1 What is a Free Lunch?

1.1 Arbitrage

The notion of arbitrage is crucial in the modern theory of Finance. It is the corner-stone of the option pricing theory due to F. Black and M. Scholes (published in 1973, Nobel prize in Economics 1997).

The underlying idea is best explained by telling a little joke: a finance professor and a normal person go on a walk and the normal person sees a 100 € bill lying on the street. When the normal person wants to pick it up, the finance professor says: don't try to do that. It is absolutely impossible that there is a 100 € bill lying on the street. Indeed, if it were lying on the street, somebody else would have picked it up already before you (end of joke).

How about financial markets? There it is already much more reasonable to assume that there are no arbitrage possibilities, i.e., that there are no 100 € bills lying around and waiting to be picked up. Let us illustrate this with an easy example.

Consider the trading of \$ versus € which takes place simultaneously at two exchanges, say in New York and Frankfurt. Assume for simplicity that in New York the \$/€ rate is 1:1. Then it is quite obvious that in Frankfurt the exchange rate (at the same moment of time) also is 1:1. Let us have a closer look why this is indeed the case. Suppose to the contrary that you can buy in Frankfurt a \$ for 0,999 €. Then, indeed, the so-called "arbitrageurs" (these are people with two telephones in their hands and three screens in front of them) would quickly act to buy \$ in Frankfurt and simultaneously sell the same amount of \$ in New York, keeping the margin in their (or their bank's) pocket. Note that there is no normalizing factor in front of the exchanged amount and the arbitrageur would try to do this on a scale as large as possible.

It is rather obvious that in the above described situation the market cannot be in equilibrium. A moment's reflection reveals that the market forces triggered by the arbitrageurs will make the \$ rise in Frankfurt and fall in New York. The arbitrage possibility will disappear when the two prices become equal. Of course "equality" here is to be understood as an approximate identity where — even for arbitrageurs with very low transaction costs — the above scheme is not profitable any more.

This brings us to a first — informal and intuitive — definition of arbitrage: an arbitrage opportunity is the possibility to make a profit in a financial market *without risk* and *without net investment of capital*. The *principle of no arbitrage* states that a mathematical model of a financial market should not allow for arbitrage possibilities.

1.2 An easy model of a financial market

To apply this principle to less trivial cases we consider a — still extremely simple — mathematical model of a financial market: there are two assets, called the bond and the stock. The bond is riskless, hence — by definition — we know what it is worth tomorrow. For (mainly notational) simplicity we neglect interest rates and assume that the price of a bond equals 1 € today as well as tomorrow, i.e.,

$$B_0 = B_1 = 1 \tag{1}$$

The more interesting feature of the model is the stock which is risky: we know its value today, say

$$S_0 = 1, \tag{2}$$

but we don't know its value tomorrow. We model this uncertainty stochastically by defining S_1 to be a random variable depending on the random element $\omega \in \Omega$. To keep things as simple as possible, we let Ω consist of two elements only, g for "good" and b for "bad", with probability $\mathbf{P}[g] = \mathbf{P}[b] = \frac{1}{2}$. We define $S_1(\omega)$ by

$$S_1(\omega) = \begin{cases} 2 & \text{for } \omega = g \\ \frac{1}{2} & \text{for } \omega = b. \end{cases} \tag{3}$$

Now we introduce a third financial instrument in our model, an *option on the stock* with strike price K: the buyer of the option has the right — but not the obligation — to buy one stock at time $t = 1$ at the predefined price K. To fix ideas let $K = 1$. A moment's reflexion reveals that the price C_1 of the option at time $t = 1$ (where C stands for *contingent* claim) equals

$$C_1 = (S_1 - K)_+, \tag{4}$$

i.e., in our simple example

$$C_1(\omega) = \begin{cases} 1 & \text{for } \omega = g \\ 0 & \text{for } \omega = b. \end{cases} \tag{5}$$

Hence we know the value of the option at time $t = 1$, *contingent on the value of the stock*. But what is the price of the option today?

The classical approach, used by actuaries for centuries, is to price contingent claims by taking expectations which leads to the value $C_0 := \mathbf{E}[C_1] = \frac{1}{2}$ in our example. Although this simple approach is very successful in many actuarial applications, it is not at all satisfactory in the present context. Indeed, the rationale behind taking the expected value is the following argument based on the law of large numbers: in the long run the buyer of an option will neither gain nor lose in the average. We rephrase this fact in a more financial lingo: the performance of an investment into the option would in the average equal the performance of the bond (for which we have assumed an interest rate zero). However, a basic feature of finance is that an investment into a risky asset should — in the average — yield a better performance than an investment into the bond (for the sceptical reader: at least, these two values should not necessarily coincide). In our "toy example" we have chosen the numbers such that $\mathbf{E}[S_1] = 1.25 > 1 = \mathbf{E}[B_1]$, so that in the average the stock performs better than the bond.

1.3 Pricing by No Arbitrage

A different approach to the pricing of the option goes like this: we can buy at time $t = 0$ a *portfolio* consisting of $\frac{2}{3}$ of stock and $-\frac{1}{3}$ of bond. The reader might be puzzled about the negative sign: investing a negative amount into a bond — "going short" in the financial lingo — means to borrow money.

One verifies that the value Π_1 of the portfolio at time $t = 1$ equals 1 or 0 in dependence of whether ω equals g or b. The portfolio "replicates" the option, i.e.,

$$C_1 \equiv \Pi_1. \tag{6}$$

We are confident that the reader now sees why we have chosen the above weights $\frac{2}{3}$ and $-\frac{1}{3}$: the mathematical complexity of determining these weights such that (6) holds true, amounts to solving two linear equations in two variables.

The portfolio Π has a well-defined price at time $t = 0$, namely $\Pi_0 = \frac{2}{3}S_0 - \frac{1}{3}B_0 = \frac{1}{3}$. Now comes the "pricing by no arbitrage" argument: equality (6) implies that we also must have

$$C_0 = \Pi_0 \tag{7}$$

whence $C_0 = \frac{1}{3}$. Indeed, suppose that (7) does not hold true; to fix ideas, suppose we have $C_0 = \frac{1}{2}$ as above. This would allow an arbitrage by buying ("going long in") the portfolio Π and simultaneously selling ("going short in")the option C. The difference $C_0 - \Pi_0 = \frac{1}{6}$ remains as arbitrage profit at time $t = 0$, while at time $t = 1$ the two positions cancel out *independently of whether the random element ω equals g or b.*

1.4 Variations of the example

Although the preceding "toy example" is extremely simple and, of course, far from reality, it contains the heart of the matter: the possibility of replicating a contingent claim, e.g. an option, by trading on the existing assets and to apply the no arbitrage principle.

It is straightforward to generalize the example by passing from the time index set $\{0, 1\}$ to an arbitrary finite discrete time set $\{0, \ldots, T\}$ by considering T independent Bernoulli random variables. This binomial model is called the Cox-Ingersoll-Ross model in finance. It is not difficult — at least with the technology of stochastic calculus, which is available today — to pass to the (properly normalized) limit as T tends to infinity, thus ending up with a stochastic process driven by Brownian motion. The so-called geometric Brownian motion is the celebrated *Black-Scholes model* which was proposed in 1965 by P. Samuelson. In fact, already in 1900 L. Bachelier used Brownian motion to price options in his remarkable thesis "Théorie de la spéculation" (member of the jury and rapporteur: H. Poincaré).

In order to apply the above no arbitrage arguments to more complex models we still need one crucial concept.

1.5 Martingale Measures

To explain this notion let us turn back to our "toy example", where we have seen that the unique arbitrage free price of our option equals $C_0 = \frac{1}{3}$. We also have seen that, by taking expectations, we obtained $\mathbf{E}[C_1] = \frac{1}{2}$ as the price of the option, which allowed for arbitrage possibilities. The economic rationale for this discrepancy was that the expected return of the stock was higher than that of the bond.

Now make the following mind experiment: suppose that the world were governed by a different probability than $\mathbf{P}$ which assigns different weights to g and b, such that under this new probability, let's call it $\mathbf{Q}$, the expected return of the stock equals that of the bond. An elementary calculation reveals that the probability measure defined by $\mathbf{Q}[g] = \frac{1}{3}$ and $\mathbf{Q}[b] = \frac{2}{3}$ is the unique solution satisfying $\mathbf{E_Q}[S_1] = S_0 = 1$. Speaking mathematically, the process S is a *martingale* under $\mathbf{Q}$, and $\mathbf{Q}$ a *martingale measure* for S.

Speaking again economically, it is not unreasonable to expect that in a world governed by $\mathbf{Q}$, the recipe of taking expected values should indeed give a price for the option which is compatible with the no arbitrage principle. A direct calculation reveals that in our "toy example" this is indeed the case,

$$\mathbf{E_Q}[C_1] = \tfrac{1}{3}. \tag{8}$$

At this stage it is, of course, the reflex of every mathematician to ask: what is precisely going on behind this phenomenon?

1.6 The Fundamental Theorem of Asset Pricing

To make a long story short: for a general stochastic process $(S_t)_{0 \le t \le T}$, modelled on a filtered probability space $(\Omega, (\mathcal{F}_t)_{0 \le t \le T}, \mathbf{P})$, the following statement *essentially* holds true. For any "contingent claim" C_T, i.e. an $\mathcal{F}_T$-measurable random variable, the formula

$$C_0 := \mathbf{E_Q}[C_T] \tag{9}$$

yields precisely the arbitrage-free prices for C_T, when $\mathbf{Q}$ runs through the probability measures on $\mathcal{F}_T$, which are equivalent to $\mathbf{P}$ and under which the process S is a martingale (*"equivalent martingale measures"*). In particular, when there is precisely one equivalent martingale measure (as is the case in the Cox-Ingersoll-Ross, the Black-Scholes and the Bachelier model), (9) gives the unique arbitrage free price C_0 for C_T. In this case we may "replicate" the contingent claim C_T as

$$C_T = C_0 + \int_0^T H_t dS_t, \tag{10}$$

where $(H_t)_{0 \le t \le T}$ is a predictable process (a *"trading strategy"*) modelling the holding in the stock S during the infinitesimal interval $[t, t + dt]$.

Of course, the stochastic integral appearing in (10) needs some care; fortunately people like K. Itô and P.A. Meyer's school of probability in Strasbourg told us very precisely how to interpret such an integral.

The mathematical challenge of the above story consists in getting rid of the word "essentially" and to turn this program into precise theorems.

Here is the central piece of the theory relating the no arbitrage arguments with martingale theory.

Fundamental Theorem of Asset Pricing: For an $\mathbf{R}^d$-valued semi-martingale $S = (S_t)_{0 \le t \le T}$ t.f.a.e.:

(i) There exists a probability measure $\mathbf{Q}$ equivalent to $\mathbf{P}$ under which S is a *sigma-martingale.*

(ii) S does not permit a *free lunch with vanishing risk.*

This Theorem was proved for the case of a probability space Ω consisting only of finitely many points by Harrison and Pliska [HP 81]. In this case one may equivalently write *no arbitrage* instead of *no free lunch with vanishing risk* and *martingale* instead of *sigma-martingale.*

In the general case it is unavoidable to speak about more technical concepts, i.e., *sigma-martingales* (which is a generalisation of the notion of a local martingale) and *free lunches.* A *free lunch* (a notion introduced by D. Kreps [K 81]) is something like an arbitrage, where — roughly speaking — agents are allowed to form integrals as in (10), then to "throw away money", and finally to pass to the limit in an appropriate topology. It was the — somewhat surprising — insight of [DS 94] that one may take the topology of *uniform convergence* (to which the term "with vanishing risk" alludes) and still get a valid theorem above.

The Fundamental Theorem may also be viewed as describing a dichotomy on the fairness of games (if we interpret stochastic processes as games of chance). Either a process is utterly unfair, in which case it allows for something like an arbitrage (more precisely, a "free lunch with vanishing risk"). If we discard this extreme case of unfairness, then one may change the odds (but not the null sets!) by passing from $\mathbf{P}$ to $\mathbf{Q}$, such that under the new measure $\mathbf{Q}$ the process is perfectly fair (more precisely, a sigma-martingale).

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