [AS 94] we conclude that $(H^n \cdot S)_{n=0}^{\infty}$ is a sequence of local martingales; inequality (115) quickly implies that this sequence is bounded in $L^1(Q)$-norm:

$$\|H^n \cdot S\|_{L^1(Q)} := \sup\{\mathbf{E}[|(H^n \cdot S)_\tau|], \tau \text{ finite stopping time}\} \leq 2, \text{ for } n \geq 0. \quad (118)$$

Let us cheat once more and assume that each $H^n \cdot S$ is in fact a uniformly integrable Q-martingale (instead of only being a local Q-martingale) and that $((H^n \cdot S)_\infty)_{n=0}^{\infty}$ is Cauchy with respect to the $L^1(Q)$-norm defined above (instead of only being bounded with respect to this norm).

Admitting the above "cheating steps" we are in a position to apply Yor's theorem to find a limiting process H to the sequence $(H^n)_{n=0}^{\infty}$ for which (116) holds true, where we even may replace the inequality by an equality. But, of course, this is only motivation, why proposition 4.12 should hold true, and we now have to find a proof without cheating.

We have taken some time for the above heuristic considerations to develop an intuition for the statement of proposition 4.12 and to motivate the general philosophy underlying its proof: *we want to prove results which are — at least more or less — known for (local) martingales S, but replacing the martingale assumption on S by the assumption that S satisfies (NFLVR).*

As a starter we give the proof of a result which shows that, under the assumption of *(NFLVR)*, the technical condition imposed on the admissible integrand H in (110) is, in fact, automatically satisfied.

Lemma 4.13 ([DS 94, theorem 3.3]) *Let S satisfy (NFLVR) and H be an admissible integrand.*

Then

$$(H \cdot S)_\infty := \lim_{t \to \infty} (H \cdot S)_t \quad (119)$$

exists and is finite, almost surely.

This result is a good illustration for our philosophy: suppose *we know already* that the assumption of 4.13 implies that S is a local martingale under some Q equivalent to P. Then the conclusion follows immediately from known results: from Ansel-Stricker [AS 94] we know that $H \cdot S$ is a super-martingale. As $H \cdot S$ is bounded from below, Doob's theorem (see, e.g., [W 91]) implies the almost sure convergence of $(H \cdot S)_t$ as $t \to \infty$ to an a.s. finite random variable.

Our goal is to replace these martingale arguments by some arguments relying only on *(NFLVR)*. The nice feature is that these arguments also allow for an economic interpretation.

Proof of Lemma 4.13 As in the usual proof of Doob's super-martingale convergence theorem we consider the number of up-crossings: to show almost sure convergence of $(H \cdot S)_t$, for $t \to \infty$, it will suffice to show that, for any $\beta < \gamma$, the P-measure of the set $\{\omega : (H \cdot S)_t(\omega) \text{ upcrosses }]\beta, \gamma[\text{ infinitely often}\}$ equals zero.

So suppose to the contrary that there is $\beta < \gamma$ such that the set

$$A = \{\omega : (H \cdot S)_t \text{ upcrosses }]\beta, \gamma[\text{ infinitely often}\} \tag{120}$$

satisfies $P[A] > 0$. The economic interpretation of this situation is the following: an investor knows at time zero that, when following the trading strategy H, with probability $P[A] > 0$ her wealth will infinitely often be less than or equal to β as well as more than or equal to γ. A smart investor will realize that this offers a free lunch with vanishing risk, as she can modify H to obtain a very rewarding trading strategy K.

Indeed, define inductively the sequence of stopping times $(\sigma_n)_{n=0}^{\infty}$ and $(\tau_n)_{n=0}^{\infty}$ by $\sigma_0 = \tau_0 = 0$ and, for $n \geq 1$,

$$\begin{aligned} \sigma_n &= \inf\{t \geq \tau_{n-1} : (H \cdot S)_t \leq \beta\}, \\ \tau_n &= \inf\{t \geq \sigma_n : (H \cdot S)_t \geq \gamma\}. \end{aligned} \tag{121}$$

The set A then equals the set where, σ_n and τ_n are finite, for each $n \in \mathbb{N}$ (as usual, the inf over the empty set is taken to be $+\infty$).

What every investor wants to do is to "buy low and sell high"; the above stopping times allow her to do that in a systematic way: define $K = H\mathbf{1}_{\{\cup_{n=1}^{\infty}]\!]\sigma_n, \tau_n]\!]\}}$, which clearly is a predictable S-integrable process. A more verbal description of K goes as follows: the investor starts by doing nothing (i.e., making a zero-investment into the risky assets $S^1, \ldots, S^d$) until the time σ_1 when the process $(H \cdot S)_t$ has dropped below β (If $\beta \geq 0$, we have $\sigma_1 = 0$)). At this time she starts to invest according to the rule prescribed by the trading strategy H; she continues to do so until time τ_1 when $(H \cdot S)_t$ first has passed beyond γ. Note that, if $\tau_1(\omega)$ is finite, our investor following the strategy K has at least gained the amount $\gamma - \beta$. At time τ_1 (if it happens to be finite) the investor clears all her positions and does not invest into the risky assets until time σ_2, when she repeats the above scheme.

One easily verifies (arguing either "mathematically" or "economically") that the process $K \cdot S$ is uniformly bounded from below and satisfies

$$(K \cdot S)_t \geq -M \qquad \text{a.s., for all } t, \tag{122}$$

where M is the uniform lower bound for $(H \cdot S)$, and

$$\lim_{t \to \infty} (K \cdot S)_t = \infty. \qquad \text{a.s. on } A. \tag{123}$$

Hence K describes a trading scheme, where the investor can lose at most a fixed amount of money, while, with strictly positive probability, she ultimately becomes infinitely rich. Intuitively speaking, this is "something like an arbitrage", and it is an easy task to formally deduce from these properties of K a "free lunch with vanishing risk": for example, it suffices to define $K^n = \frac{1}{n} K \mathbf{1}_{]\!]0, \tau_n \wedge T_n]\!]}$, for a sequence of (deterministic) times $(T_n)_{n=0}^{\infty}$, to let $f_n = (K^n \cdot S)_\infty = (K^n \cdot S)_{\tau_n \wedge T_n}$ and to define $g_n = f_n \wedge (\gamma - \beta)\mathbf{1}_B$ where $B = \bigcap_{n=0}^{\infty} \{\tau_n \leq T_n\}$. If $(T_n)_{n=1}^{\infty}$ tends to infinity sufficiently fast, we have $P[B] > 0$, and one readily verifies that $(g_n)_{n=1}^{\infty}$ converges uniformly to $(\gamma - \beta)\mathbf{1}_B$.

Summing up, we have shown that *(NFLVR)* implies that, for $\beta < \gamma$, the process $H \cdot S$ almost surely upcrosses the interval $]\beta, \gamma[$ only finitely many times. Whence $(H \cdot S)_t$ converges almost surely to a random variable $(H \cdot S)_\infty$ with values in $\mathbb{R} \cup \{\infty\}$. The fact that $(H \cdot S)_\infty$ is a.s. finitely valued follows from another application (similar but simpler than above) of the assumption of *(NFLVR)*, which we leave to the reader. ■

After all these preparations we finally start to sketch the main arguments underlying the proof of proposition 4.12. The strategy is to obtain from assumption (115) and from suitable modifications of the original sequence $(H^n)_{n=0}^\infty$, still denoted by $(H^n)_{n=0}^\infty$, more information on the convergence of the sequence of *processes* $(H^n \cdot S)_{n=0}^\infty$. Eventually we shall be able to reduce to the case where $(H^n \cdot S)_{n=0}^\infty$ converges in the semi-martingale topology; at this stage Memin's theorem will give us the desired limiting trading strategy H.

So, what can we deduce from assumption (115) and the a.s. convergence of $(f_n)_{n=0}^\infty = ((H^n \cdot S)_\infty)_{n=0}^\infty$ for the convergence of the sequence of processes $(H^n \cdot S)_{n=0}^\infty$? The unpleasant answer is: a priori, we cannot deduce anything. To see this, recall the "suicide" strategy H which we have discussed in the context of inequality (114) above: it designs an admissible way to lose one €. Speaking mathematically, the corresponding stochastic integral $H \cdot S$ starts at $(H \cdot S)_0 = 0$, satisfies $(H \cdot S)_t \geq -1$ a.s., for all $t \geq 0$, and $(H \cdot S)_\infty = -1$. But clearly this is not the only admissible way to lose one € and there are many other trading strategies K on the process S having the same properties. A trivial example is, to first wait without playing for a fixed number of games of the roulette, and to start the suicide strategy only after this waiting period; of course, this is a (slightly) different way of losing one €.

Speaking mathematically, this means that — even when S is a martingale, as it is the case in the example of the suicide strategy — the condition $(H \cdot S)_t \geq -1$. a.s., for all $t > 0$, and the final outcome $(H \cdot S)_\infty$ do not determine the process $H \cdot S$. In particular there is no hope to derive from (115) and the a.s. convergence of the sequence of random variables $((H^n \cdot S)_\infty)_{n=0}^\infty$ a convergence property of the sequence of processes $(H^n \cdot S)$.

The idea to remedy the situation is to remark the following fact: the suicide strategy is a silly investment and obviously there are better trading strategies, e.g., not to gamble at all. By discarding such "silly investments", we hopefully will be able to improve the situation.

Here is the way to formalize the idea of discarding "silly investments": Denote by D the set of all random variables h such that there is a random variable $f \geq h$ and a sequence $(H^n)_{n=0}^\infty$ of admissible trading strategies satisfying (115), and such that $(H^n \cdot S)_\infty$ converges a.s. to f. We call f_0 a maximal element of D if the conditions $h \geq f_0$ and $h \in D$ imply that $h = f_0$.

For example, in the context of the "suicide strategy", $f \equiv -1$ is an element of D, but not a maximal element. A maximal element dominating f is, for example, $f_0 \equiv 0$.

More generally, it is not hard to prove under the assumptions of proposition 4.12 that, for a given $f = (H \cdot S)_\infty \geq -1$, where H is an admissible integrand, there is a maximal element $f_0 \in D$ dominating f (see [DS 94, lemma 4.3]).

The point of the above concept is that, in the proof of proposition 4.12, we may assume without loss of generality that f is a maximal element of D. Under this additional assumption it is indeed possible to derive from the a.s. convergence of the sequence of random variables $((H^n \cdot S)_\infty)_{n=0}^\infty$ some information on the convergence of the sequence of processes $((H^n \cdot S)_{t \geq 0})_{n=0}^\infty$.

As the proof of this result is another nice illustration of our general approach of replacing "martingale arguments" by "economically motivated arguments" relying on the assumption *(NFLVR)*, we sketch the argument.

Lemma 4.14 ([DS 94, lemma 4.5]) *Under the assumptions of proposition 4.12 suppose, in addition, that f is a maximal element of D.*

Then the sequence of random variables

$$F_{n,m} = \sup_{t \geq 0} |(H^n \cdot S)_t - (H^m \cdot S)_t| \tag{124}$$

tends to zero in probability, as $n, m \to \infty$.

Proof Suppose to the contrary that there is $\alpha > 0$, and sequences $(n_k, m_k)_{k \geq 1}$ tending to ∞ s.t., for each k, we have $\mathbb{P}[\sup_{t \geq 0}((H^{n_k} \cdot S)_t - (H^{m_k} \cdot S)_t) > \alpha] \geq \alpha$.

Define the stopping times τ_k as

$$\tau_k = \inf\{t : (H^{n_k} \cdot S)_t - (H^{m_k} \cdot S)_t \geq \alpha\}, \tag{125}$$

so that we have $\mathbb{P}[\tau_k < \infty] \geq \alpha$.

Define L^k as $L^k = H^{n_k} \mathbf{1}_{[\![0,\tau_k]\!]} + H^{m_k} \mathbf{1}_{]\!]\tau_k,\infty[\![}$. Clearly the process L^k is predictable and $L^k \cdot S \geq -1$.

Translating the formal definition into prose: the trading strategy L^k consists in following the trading strategy H^{n_k} up to time τ_k, and then switching to H^{m_k}. The idea is that L^k produces a sensibly better final result $(L^k \cdot S)_\infty$ than either $(H^{n_k} \cdot S)_\infty$ or $(H^{m_k} \cdot S)_\infty$, which will finally lead to a contradiction to the maximality assumption on f.

Why is L^k "sensibly better" than H^{n_k} or H^{m_k}? For large k, the random variables $(H^{n_k} \cdot S)_\infty$ as well as $(H^{m_k} \cdot S)_\infty$ will both be close to f in probability; for the sake of the argument, assume that both are in fact equal to f (keeping in mind that the difference is "small with respect to convergence in probability"). A moment's reflection reveals that this implies that the random variables $(L^k \cdot S)_\infty$ equal f plus the random variable $((H^{n_k} \cdot S)_{\tau_k} - (H^{m_k} \cdot S)_{\tau_k}) \mathbf{1}_{\{\tau_k < \infty\}}$. The latter random variable is non-negative and with probability α greater than or equal to α; this means that this difference between f and $(L^k \cdot S)_\infty$ is not "small with respect to convergence in probability"; this is, what we had in mind when saying that L^k is a "sensible" impovement as compared to H^{n_k} or H^{m_k}.

Modulo some technicalities, which are worked out in [DS 94, lemma 4.5], this gives the desired contradiction to the maximality assumption on f, thus finishing the (sketch of the) proof of lemma 4.14. ∎

Lemma 4.14 is our first step towards a proof of proposition 4.12: it gives some information on the convergence of the sequence of processes $(H^n \cdot S)_{n=0}^\infty$ in terms of the maximal functions defined in (124). But the assertion that these maximal functions tend to zero in probability is still much weaker than the convergence of $(H^n \cdot S)_{n=0}^{\infty}$ with respect to the semi-martingale topology, which we finally need in order to be able to apply Memin's theorem. There is still a long way to go!

But it is time to finish this "guided tour" and to advise the interested reader to find the remaining part of the proof on pages 482–494 of [DS 94]. We hope that we have succeeded to give some motivation for the proof and for the "economically motivated" arguments underlying it.

To finish this section we return to the issue, that we always have assumed that the process S is *locally bounded*. What happens if we drop this — technically very convenient — assumption?

Before starting to answer this question, we remark that it is not only of "academic" interest. It is also important from the point of view of applications: once one leaves the framework of continuous processes S — and there are good empirical reasons to do so — it is also natural to allow for the jumps to be unbounded. As a concrete example we mention the family of ARCH (Auto Regressive Conditional Heteroskedastic) processes and their relatives (GARCH, EGARCH etc.), which are very popular in the econometric literature. These are processes in discrete time where the conditional distribution of the jumps is Gaussian. In particular, these processes are not locally bounded. There are many other examples of processes which fail to be locally bounded, used in the modelling of financial markets.

The answer to the above question is as we expect it to be: *mutatis mutandis* the fundamental theorem of asset pricing 4.10 and the related theorems obtained in its proof carry over to the case of not necessarily locally bounded $\mathbb{R}^{d+1}$-valued semi-martingales S. Not coming as a surprise, the techniques of the proofs have to be refined: in particular, we cannot entirely reduce the situation to the study of the space $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, and the weak-star and norm topology of this space: there is no possibility any more to reduce to the case of (one-sided) bounded stochastic integrals and we therefore have to use larger spaces than $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Yet it turns out — and this is slightly surprising — that the duality between $L^\infty(P)$ and $L^1(P)$ still remains the central issue of the proof.

Here is the statement of the extension of the fundamental theorem of asset pricing as obtained in [DS 98].

Theorem 4.15 ([DS 98, corr.1.2]) *The following assertions are equivalent for an $\mathbb{R}^{d+1}$-valued semi-martingale model $S = (S_t)_{t \geq 0}$ of a financial market:*

(i) (ESMM), i.e., there is a probability measure Q equivalent to P such that S is a sigma-martingale under Q.

(ii) (NFLVR), i.e., S satisfies the condition of no free lunch with vanishing risk.

There is a slight change in the statement of the theorem as compared to the statement of theorem 4.10: the term "local martingale" in the definition of *(EMM)* was replaced by the term "sigma-martingale" thus replacing the acronym *(EMM)* by *(ESMM)*. On the other hand, condition (ii) remained completely unchanged.

The notion of a sigma-martingale is a generalisation of the notion of a local martingale:

Definition 4.16 *[DS 98] An $\mathbb{R}^n$-valued semi-martingale $S = (S_t)_{t\geq 0}$ is called a* sigma-martingale *if there is a predictable process $g = (g_t)_{t\geq 0}$, taking its values in $]0,1]$, such that the stochastic integral $g \cdot S$ is a martingale.*

It is easy to verify that a local martingale satisfies the above condition. More delicate is the fact that there are examples of sigma-martingales which fail to be local martingales: this was shown in a famous and ingenious example by M. Emery [E 80].

It is shown in [DS 98] that the notion of sigma-martingales makes good sense economically in the present context. Indeed, the "only if" implication of lemma 4.3 above extends to not necessarily locally bounded semi-martingales, if we replace the term local martingale by the term sigma-martingale. For this as well as for the (rather technical) proof of theorem 4.15 we refer to [DS 98].

5 Some Applications of the Fundamental Theorem of Asset Pricing

The crucial message of theorem 4.10 and the results obtained in the course of the proof is not only, that the version of the FTAP, as obtained by Harrison-Pliska for the case of finite Ω (theorem 2.8 above) and subsequently extended by several authors (we refer to [DS 94] for references on the literature), carries over — mutatis mutandis — to the general semi-martingale setting. For the applications, the additional information provided by theorem 4.11 pertaining to the weak-star closedness of the set C turns out to be at least as relevant.

As a typical example we show that, once the weak-star closedness of C is established by theorem 4.11, it is straight forward to deduce the extension of theorem 2.11 on *Pricing by No Arbitrage* from the setting of finite Ω to the present semi-martingale setting.

We start with the analogue of proposition 2.10: for the sake of coherence we again place us into the setting of locally bounded processes as in the previous section; but we remark that the subsequent results also extend to the non locally bounded case (see [DS 98]).

Proposition 5.1 *Suppose that the locally bounded $\mathbb{R}^{d+1}$-valued semi-martingale $S = (S_t)_{t\geq 0}$ satisfies (NFLVR). Then the polar of C, taken with respect to the duality between $L^\infty(\mathbb{P})$ and $L^1(\mathbb{P})$, and identifying a $\mathbb{P}$-absolutely continuous measure Q with its Radon-Nikodym derivative $\frac{dQ}{dP}$, is equal to cone$(\mathcal{M}^a(S))$, and $\mathcal{M}^e(S)$ is dense in $\mathcal{M}^a(S)$ with respect to the norm topology of $L^1(\mathbb{P})$. Hence the following assertions are equivalent for an element $g \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$:*

(i) $g \in C$,

(ii) $\mathbf{E}_Q[g] \leq 0$, *for all* $g \in \mathcal{M}^a(S)$,

(iii) $\mathbf{E}_Q[g] \leq 0$, *for all* $g \in \mathcal{M}^e(S)$,

Proof First note that, similarly as in lemma 2.7, a probability measure Q, absolutely continuous with respect to $\mathbb{P}$, is in $\mathcal{M}^a(S)$ iff $\mathbf{E}_Q[g] \leq 0$, for all $g \in C$: the necessity of this condition was shown in (114); for the sufficiency we use the local boundedness of S and lemma 4.3 to obtain that the condition $\mathbf{E}_Q[g] \leq 0$, for $g \in C$, implies in particular that S is a local martingale under Q. In other words, the polar of C equals cone$(\mathcal{M}^a(S))$.

The bipolar theorem [Sch 66] therefore implies that an element g of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is in the *weak-star closure* of C iff condition (ii) is satisfied. By theorem 4.11 we know that C is already weak-star closed, hence (i) is equivalent to (ii).

The density of $\mathcal{M}^e(S)$ in $\mathcal{M}^a(S)$ and therefore the equivalence of (ii) and (iii) follows by the same argument as in proposition 2.10 above. ■

We now carry the argument underlying theorem 2.11 over to the present setting.

To maintain in line with the formulation of theorem 2.11, it is convenient to introduce some notation. A given $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is called *super-hedgeable* (resp. *strictly super-hedgeable*) at price $a \in \mathbb{R}$, if there is an admissible trading strategy H

s.t. $f \le a + (H \cdot S)_\infty$ (resp. s.t., in addition, we have $P[f < a + (H \cdot S)_\infty] > 0$). In other words, f is super-hedgeable (resp. strictly super-hedgeable) at price a, if $f - a$ is in C (resp. if, in addition, $f - a$ is not a maximal element of C). Accordingly, we say that f is *sub-hedgeable*, (resp. *strictly sub-hedgeable*) at price a if $-(f - a)$ is in C (resp. if, in addition, $-(f - a)$ is not maximal in C). A real number a is called an *arbitrage free price* for f, if f is neither strictly super- nor strictly sub-hedgeable at price a.

Denoting by S_+ the set of prices a at which f is super-hedgeable, it is rather obvious that S_+ is an interval, its upper bound being equal to ∞, and its lower bound being an element of the interval [ess inf (f), ess sup (f)]. It is less obvious that S_+ is closed, but this fact is a straightforward consequence of theorem 4.11: If $(f - a_n) \in C$, for each n, and $\lim_{n\to\infty} a_n = a$, then $f - a \in C$. Hence there is $\beta \in \mathbb{R}$ s.t. $S_+ = [\beta, \infty[$.

Denoting by S_- the set of prices at which S is sub-hedgeable we similarly obtain that $S_- =]-\infty, \alpha]$, for some $\alpha \in \mathbb{R}$. As, for any $Q \in \mathcal{M}^e(S)$, we have $\mathbf{E}_Q[f] \le \beta$ and $\mathbf{E}_Q[f] \ge \alpha$, (apply (114)), we observe that $\alpha \le \beta$, as soon as $\mathcal{M}^e(S) \ne \emptyset$.

Using the notation (30) and (31) we also have $\beta = \overline{\pi}(f)$ and $\alpha = \underline{\pi}(f)$. Indeed, we just have remarked the inequalities $\overline{\pi}(f) \le \beta$ and $\underline{\pi}(f) \ge \alpha$. Conversely, we know from theorem 4.11 and proposition 5.1 that, for $a < \beta$, we may find $Q \in \mathcal{M}^e(S)$ such that $\mathbf{E}_Q[f - a] > 0$, as $f - a$ is not in C. Hence for $\overline{\pi}(f) := \sup\{\mathbf{E}_Q[f] : Q \in \mathcal{M}^e(S)\}$, we obtain the inequality $\overline{\pi}(f) \ge \beta$; the same argument implies that $\underline{\pi}(f) \le \alpha$.

Having established $\alpha = \underline{\pi}(f)$ and $\beta = \overline{\pi}(f)$, we need a little extra argument for the proper treatment of the boundary cases α and β.

Lemma 5.2 *Under the above assumptions suppose in addition that $\alpha < \beta$. Then f is strictly super-hedgeable at price β and strictly sub-hedgeable at price α. Hence, for $Q \in \mathcal{M}^e(S)$, we have $\mathbf{E}_Q[f] \in]\alpha, \beta[$.*

Proof We know that f is super-hedgeable at price β, i.e., there is an admissible trading strategy H such that $f \le \beta + (H \cdot S)_\infty$. To show that f is, in fact, strictly super-hedgeable at price β, define the stopping time τ by

$$\tau = \inf\{t : (H \cdot S)_t \ge 1 + \operatorname{ess\,sup}(f)\}. \tag{126}$$

Clearly $\widehat{H} := H\mathbf{1}_{[\![0,\tau]\!]}$ also is a super-hedging strategy for f.

Now we distinguish two cases: either $P[\tau < \infty] > 0$. Then the trading strategy $\widehat{H}$ strictly super-hedges f. Or $P[\tau < \infty] = 0$; in this case we have that $H = \widehat{H}$ and that $H \cdot S$ is a bounded process; therefore $H \cdot S$ is a uniformly integrable martingale under each $Q \in \mathcal{M}^a(S)$. This implies that the original strategy H defines a strict super-hedge for f, i.e., $P[f < \beta + (H \cdot S)_\infty] > 0$. Indeed, otherwise we would have that $\mathbf{E}_Q[f] = \mathbf{E}_Q[\beta + (H \cdot S)_\infty] = \beta$, for each $Q \in \mathcal{M}^e(S)$, in contradiction to the assumption $\alpha < \beta$.

Summing up, we have shown that f is strictly super-hedgeable at price β; applying the same argument to $-f$ we see that f is strictly sub-hedgeable at price α.

The final statement of the lemma is now obvious. ■

Taking up again the discussion preceding lemma 5.2, we distingnish two cases: either $\alpha < \beta$, in which case lemma 5.2, tells us that the arbitrage-free prices for f

consist of the open interval $]\alpha, \beta[$. We then also have that $]\alpha, \beta[=]\underline{\pi}(f), \overline{\pi}(f)[$. In the case $\alpha = \beta$ we have that there is an admissible trading strategy H such that $f \leq \alpha + (H \cdot S)_\infty$. Fixing an arbitrary $Q \in \mathcal{M}^e(S)$, we must have $\mathbf{E}_Q[f] = \alpha$, so that $H \cdot S$ must be a uniformly integrable martingale under Q (it is a Q-super-martingale verifying $\mathbf{E}_Q[(H \cdot S)_\infty] = \mathbf{E}_Q[(H \cdot S)_0] = 0$). Hence $(H \cdot S)_t = \mathbf{E}_Q[f - a | \mathcal{F}_t]$, which shows in particular that the process $H \cdot S$ is bounded. Therefore H as well as $-H$ are admissible trading strategies.

Summing up: we have proved the subsequent extension of theorem 2.11 (compare [DS 95, theorem 5.7]) to the present semi-martingale setting, which carries over almost verbatim from the setting of finite Ω.

Theorem 5.3 (Pricing by No-Arbitrage) *Assume that the locally bounded semi-martingale $S = (S_t)_{t \geq 0}$ satisfies (NFLVR) and let*

$$\overline{\pi}(f) = \sup \{\mathbf{E}_Q[f] : Q \in \mathcal{M}^e(S)\}, \tag{127}$$

$$\underline{\pi}(f) = \inf \{\mathbf{E}_Q[f] : Q \in \mathcal{M}^e(S)\}, \tag{128}$$

Either $\underline{\pi}(f) = \overline{\pi}(f)$, in which case $f = \pi(f) + (H \cdot S)_\infty$, where $\pi(f) = \overline{\pi}(f) = \underline{\pi}(f)$ and H is a predictable process such that the process $H \cdot S$ is bounded.

Or $\underline{\pi}(f) < \overline{\pi}(f)$, in which case $\{\mathbf{E}_Q[f] : Q \in \mathcal{M}^e(S)\}$ equals the open interval $]\underline{\pi}(f), \overline{\pi}(f)[$, which in turn equals the set of arbitrage-free prices for the contingent claim f.

In the formulation of the above theorem we have restricted ourselves to the case of bounded random variables $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. One may also extend it — mutatis mutandis — to the case of functions f which are uniformly bounded from below or, more generally, bounded from below by some fixed random variable w having appropriate integrability conditions (see, e.g., [J 92], [AS 94] and [DS 98]).

Let us briefly review some other applications of theorem 4.11. A rather subtle consequence, requiring quite a bit of additional work, is the subsequent extension of the optional decomposition theorem 2.15 to a general semi-martingale setting as given by D. Kramkov([K 96]):

Theorem 5.4 ***emph(Optional Decomposition)*** *Let $S = (S_t)_{t \geq 0}$ be a locally bounded $\mathbb{R}^{d+1}$-valued semi-martingale satisfying (NFLVR), and let $V = (V_t)_{t \geq 0}$ be a non-negative, adapted, càdlàg process, defined on the filtered stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.*

The following assertions are equivalent:

(i) V is a super-martingale, for each $Q \in \mathcal{M}^e(S)$.

(i') V is a super-martingale, for each $Q \in \mathcal{M}^a(S)$.

(ii) V may be decomposed into $V = V_0 + H \cdot S - C$, where H is an admissible trading strategy and $C = (C_t)_{t \geq 0}$ is an increasing, càdlàg, adapted process starting at $C_0 = 0$.

The above theorem extends the "baby version" for finite Ω presented in theorem 2.15 above. A first non-trivial version of this theorem was given by N. El Karoui and M.-C. Quenez [KQ 95] in the context of a filtration generated by an n-dimensional Brownian motion, using techniques from stochastic control. The version stated above was proved by D. Kramkov [K 96]. Subsequently H. Föllmer and Y. Kabanov [FK 98] extended the result to the case of non locally bounded semi-martingales; their method uses a Lagrange multiplier technique and does not rely on theorem 4.11. Finally, F. Delbaen and the present author [DS 99] also removed the assumption of non-negativity of V; their proof is similar in spirit to Kramkov's original one and heavily relies on theorem 5.3. We shall now present the basic idea of this proof.

Sketch of proof of theorem 5.4 As in theorem 2.15 above we only have to show the implication (i) $\Rightarrow$ (ii). Fix an increasing sequence of finite meshes $M^n = \{0, t_1^n, \ldots, t_{N_n}^n\}$, such that $\cup_{n=1}^{\infty} M^n$ is dense in $\mathbb{R}_+$. For example, we may take $N_n = n2^n$ and $t_i^n = \frac{i}{2^n}$, for $i = 1, \ldots, N_n$.

For fixed $n \in \mathbb{N}$ and $i = 1, \ldots, N_n$, we proceed similarly as in the proof of theorem 2.15 above: we consider the process $(S_t)_{t_{i-1}^n \le t \le t_i^n}$ and apply theorem 5.3: the condition

$$\mathbf{E}_Q[V_{t_i^n} | \mathcal{F}_{t_{i-1}^n}] \le V_{t_{i-1}^n}, \qquad \text{for } Q \in \mathcal{M}^e(S), \tag{129}$$

implies that there is an admissible predictable process $(H_t^{n,i})_{t_{i-1}^n < t \le t_i^n}$, supported by $]t_{i-1}^n, t_i^n]$, such that

$$V_{t_i^n} \le V_{t_{i-1}^n} + (H^{n,i} \cdot S)_{t_i^n}. \tag{130}$$

In fact, we have to apply theorem 5.3 conditionally with respect to the sigma-algebra $\mathcal{F}_{t_{i-1}^n}$; but this conditional extension of theorem 5.3 does not present any difficulty.

Fixing $n \in \mathbb{N}$, letting $H^n := \Sigma_{i=1}^{N_n} H^{n,i}$ and, defining $\Delta C_i^n := V_{t_{i-1}^n} + (H^{n,i} \cdot S)_{t_i^n} - V_{t_i^n}$ for $i = 1, \ldots, N_n$, we obtain the following objects: an admissible trading strategy $H^n = (H_t^n)_{t \ge 0}$, indexed by $\mathbb{R}_+$, and an adapted increasing process $C^n = (C_t^n)_{t \in M^n}$, indexed by the finite time index set M^n, such that

$$V_t = (H^n \cdot S)_t - C_t^n, \qquad \text{for } t \in M_n. \tag{131}$$

This is not yet quite what we want to have, as we want to find a predictable process $H = (H_t)_{t \ge 0}$ and an adapted càdlàg process $C = (C_t)_{t \ge 0}$, indexed by $t \in \mathbb{R}_+$, such that (131) holds true for all $t \in \mathbb{R}_+$.

But it is clear what we have to do to achieve this goal: we have to *pass to the limit* of the sequence $(H^n)_{n=0}^{\infty}$. Hence, again, we face our usual problem: how to pass from a sequence $(H^n)_{n=0}^{\infty}$ of admissible integrands to a limit H? Similarly as in the context of the proof of the fundamental theorem of asset pricing, the only essential information on the sequence of admissible trading strategies $(H^n)_{n=1}^{\infty}$ is that they have a uniform lower bound: indeed, one easily deduces from the assumption $V \ge 0$ that $H^n \cdot S \ge -V_0$, for all $n \in \mathbb{N}$.

Hence the basic problem of the proof of the present theorem is very similar in spirit to the theme of the proof of theorem 4.11. It turns out that, refining some of these arguments, it is indeed possible to find a limiting strategy H above. For the details we refer to [K 96] or [DS 99]. ■

As a final application of theorem 4.11 we mention the topic of utility optimization in financial markets. Roughly speaking, one fixes a utility function U on $\mathbb{R}$, i.e. an increasing, strictly concave function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, and an initial endowment $x \in \mathbb{R}$. A typical problem consists in finding, for a fixed horizon T, a trading strategy H maximizing

$$\mathbf{E}[U(x + (H \cdot S)_T)]. \tag{132}$$

We cannot go in detail into this rather extensive theory here and refer, e.g., to the survey paper [S 01]. We only mention that the modern way to deal with the problem of maximizing (132) is to use the duality theory of convex optimization in infinite-dimensional spaces. The crucial property in order to make this theory work, again, is the polar relation between the sets C and $\mathcal{M}^a(S)$ as stated in theorem 5.2. The heart of the matter therefore again is the weak-star closedness of C as stated in theorem 4.11.

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