Operators from L^1 to Banach Spaces and Subsets of L^∞

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Abstract: We investigate some weakenings of the notion of Radon-Nikodym property, such as the point of continuity property [B-R], the property of (strong) regularity [G-G-M] and the weak-star Radon-Nikodym property [T] for operators T from $L^1[0,1]$ to a Banach space X. We characterise these classes of operators in terms of properties of the set $M = T^*(\text{ball}\ (X^*))$ in $L^\infty[0,1]$ similar to Grothendieck's characterisation of Riesz-representable operators in terms of equimeasurable subsets M of $L^\infty[0,1]$. We also characterise strong regularity in terms of a kind of multidimensional dentability and show that a Banach space X is strongly regular iff every operator from $L^1[0,1]$ to X is strongly regular. Finally we show that regular operators from $L^1[0,1]$ to X have a Pettis-derivative with values in X^{**} thus extending a theorem of N. Ghoussoub, G. Godefroy and B. Maurey.

<u>Introduction:</u> Some eight years ago J. Bourgain introduced in the remarkable paper [B1] the notion of the "convex point-of-continuity property" (CPCP) (under the name property (*); in [R2] or [S6] one may find different presentations of the content of [B1]). This concept - a weakening of the Radon-Nikodym property - was the starting point of a series of investigations trying (sucessfully) to obtain a deeper understanding of the Radon-Nikodym property in terms of geometrical and topological notions by J. Bourgain and H.P. Rosenthal [B-R], G. Edgar and R. Wheeler [E-W] and the papers of N. Ghoussoub and B. Maurey (e.g. [G-M]).

J. Bourgain had observed in [B1] that a weakly relatively open subset of a bounded, convex set contains a convex combination of slices (lemma 1.2a below). This fact is fundamental for all the later development. N. Ghoussoub, G. Godefroy and B. Maurey [G-G-M] used this lemma to develope and elaborate the notions of strongly regular and regular sets in a Banach space.

Parallel to this development new results about Pettis-integrability and weak versions of the Radon-Nikodym property were achieved (see the comprehensive memoir of M. Talagrand [T] and the references and comments (p. 208 ff.) given there): In particular, D.H. Fremlin [F] introduced the notion of stable sets of functions and related it to Pettis-integrability (under certain set-theoretical assumptions).

Finally, these two directions of investigation were linked by N. Ghoussoub, G. Godefroy and B. Maurey [G-G-M] who proved - under some separability assumption - that a regular Banach space has the weak-star Radon-Nikodym property as introduced in [T].

Here lies the starting point for the present paper which builds mainly on [G-G-M] and [T]: We investigate systematically the notions which became important in these investigations in terms of operator T from $L^1[0,1]$ to X. We also characterise the properties of these operators in terms of the subsets $M = T^*(\text{ball }(X^*))$ of $L^\infty[0,1]$. The pattern for these results is the theorem of Grothendieck that T is representable by a Bochner-derivative iff M is equimeasurable (theorem 1.15 below). We get analogous results for all the notions investigated here (theorem 3.4 and 6.5 below).

In chapter 1 we give the definitions and a review of results concerning the RNP such as a detailed discussion of Grothendieck's theorem (1.15). We also obtain some general results about the classes of operators between arbitrary Banach spaces X and Y related to our notions; they will not be needed for the subsequent results and may be skipped at a first reading.

In chapter 2 we investigate the geometrical and topological structure of the positive face F of the unit ball of $L^1[0,1]$. The remarkable properties of F (a kind of converse to Bourgain's lemma 1.2a holds true for this set) will be basic for the subsequent results.

In chapter 3 we show that the notions of "point of continuity", "convex point of continuity" and "strong regularity" coincide for operators from $L^1[0,1]$ to X and may be characterised in terms of subsets of $L^\infty[0,1]$ of "small oscillation".

In chapter 4 we give a number of examples and remarks. In particular, example 4.6 will be the source of motivation for the results of chapter 5 and 7.

In chapter 5 we obtain one of the major results of the paper: A Banach space X is strongly regular iff every operator $T : L^1[0,1] \to X$ is strongly regular (theorem 5.2). For the proof we shall characterise strong regularity as a kind of "multi-dimensional dentability" (lemma 5.1).

In chapter 6 we characterise "regular operators" $T : L^1[0,1] \to X$ in terms of subsets of $L^\infty[0,1]$ of "regular oscillation". We also clarify that regularity corresponds to a kind of weak-star to weak continuity just in the same way as strong regularity corresponds to weak to norm continuity.

In chapter 7 we relate the theory developped in the previous chapters to Fremlin's notion of stable sets and to a condition due to J. Bourgain [B3]. In particular we reprove the theorem of N. Ghoussoub, G. Godefroy and B. Maurey that a regular Banach space has the weak-star Radon-Nikodym property (without separability assumptions).

This paper owes much to many: In particular I want to thank N. Ghoussoub for suggesting the notion of a regular operator and to F. Lust-Piquard, B. Maurey and G. Mokobodzki for conversations on these topics during a stay in Paris in November 1985. Thanks go also to J.B. Cooper, V. Losert, H.P. Rosenthal, Ch. Stegall and A. Wessel for stimulating discussions and suggestions.

We have not striven for maximal generality. For example we have restricted us throughout the paper to the space $L^1([0,1],m)$, m denoting Lebesgue-measure, as we feel this space to be the natural framework to consider the Radon-Nikodym property and related notions. However, all the proofs carry over to spaces $L^1(\mu)$ where μ is any (not purely atomic) probability measure.

There is a difference between our approach and that of [T]: M. Talagrand presents the results of [T] in terms of sets of measurable functions, i.e. subsets of $\mathcal{L}^{\infty}(\mu)$, while we consider here sets of equivalence classes of functions, i.e. subsets of $L^{\infty}(\mu)$. As usual, we shall not distinguish between a function and its equivalence class if there is no danger of confusion; for example we shall write, for $g \in L^\infty[0,1]$ and A a measurable subset of $[0,1]$, $m(A) > 0$,

$$\operatorname{osc}(g|A) \tag{1}$$

meaning the essential oscillation of f over the - strictly speaking - equivalence class of sets equalling A almost everywhere.

However there will be situations, in particular in chapter 7, where we have to distinguish rather pedantically between functions and their equivalence classes.

We shall write F for the positive face of $L^1[0,1]$, i.e.

$$F = \{f \in L^1[0,1] : f \geqq 0 \text{ and } \|f\| = 1\}$$

and for $A \subset [0,1]$, $m(A) > 0$

$$F_A = \{f \in L^1[0,1] : f \geqq 0 \text{ and } \|f\| = \|f\chi_A\| = 1\}.$$

With this notation one may give a more precise definition of (1), namely for $g \in L^\infty[0,1]$,

$$\operatorname{osc}(g|A) = \sup\{\langle f_1 - f_2, g\rangle : f_1, f_2 \in F_A\}.$$

All Banach spaces will be over the reals and all operators will be linear and continuous.

After writing up this paper we learned about the preprint of A. Wessel [W], where some results similar to those of chapter 3 are obtained and related to non-Dunford-Pettis-operators.

1. The Radon-Nikodym property and related notions

We recall some definitions related to the Radon-Nikodym property, which were introduced in the last years and seem important for a deeper understanding of the topic. The characterisation a) of the Radon-Nikodym property in terms of diameters of slices essentially goes back to M.A. Rieffel [R1]:

1.1. Definition:

a) A closed, convex, bounded subset C of a Banach space X has the "Radon-Nikodym property" (RNP) if for every subset $D \subset C$ and $\varepsilon > 0$ there is a slice S of D of diameter less than ε (see, e.g., [D-U] or [B4]).

Recall that a slice of D is a set of the form

$$S = S(x^*, \beta) = \{x \in D : \langle x, x^* \rangle > M_{x^*} - \beta\}$$

where $\beta > 0$, $x^* \in X^*$, $\|x^*\| = 1$ and

$$M_{x^*} = \sup \{\langle x, x^* \rangle : x \in D\}.$$

The diameter of a non-empty subset A of a Banach space X is defined as

$$\operatorname{diam}(A) = \sup \{\|x-y\| : x, y \in A\}.$$

b) (c.f. [B-R]) A weakly closed, bounded (not necessarily convex) subset $C \subset X$ is said to have the "Point-of-

Continuity-property" (PCP) if for every subset $D \subset C$ and $\varepsilon > 0$ there is a relatively weakly open subset $U \subset D$ with $\operatorname{diam}(U) < \varepsilon$.

c) (c.f. [B1], [E-W], [G-G-M] and [G-M-S]): A closed, bounded, convex subset $C \subset X$ is said to have the "convex-Point-of-Continuity-property" (CPCP) if for every convex subset $D \subset C$ and $\varepsilon > 0$ there is a relatively weakly open subset $U \subset D$ with $\operatorname{diam}(U) < \varepsilon$.

d) (c.f. [B1] implicitely and [G-G-M] explicitely): A closed, bounded, convex $C \subset X$ is said to be strongly regular (SR) if for every $D \subset C$ and $\varepsilon > 0$ there are slices $S_1,\ldots,S_n$ of D s.t.

$$\operatorname{diam}(n^{-1}(S_1+\ldots+S_n)) < \varepsilon .$$

H.P. Rosenthal [R2] proposed the name "small-combinations-of-slices-property" (SCSP) for this concept.

e) (c.f. [G-G-M]): A closed, convex, bounded $C \subset X$ is called "regular" (R) if for $\varepsilon > 0$ and $x^{***} \in X^{***}$ there is a slice S of C such that for $\widetilde{S}$, the weak-star-closure of S in X^{**}, we have

$$\operatorname{osc}(x^{***} | \widetilde{S}) < \varepsilon$$

where osc denotes the oscillation of x^{***} on $\widetilde{S}$, i.e.

$$\operatorname{osc}(x^{***},\widetilde{S}) = \sup\{|\langle x^{***},x_1^{**}\rangle - \langle x^{***},x_2^{**}\rangle| : x_1^{**},x_2^{**} \in \widetilde{S}\}.$$

f) (c.f. [T] 7-1-7): A closed, bounded, convex $C \subset X$ has the "weak-star-Radon-Nikodym-property" (W^*-RNP) if for every operator

$$T : L^1[0,1] \to X$$

such that $T(F) \subset C$ there is a Pettis-integrable (with respect to X^{***}(!)) function $F : [0,1] \to X^{**}$ representing T via the formula

$$\langle Tf, x^{***} \rangle = \int_{[0,1]} f(t) . \langle F(t), x^{***} \rangle dt$$

for $f \in L^1[0,1]$ and $x^{***} \in X^{***}$. F denotes the positive face of the unit-ball of $L^1[0,1]$, i.e.

$$F = \{f \in L^1[0,1] : f \geq 0 \text{ and } \|f\|_1 = 1\}.$$

g) (c.f. [T] 7-2-3): A closed, convex, bounded $C \subset X$ has the "compact-range-property" (CRP) if for every operator

$$T : L^1[0,1] \to X$$

such that $T(F) \subset C$, the operator T is Dunford-Pettis, i.e. $T\{f : \|f\|_\infty \leq 1\}$ is relatively compact in X.

1.2 Remark: This was a long list! But it seems unavoidable to deal with these concepts if one wants to penetrate deeper into the problems related with the concept of (RNP).

We have given the "local" definitions. A Banach space X

has the properties (a) - (g) iff the unit-ball of X has the corresponding property.

There is an increasing order of generality of the above concepts:

(a) versus (b): It was shown in [B-R] that (RNP) $\Rightarrow$ (PCP) while (PCP) $\not\Rightarrow$ (RNP).

(b) versus (c): (PCP) $\Rightarrow$ (CPCP) is obvious while (CPCP) $\not\Rightarrow$ (PCP) was shown in [G-M-S].

(c) versus (d): That (CPCP) $\Rightarrow$ (SR) was shown in [B1] and this crucial observation was the starting point for most of the investigations we are dealing here with. Because of its importance we shall state this lemma explicitely below (1.2a). Whether (SR) $\Rightarrow$ (CPCP) holds true seems to be open.

(d) versus (e): That (SR) $\Rightarrow$ (R) was observed in [G-G-M] while the validity of the converse is open too.

(e) versus (f): It was proved - under some separability assumption - in [G-G-M] that (R) $\Rightarrow$ (W^*RNP), thus relating the geometric concepts (a) - (e) to the work of Talagrand [T] on the Pettis integral (see also theorem 7.9 below).

(f) versus (g): The memoir [T] contains a wealth of information on (W^*RNP), (CRP) and related notions in particular, it is shown that (W^*RNP) $\Rightarrow$ (CRP) while the converse does not hold true.

1.2a Lemma ([B1], for a proof see [G-G-M], lemma II.1): Let C be a convex, bounded subset of X and U a relatively weakly open subset of C. Then there are slices $(S_1, \ldots, S_n)$ in C such that

$$n^{-1} \sum_{i=1}^{n} S_i \subset U.$$

1.3 Extension of the above properties to operators: There is an obvious extension of the above notions to operators; it is chosen in such a way that a Banach space X has the respective property iff the identity operator has this property:

1.4 Definition: (A) A (continuous, linear) operator T from a Banach space X to a Banach space Y is called an (RNP)-operator if for every closed, convex, bounded $C \subset X$ and $\varepsilon > 0$ there is a slice S of C such that

$$\operatorname{diam}(T(S)) < \varepsilon.$$

In an obvious way we define the notions (B) to (G) corresponding for (b) to (g) above. For illustration we do this for the notion of regularity:

(E): An operator $T : X \to Y$ is called regular ($\underline{R}$) if for every closed, convex, bounded $C \subset X$ and $\varepsilon > 0$ and $y^{***} \in Y^{***}$ there is a slice S of C such that

$$\operatorname{osc}\left(y^{***} \mid \overline{T(S)}^{\sigma(Y^{**}, Y^{*})}\right) < \varepsilon.$$

<u>1.5 Remark:</u> This generalisation of the above properties of spaces (or sets) to properties of operators is purely formal and there is little interesting that we can tell in this general setting.

Note, however, that there is no hope for any factorisation result of the above properties: It has been shown in [G-J] that there is an (RNP)-operator T from a Banach lattice X to c such that every Banach space Y, through which there is a factorisation of T contains c_o. Hence Y fails (CRP), the most general of the conditions (a) - (h) above.

Also note that the above properties define operator ideals (c.f. [P]). In particular if $T : X \to Y$ and either X or Y has one of the properties (a) - (g) above then T has the corresponding property too.

The definition of (RNP)-operators is usually given in terms of representation of operators from $L^1[0,1]$ (c.f. [P], 24.2). The subsequent proposition shows that our definition (A) is equivalent.

<u>1.6 Proposition:</u> $T : X \to Y$ is an (RNP)-operator (as defined above) iff for every continuous $R : L^1[0,1] \to X$ the operator RT is representable by a Bochner-integrable function (c.f. [D-U], III.2).

Proof: If RT is not representable, a routine exhaustion argument implies (see, e.g. [S4]) that there is a subset $A \subset [0,1]$ of positive measure and $\alpha > 0$ such that for every slice S of F_A we have

$$\operatorname{diam} (RT(S)) > \alpha.$$

Hence $C = \overline{T(F_A)}$ satisfies the assumption of definition (A) above .

For the converse suppose there is $C \subset X$ such that for each slice S of C $\operatorname{diam} (T(S)) > 2\alpha$. We have to construct an operator $R : L^1[0,1] \to X$, which is done by constructing a bush; this construction is routine by now (c.f. [D-U]) hence we don't give the details but only the crucial lemma (compare also the construction of proposition 1.11 and theorem 5.2 below):

1.7 Lemma: Let $T : X \to Y$ be an operator, C closed, convex, bounded in X and $\alpha > 0$ such that for every slice S of C,

$$\operatorname{diam} (T(S)) > 2\alpha.$$

Given $x_o \in C$ and $\varepsilon > 0$ there are $x_1,\dots,x_n$ such that

$$\|x_o - n^{-1}(x_1+\dots+x_n)\|_X < \varepsilon$$

and

$$\|T(x_o - x_i)\|_Y \geq \alpha, \quad i=1,\dots,n.$$

Proof: Let

$$U = \{x \in C : \|T(x_o - x)\| < \alpha\}$$

and let

$$D = \overline{co}\ (C \setminus U).$$

If $x_o \in D$ then we clearly may find $x_1, \ldots, x_n$ in $C \setminus U$ as in the assertion of the lemma and we are finished.

If $x_o \notin D$ then by the separation theorem there is a slice S of C, determined by some $x^* \in X^*$, such that $S \subset U$. However this would imply that

$$\operatorname{diam}\ (T(S)) \leqq 2\alpha,$$

a contradiction proving the lemma.

□

1.8 Remark: A word of warning seems necessary. If $T : X \to Y$ is a (PCP)- (resp. (CPCP)-) operator, C a closed, bounded (resp. convex) subset of X and $\varepsilon > 0$ then

$$C_\varepsilon = \{x \in C : \exists \text{ a weak neighbourhood } U \text{ of } x \text{ s.t. } \operatorname{diam} T(U \cap C) < \varepsilon\}$$

is a weakly open and weakly dense subset of C.

However, in this general setting we do not have any argument to ensure that

$$\bigcap_{\varepsilon>0} C_\varepsilon \neq \emptyset,$$

i.e. that there are "points of continuity" $x \in C$ such that T is weak-to-norm continuous at x. We do not dispose of an example where the above intersection is empty, but there seems no reason, why this should not happen in general (see however theorem 3.4 below for the case, where the domain space is $L^1[0,1]$). Nevertheless we believe, that the definition 1.4 above is the proper generalisation of the concepts of (PCP) and (CPCP) to operators, although the names might be somewhat misleading.

For the case of (SR) however we do have a positive result even in this general setting.

<u>1.9 Definition</u> ([R2], remark after 3.1): Let C be a closed, convex, bounded subset of X and $T : X \to Y$. Following H.P. Rosenthal we call $x \in C$ an <u>SCS-point under T</u> if, for $\varepsilon > 0$ there is a convex combination of slices of C

$$W = n^{-1}(S_1+\ldots+S_n)$$

such that $x \in W$ and $\operatorname{diam}(T(W)) < \varepsilon$.

<u>1.10 Remark:</u> The above notion is slightly stronger than the related notion of "strongly regular points" ([G-G-M],III.1), where one only requires that x is in the closure of W.

The following result is a variation of a refinement, due to R.P. Rosenthal, of a result of N. Ghoussoub, G. Godefroy and B. Maurey.

1.11 Proposition: Let $T : X \to Y$ be a strongly regular operator and C a closed, convex, bounded subset of X. The set of SCS-points of C under T is a convex, $\|.\|$-dense subset of C.

For the proof we need a lemma:

1.12 Lemma: Under the assumption of 1.11 let $x_o \in C$ and $\varepsilon > 0$. There are $x_1,\dots,x_n \in C$ and relative weak neighbourhoods $U_1,\dots,U_n$ in C such that

(i) $\|x_o - n^{-1}(x_1+\dots+x_n)\|_X < \varepsilon$

(ii) $\operatorname{diam}_{\|.\|_Y} (T(n^{-1}(U_1+\dots+U_n)) < \varepsilon$.

Proof: We shall show that we may choose the U_i above even as slices of C.

Let C_ε be the $x \in C$ such that there is a convex combination $W = n^{-1}(S_1+\dots+S_n)$ of slices of C such that $x \in W$ and $\operatorname{diam}(T(W)) < \varepsilon$. It follows easily from the assumption that T is an (SR) operator that C_ε is weakly dense in C (see, e.g. the arguments given in [G-G-M]). It is clear that C_ε is convex, hence it is norm-dense in C. The lemma is proved.

□

Proof of proposition 1.11: (the argument is a variation of [R2], remark 3.8.2): Let $x_o \in C$ and $\delta > 0$. We have to show that there is an SCS-point ξ_o of C under T such that $\|x_o-\xi_o\| < \delta$. We shall do this by constructing an "approximate bush" starting at x_o; the origin of the associated "exact bush" will then be the desired ξ_o.

We proceed by induction on n: For $n = 1$ apply lemma 1.12 to find k_1 and $x_1,\dots,x_{k_1}$ and relative weak neighbourhoods $U_1,\dots,U_{k_1}$ such that

(i) $\|x_o-k_1^{-1}(x_1+\dots+x_{k_1})\| < \delta/2$

(ii) $\operatorname{diam}(T(k_1^{-1}(U_1+\dots+U_{k_1}))) < 1/2$.

For $n = 2$ find $\delta_1 > 0$ such that, for $i_1 = 1,\dots,k_1$,

$$\bar{B}(x_{i_1},\delta_1) \cap C \subset U_{i_1},$$

where $\bar{B}(x_{i_1},\delta_1)$ denotes the closed ball of radius δ_1 around x_{i_1}.

Let $\eta_1 = \min(\delta_1/2,\delta/4)$ and apply 1.12 to find $k_2 \in \mathbb{N}$ and $x_{i_1,i_2} \in C$, $1 \leqq i_1 \leqq k_1$, $1 \leqq i_2 \leqq k_2$ and relative weak neighbourhoods U_{i_1,i_2} such that

(i) $\|x_{i_1}-k_2^{-1}\sum_{i_2=1}^{k_2} x_{i_1,i_2}\| < \eta_1$, $\quad 1 \leqq i_1 \leqq k_1$

and

(ii) $\operatorname{diam}(T((k_1k_2)^{-1}\sum_{i_1=1}^{k_1}\sum_{i_2=1}^{k_2} U_{i_1,i_2})) < 1/2$

For the general step suppose $k_1,\dots,k_n$, $x_{i_1},\dots,x_n$, $S_{i_1,\dots,i_n}$ and $\delta_1,\dots,\delta_{n-1}$ constructed. Find $\delta_n > 0$ such that

$$\bar{B}(x_{i_1,\dots,i_n},\delta_n) \cap C \subset U_{i_1,\dots,i_n}, \quad \begin{array}{l} i \leqq i_1 \leqq k_1,\dots \\ \dots,1 \leqq i_n \leqq k_n \end{array}$$

and let $\eta_n = \min(\delta_n/2, \delta_{n-1}/4, \dots, \delta/2^{n+1})$ and apply 1.12 to find $k_{n+1} \in \mathbb{N}$ and $x_{i_1,\dots,i_{n+1}} \in C$, for $1 \leqq i_1 \leqq k_1$, $\dots, 1 \leqq i_{n+1} \leqq k_{n+1}$ and relative weak neighbourhoods $U_{i_1,\dots,i_{n+1}}$ such that

(i) $$\|x_{i_1,\dots,i_n} - k_{n+1} \sum_{i_{n+1}=1}^{k_{n+1}} x_{i_1,\dots,i_{n+1}}\| < \eta_n$$

for $1 \leqq i_1 \leqq k_1, \dots, 1 \leqq i_n \leqq k_n$

(ii) $$\operatorname{diam}\left(T\left((k_1\dots k_{n+1})^{-1} \sum_{i_1=1}^{k_1} \dots \sum_{i_{n+1}=1}^{k_{n+1}} U_{i_1,\dots,i_{n+1}}\right)\right) < 1/n.$$

This finishes the inductive construction of the "approximate bush". To get the "exact bush" let

$$\xi_o = \lim_{n\to\infty} (k_1,\dots,k_n)^{-1} \sum_{i_1=1}^{k_1} \dots \sum_{i_n=1}^{k_n} x_{i_1,\dots,i_n}$$

and for $1 \leqq i_1 \leqq k_1, \dots, 1 \leqq i_p \leqq k_p$

$$\xi_{i_1,\dots,i_p} = \lim_{n\to\infty} (k_{p+1}\dots k_n)^{-1} \sum_{i_{p+1}=1}^{k_{p+1}} \dots \sum_{i_n=1}^{k_n} x_{i_1,\dots,i_n}.$$

It follows from the construction that $\xi_{i_1,\dots,i_p} \in U_{i_1,\dots,i_p}$, hence ξ_o is an SCS-point of C under T. The observation, that $\|x_o - \xi_o\| < \delta$ finishes the proof.

□

1.13 Operators from $L^1[0,1]$ to Banach spaces X: After these few general results about operators between Banach spaces X and Y related to the properties studied here, we now focus our attention to operators from $L^1[0,1]$ to X. For these operators we can prove much better results than in the above general setting.

Note that an operator $T : L^1[0,1] \to X$ is (CRP) according to our definition 1.2 iff T is Dunford-Pettis and in this case we shall of course use the classical name.

We first deal with the characterisation of (RNP)-operators in terms of equimeasurable sets. This fundamental idea is due to A. Grothendieck [G]. Theorem 1.15 below is essentially known (in fact: essentially due to A.Grothendieck), although the characterisation 1.15 (ii) and its proof do not seem to have been noticed before. We present this theorem in some detail as it will be the pattern for our further investigations.

1.14 Definition [G]: A bounded subset M of $L^\infty[0,1]$ is called equimeasurable if, for $\varepsilon > 0$, there is $A \subset [0,1]$, $m(A) > 1-\varepsilon$ such that the restriction

$$M|_A = \{g.\chi_A : g \in M\}$$

is relatively $\|.\|_\infty$-compact.

1.15 Theorem: Let $T : L^1[0,1] \to X$ be a (continuous, linear) operator. T.f.a.e.

(i) $T^*(\text{ball } (X^*))$ is equimeasurable;

(ii) T is an (RNP)-operator (in the sense of definition 1.2);

(iii) T is representable, i.e. there is $F : [0,1] \to X$ such that, for $f \in L^1[0,1]$

$$T(f) = \text{Bochner} - \int_{[0,1]} F(t).f(t)\,dm$$

(iv) For $A \subset [0,1]$, $m(A) > 0$ and $\varepsilon > 0$ there is a slice S of F_A such that $\text{diam}\,(T(S)) < \varepsilon$.

Proof:

(i) $\Leftrightarrow$ (iii) is the wellknown equivalence due to Grothendieck ([G], see also [S4]).

(i) $\Rightarrow$ (ii): We give a geometric proof using the classical Krein-Milman theorem and the following - also classical - fact: If K is convex, compact in a locally convex space E and x_o is extreme in K then the slices containing x_o form a relative neighbourhood base of x_o in K. Suppose first $T^*(\text{ball } (X^*))$ is compact in $L^\infty[0,1]$, i.e. T is a compact operator. For $\varepsilon > 0$ and a convex, bounded (say $\|C\| = \sup\{\|f\| : f \in C\} = 1$) subset $C \subset X$ denote $\widetilde{C}$ the closure of C in $(L^1[0,1]^{**}, \sigma(L^1[0,1]^{**}, L^\infty[0,1]))$. By Krein-Milman there is an extreme point $x_o^{**} \in \widetilde{C}$ with

$\|x_o^{**}\| > 1-\varepsilon$. As $T^{**} : L^1[0,1]^{**} \to X$ is weak-star to norm continuous by the above remark there is a weak-star slice $\tilde{S}$ of $\tilde{C}$ such that

$$\text{diam } (T^{**}(S)) < \varepsilon.$$

As $\{x^{**} \in \tilde{C} : \|x^{**}\| > 1-\varepsilon\}$ is a relative weak-star neighbourhood of $x_o^{**} \in \tilde{C}$ we may assume that for $x^{**} \in \tilde{S}$ we have $\|x^{**}\| > 1-\varepsilon$. The slice S of C given by $S = \tilde{S} \cap C$ satisfies $\text{diam } (T(S)) < \varepsilon$ thus proving (ii) in the compact case.

Now assume only that $M = T^*(\text{ball } (X^*))$ is equimeasurable and let $(A_n)_{n=1}^{\infty}$ be subsets of $[0,1]$, $m(A_n) \to 1$ such that M restricted to A_n is compact in $L^{\infty}[0,1]$. Given C as above denote C_n the restriction of C to A_n and note that

$$\lim_{n\to\infty} \|C_n\| = \lim_{n\to\infty} \{ \|f : \chi_{A_n}\|_1 : f \in C\} = 1.$$

Let n_o be such that $\|C_{n_o}\| > 1-\varepsilon$. By the first part of the proof, applied to

$$T|_{A_{n_o}} : L^1(A_{n_o}, m|_{A_{n_o}}) \to X$$

we find a slice $S = S(g,\beta)$ of C_{n_o}, with g in the unit sphere of $L^{\infty}(A_{n_o})$, such that

$$\text{diam } (T|_{A_{n_o}}(S)) < \varepsilon$$

and

$$f \in S \Rightarrow \|f\|_1 > 1-\varepsilon.$$

Considering g as an element of $L^\infty([0,1])$ - taking the value 0 outside A_{n_o} - let

$$S' = S'(g,\beta) = \{f \in C : \langle f,g\rangle > M_g - \beta\}$$

be the slice corresponding to S but taken in C. Clearly

$$S' \subset S + \varepsilon \,.\, \text{ball}\,(L^1[0,1])$$

hence

$$\text{diam}\,(T(S')) < \varepsilon + 2\varepsilon\|T\|$$

which proves (ii).

(ii) $\Rightarrow$ (iv): obvious.

(iv) $\Rightarrow$ (i): We shall make use of the propositions 2.2 and 2.2a below.

Let $\varepsilon > 0$ and $n \in \mathbb{N}$ and apply 2.2, 2.2a and an exhaustion argument to find disjoint compact subsets $K_1^n,\dots,K_{m_n}^n$ such that, for $g \in M$ and $1 \leqq j \leqq m_n$,

$$\text{osc}\,(g\,|\,K_{m_n}^n) < n^{-1}$$

and

$$m\Big(\sum_{j=1}^{m_n} K_j^n\Big) > (1 - \varepsilon/2^n).$$

We may assume that K_j^n is the support of the restriction of m to K_j^n.

Let

$$K = \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{m_n} K_j^n,$$

which is a compact set of measure greater than $1-\varepsilon$. We shall show that M restricted to K is relatively $\|.\|_\infty$-compact. For $n \in \mathbb{N}$, let

$$\delta_n = \min \{\text{dist } (K_i^n, K_j^n) : 1 \leqq i \neq j \leqq k_n\}.$$

Note that for $t_1, t_2 \in K$, $|t_1-t_2| < \delta_n$, t_1 and t_2 lie in the same K_j^n hence, for $g \in M$,

$$|g(t_1)-g(t_2)| < n^{-1}.$$

Here we have identified the restriction of the equivalence class g to K with its (unique) continuous representant on K. A glance at the Ascoli-Arzela theorem proves (i) and therefore 1.15.

□

<u>1.16 Remark:</u> I could not resist to give the "topological" argument for (iv) ⇒ (i) above. This was meant as a little "hommage à Grothendieck" and the french style of measure theory. However, as the alert reader has noticed, the proof may just as well be given in the abstract setting without using compactness. In particular the above result has nothing to do with the special topological structure of the measure space ([0,1],m).

2. The geometrical and topological structure of the positive face of the unit ball of $L^1[0,1]$

We shall see in the next chapter that most of the properties of operators from $L^1[0,1]$ to X we are here dealing with, find their explanation in the geometrical properties of the positive face F of L^1 (e.g., the fact that the properties (PCP), (CPCP) and (SR) are equivalent for these operators). Hence we assemble in this section the necessary technical tools.

2.1 Notation: Recall that F denotes the positive face of the unit ball of $L^1[0,1]$ i.e.

$$F = \{f \in L^1[0,1] : f \geqq 0 \text{ and } \|f\| = 1\}.$$

Given $A \subseteq [0,1]$, $m(A) > 0$ denote

$$F_A = \{f \in L^1[0,1] : f \geqq 0 \text{ and } \|f\| = \|f.\chi_A\| = 1\},$$

which is of course a closed, convex, bounded subset of $L^1[0,1]$.

If $B \subset A$, $m(B) > 0$ and $\beta > 0$ denote

$$S_{B,\beta} = S_{B,\beta}(F_A) = \{f \in F_A : \|f.\chi_B\| > 1-\beta\}.$$

Note that $S_{B,\beta}$ is a slice (determined by $\chi_B \in L^\infty[0,1]$) of F_A and the family of these slices forms a fundamental system of the slices of F_A:

<u>2.2 Proposition:</u> Let S be a slice of F_A. Then there is a compact $K \subset A$, $\mu(K) > 0$ and $\beta > 0$ such that

$$\overline{S_{K,\beta}} \subset S.$$

<u>Proof:</u> Suppose S is of the form

$$S = S_{g,\gamma} = \{f \in F_A : \langle g,f \rangle > M_g - \gamma\}$$

where $g \in L^\infty[0,1]$, $\|g\|_\infty = 1$, and $1 > \gamma > 0$. Note that

$$M_g = \sup \{\langle g,f \rangle : f \in F_A\}$$

is just the essential supremum of g on A.

We may find a compact $K \subset A$, $m(K) > 0$ such that the essential infimum of g over K is bigger than $M_g - \gamma/3$. Let $\beta = \gamma/3$ and note that

$$\overline{S_{K,\beta}} \subset S_{g,\gamma}$$

Indeed, for $f \in \overline{S_{K,\beta}}$ we have

$$|\langle f,g\rangle| \geq |\int_K f(t).g(t)dt| - |\int_{A\setminus K} f(t)g(t)dt|$$

$$\geq (M_g - \gamma/3).(1-\gamma/3) - \gamma/3$$

$$\geq M_g - 3\gamma/3 + \gamma^2/9 > M_g - \gamma.$$

□

The next easy proposition will be the pattern for the crucial proposition 2.7:

<u>2.2a Proposition:</u> Let $T : L^1[0,1] \to X$ be an operator, $A \subset [0,1]$, $m(A) > 0$, $\beta > 0$ and let S be the slice $S_{A,\beta}$ taken in F. Denoting $M = T^*(\text{ball } X^*)$ we have

$$\sup\{\text{osc}(g|A) : g \in M\} \leq \text{diam } T(S_{A,\beta}) \leq$$

$$\leq \sup\{\text{osc}(g|A) : g \in M\} + 4\beta\|T\|.$$

<u>Proof:</u> We have (by definition)

$$\sup\{\text{osc}(g|A) : g \in M\} = \sup\{\langle T^*x^*, f_1 - f_2\rangle :$$

$$: f_1, f_2 \in F_A, \|x^*\| \leq 1\}$$

$$= \text{diam}(T(F_A)).$$

The inequalities now follow from

$$F_A \subset S_{A,\beta} \subset F_A + 2\beta(\text{ball }(L^1[0,1])).$$ □

2.3 Notation: We now turn our attention from slices to general weak neighbourhoods.

First, let C be any closed, convex, bounded subset of $L^1[0,1]$, $P = (A_1,\dots,A_n)$ a partition of $[0,1]$ into sets of strictly positive measure and $\varepsilon \geqq 0$. Define, for $f \in C$,

$$U_{P,\varepsilon}(f) = \{g \in C : \sum_{i=1}^{n} |\int_{A_i} (f-g)\,dm| \leqq \varepsilon\}$$

and

$$U^o_{P,\varepsilon}(f) = \{g \in C : \sum_{i=1}^{n} |\int_{A_i} (f-g)\,dm| < \varepsilon\}.$$

Clearly, for $\varepsilon > 0$, the above sets are relative weak neighbourhoods of f in C and, for $\varepsilon = 0$, the first set is a closed, convex subset of C containing f. It is an exercise left to the reader to verify that $U_{P,\varepsilon}(f)$ forms a relative weak neighbourhood base of f in C, when P runs through the finite partitions of $[0,1]$ into sets of positive measure and ε runs through $]0,1]$.

If the closed, convex, bounded set C is just F we shall write $V_{P,\varepsilon}(f)$ (resp. $V^o_{P,\varepsilon}(f)$) for $U_{P,\varepsilon}(f)$ (resp. $U^o_{P,\varepsilon}(f)$).

Note that $V^o_{P,\varepsilon}(f)$ is a typical example of a convex combination of slices, namely

$$V^o_{P,\varepsilon}(f) = \sum_{i=1}^{n} (\int_{A_i} f\,dm)\, S_{A_i,\varepsilon/2}.$$

It is instructive to verify directly this formula - which will be a special case of proposition 2.5 below.

<u>2.4 Proposition:</u> For $f \in F$ and $\varepsilon \geqq 0$ the following formula holds true

$$V_{P,\varepsilon}(f) = [V_{P,0}(f) + \varepsilon.\text{ball}\,(L^1[0,1])] \cap F \qquad (1)$$

<u>Proof:</u> If $g + \varepsilon h$ belongs to the right hand side, where $g \in F$, $h \in \text{ball}\,(L^1[0,1])$ then clearly

$$\sum_{i=1}^{n} \left|\int_{A_i} (f-(g+\varepsilon h)dm\right| = \sum_{i=1}^{n} \left|\int_{A_i} \varepsilon h dm\right| \leqq \varepsilon.$$

Conversely, given $g \in V_{P,\varepsilon}(f)$, define

$$g_1 = \sum_{i=1}^{n} g.\chi_{A_i} \left(\int_{A_i} fdm / \int_{A_i} gdm\right).$$

Clearly, $g_1 \in V_{P,0}(f)$ and

$$\|g-g_1\|_1 = \sum_{i=1}^{n} \int_{A_i} \left|g.\chi_{A_i} \left(\int_{A_i} fdm / \int_{A_i} gdm - 1\right)\right| d\mu =$$

$$= \sum_{i=1}^{n} \left|\int_{A_i} fdm - \int_{A_i} gdm\right| \leqq \varepsilon.$$

□

<u>2.5 Proposition (compare [S2], lemma 2.5):</u> For $f_1, f_2 \in F$ and $\lambda \in]0,1[$ the following formula holds true

$$\lambda V_{P,\varepsilon}(f_1) + (1-\lambda) V_{P,\varepsilon}(f_2) = V_{P,\varepsilon}(\lambda f_1 + (1-\lambda) f_2). \qquad (2)$$

<u>Proof:</u> Suppose first $\varepsilon = 0$. An element $g_1 \in F$ belongs to $V_{P,0}(f_1)$ iff

$$\int_{A_i} (f_1 - g_1)\,dm = 0. \qquad i = 1,\dots,n$$

From this remark it is obvious that an element $\lambda g_1 + (1-\lambda)g_2$ of the left hand side of (2) belongs to the right hand side too.

If g belongs to the right hand side of (2) then

$$g_1 = \lambda^{-1} \sum_{i=1}^{n} (g.\chi_{A_i}).(\int_{A_i} f_1 dm / \int_{A_i} g dm)$$

and

$$g_2 = (1-\lambda)^{-1} \sum_{i=1}^{n} (g.\chi_{A_i}).(\int_{A_i} f_2 dm / \int_{A_i} g dm)$$

furnishes the desired decomposition $g = \lambda g_1 + (1-\lambda) g_2$.

The case $\varepsilon > 0$ now follows from 2.4.

□

<u>2.6 Remark:</u> Formula (2) shows a remarkable geometrical property of F (which also holds true for each F_A): A convex combination of a finite number of relatively weakly open subsets (in particular slices) of F is still weakly open. Hence a converse of Bourgain's lemma (1.2a above) holds true for F. This is the reason behind the fact that the notions of (PCP), (CPCP) and (SR) operators coincide for operators from L^1 to X as we shall see in theorem 3.4 below.

Note that in general a convex combination of slices need not be relatively weakly open, as is shown by the following very easy example: Let C be the unit ball of ℓ^2

$$S_1 = \{x \in \ell^2 : (x,e_1) > \sqrt{3/4}\}$$

and

$$S_2 = \{x \in \ell^2 : (x,-e_1) > \sqrt{3/4}\}.$$

Clearly for every $x \in 1/2(S_1+S_2)$ we have $\|x\| \leq \sqrt{2}/2$ (this is a very rough estimate).

As any relatively weakly open subset U of C contains an element of norm 1 we conclude that the relative weak interior of $1/2(S_1+S_2)$ in C is empty.

We can now prove the crucial result of this section:

<u>2.7 Proposition:</u> Let $T : L^1[0,1] \to X$ be an operator, $P = (A_1,\ldots,A_n)$ a partition of $[0,1]$ into sets of positive measure and $f \in F$. Then

$$\operatorname{diam}\,(T(V_{P,0}(f))) = \sup_{x^*\in \mathrm{ball}(X^*)} \{\sum_{i=1}^{n} [(\int_{A_i} f dm).\mathrm{osc}(T^*x^*|A_i)]\}$$

where diam denotes the diameter with respect to the norm of X.

Hence, for $\varepsilon > 0$,

$$\sup_{\|x^*\|\leq 1} \left\{ \sum_{i=1}^{n} \left[\left(\int_{A_i} f dm\right) . \operatorname{osc}(T^* x^* | A_i) \right] \right\} \leq \operatorname{diam}\, (T(V_{P,\varepsilon}(f)))$$

$$\leq \sup_{\|x^*\|\leq 1} \left\{ \sum_{i=1}^{n} \left[\left(\int_{A_i} f dm\right) . \operatorname{osc}(T^* x^* | A_i) \right] \right\} + 2\varepsilon . \|T\|.$$

Proof: Clearly

$$\operatorname{diam}\, (T(V_{P,O}(f))) = \sup \{ \|T(g_1 - g_2)\| : g_1, g_2 \in V_{P,O}(f) \}$$

$$= \sup_{\|x^*\|\leq 1} \{ | \langle g_1 - g_2, T^* x^* \rangle | : g_1, g_2 \in V_{P,O}(f) \}.$$

Hence the first formula follows immediately from the observation, that for any set A with $m(A) > O$ and $f \in L^{\infty}(m)$

$$\operatorname{osc}\, (f | A) = \sup \{ \langle g_1 - g_2, f \rangle : g_1, g_2 \in F_A \}.$$

Finally, the last inequality follows immediately from proposition 2.4.

□

3. Sets of small oscillation and strongly regular operators

3.1 Definition: A subset M of $L^\infty[0,1]$ is a set of small oscialtion (SO) if, for $\varepsilon > 0$, there is a finite partition $P = (A_1,\dots,A_n)$ of $[0,1]$ into sets of positive measure s.t., for $g \in M$,

$$\sum_{i=1}^{n} m(A_i).\mathrm{osc}\,(g|A_i) < \varepsilon.$$

It is an easy and instructive exercise to verify that an equimeasurable set is (SO). Theorem 3.4 below shows that (PCP)-, (CPCP)- and (SR)-operators correspond to (SO)-sets in the same way as (RNP)-operators correspond to equimeasurable sets.

3.2 Proposition: A bounded subset M of $L^\infty[0,1]$ is (SO) iff for $A \subset [0,1]$, $m(A) > 0$ there is $B \subset A$, $m(B) > 0$ such that M restricted to B is (SO).

In fact, it suffices to require that, for $\varepsilon > 0$ and $A \subset [0,1]$, $m(A) > 0$, there is $f \in L^1[0,1]$, $\chi_A \geqq f \geqq 0$, and a partition $P = (A_1,\dots,A_n)$ of $[0,1]$ such that

$$\sup\Big\{\sum_{i=1}^{n}\int_{A_i} f dm.\mathrm{osc}\,(g|A_i) : g \in M\Big\} \leqq \varepsilon.\|f\|_1. \qquad (1)$$

Proof: The condition is obviously necessary. In order to show that the above condition implies that M is (SO) we may assume $\|M\|_\infty \leqq 1$. Fix $\varepsilon > 0$.

Assume there are $f^1, f^2 \in L^1[0,1]$ and P^1 and P^2 partitions of $[0,1]$ such that for both (f^1,P^1) and (f^2,P^2) formula (1) holds true. Let P be the partition $[0,1]$ generated by P^1 and P^2 and $f = f^1+f^2$; then (f,P) satisfy inequality (1) too.

Let $(f^j,P^j)_{j=1}^{\infty}$ be a family of pairs as above such that (f^j,P^j) satisfy (1) and maximal with respect to the condition

$$\sum_{j=1}^{\infty} f^j(t) \leqq 1 \qquad \text{for } t \in [0,1].$$

It follows easily from the assumption that

$$\sum_{j=1}^{\infty} f^j(t) = 1 \qquad \text{for a.e. } t \in [0,1].$$

Hence there is $k \in \mathbb{N}$ such that

$$\|1 - \sum_{j=1}^{k} f^j\|_1 < \varepsilon.$$

Let $P = (A_1,\dots,A_n)$ be the partition generated by $P^1,\dots,P^j$. For $g \in M$ we may estimate

$$\sum_{i=1}^{n} m(A_i).\mathrm{osc}\,(g|A_i) \leqq$$

$$\leqq \sum_{i=1}^{n} \int_{A_i} \Big(\sum_{j=1}^{k} f^j\Big)dm.\mathrm{osc}\,(g|A_i) + 2.\sum_{i=1}^{n} \Big(m(A_i) - \int_{A_i} \Big(\sum_{j=1}^{k} f^j\Big)dm\Big) \leqq$$

$$\leqq \varepsilon + 2\varepsilon = 3\varepsilon$$

which readily shows that M is (SO).

□

<u>3.3 Proposition:</u> Let $T : L^1[0,1] \to X$ be an operator, $f \in F$ and suppose that the restriction of T to F is weak-to-norm continuous at f. Let $B = \{f > 0\}$. Then the restriction of $M = T^*(\text{ball }(X^*))$ to B is of small osciallation.

<u>Proof:</u> It follows from proposition 3.2 above that it suffices to show, for $\alpha > 0$, that the restriction of M to the set $B_\alpha = \{f > \alpha\}$ is (SO).

Fix $\alpha > 0$ and $\delta > 0$ and find a partition $P = (A_1,\dots,A_n)$ and $\varepsilon > 0$ such that

$$\text{diam }(T(V_{P,\varepsilon}(f))) < \delta.$$

There is no loss of generality in assuming that B belongs to the σ-algebra generated by P, i.e. there is a subset $I \subset \{1,\dots,n\}$ such that

$$B_\alpha = \bigcup_{i\in I} A_i.$$

From proposition 2.7 we infer that

$$\sup_{\|x^*\|\leq 1} \sum_{i\in I} m(A_i).\text{osc }(T^*x^*|A_i) \leq$$

$$\leq \sup_{\|x^*\|\leq 1} \alpha^{-1}. \sum_{i\in I} (\int_{A_i} f dm).\text{osc }(T^*x^*|A_i) \leq$$

$$\leq \alpha^{-1} \text{ diam }(T(V_{P,\varepsilon}(f))) \leq \alpha^{-1}.\delta.$$

As $\delta > 0$ is arbitrary we finished the proof.

□

We are now ready to state the general theorem.

<u>3.4 Theorem:</u> Let $T : L^1[0,1] \to X$ be an operator. T.f.a.e.

(i) $T^*(\text{ball }(X^*))$ is (SO);

(ii) T is (PCP);

(ii') T is (CPCP);

(ii") T is (SR);

(iii) T restricted to F is weak-to-norm continuous at every $f \in F$;

(iii') T restricted to F is weak-to-norm continuous at $f \equiv 1$;

(iii") For $\varepsilon > 0$ there are $f_1,\dots,f_n \in F$ s.t. T restricted to F is weak-to-norm continuous at each f_i and $m(\bigcup_{i=1}^{n} \{f_i > 0\}) > 1-\varepsilon$;

(iv) For $A \subset [0,1]$, $m(A) > 0$ and $\varepsilon > 0$ there is a relatively weakly open subset U of F_A such that $\text{diam }(T(U)) < \varepsilon$.

<u>Proof:</u>

(i) $\Rightarrow$ (iii): Let $f \in F$ and $\delta > 0$ and suppose first that $f \in L^\infty[0,1]$, say $\|f\|_\infty \leq M$. Find a partition $P = (A_1,\dots,A_n)$ such that for every $\|x^*\| \leq 1$

$$\sum_{i=1}^{n} m(A_i).\text{osc }(T^*x^* | A_i) < \delta/2M.$$

Consider $V_{P,\varepsilon}(f)$ for $\varepsilon < \delta/4\|T\|$. Noting that

$$\int_{A_i} f dm \leq M.m(A_i)$$

we may apply proposition 2.7 to see that

$$\operatorname{diam}\,(T(V_{P,\varepsilon}(f))) \leqq M.(\delta/2M) + 2\varepsilon.\|T\| < \delta.$$

If $f \in F$ is arbitrary, write $f = \lambda f_1+(1-\lambda)f_2$ where $f_1, f_2 \in F$, $\lambda < \delta/2\|T\|$ and f_2 is in $L^\infty(m)$. By the previous argument we may find a partition P and $\varepsilon > 0$ s.t.

$$\operatorname{diam}\,(T(V_{P,\varepsilon}(f_2))) < \delta.$$

On the other hand we trivially have

$$\operatorname{diam}\,(T(V_{P,\varepsilon}(f_1))) \leqq 2.\|T\|.$$

By proposition 2.5 we obtain

$$\operatorname{diam}\,(T(V_{P,\varepsilon}(f))) =$$

$$= \operatorname{diam}\,(\lambda.T(V_{P,\varepsilon}(f_1)))+ (1-\lambda).T(V_{P,\varepsilon}(f_2))) \leqq$$

$$\leqq \lambda.\operatorname{diam}\,(T(V_{P,\varepsilon}(f_1)))+ (1-\lambda).\operatorname{diam}\,(T(V_{P,\varepsilon}(f_2))) <$$

$$< \qquad \delta \qquad + \qquad \delta \qquad = \quad 2\delta.$$

Hence we have obtained a relative weak neighbourhood $V_{P,\varepsilon}(f)$ of f in F s.t. $T(V_{P,\varepsilon}(f))$ has diameter less than 2δ, which readily shows that T restricted to f is weak-to-norm continuous at f.

(iii) $\Rightarrow$ (iii') $\Rightarrow$ (iii") are obvious and
(iii") $\Rightarrow$ (i) follows from the preceding propositions 3.2 and 3.3.

(i) $\Rightarrow$ (ii): Let C be a bounded subset of $L^1[0,1]$. We may and do suppose $\|C\| = \sup\{\|f\| : f \in C\} = 1$ as well as $\|T\| = 1$.

We have to show that for $\delta > 0$ there is a relatively weakly open subset U of C such that $\operatorname{diam}(T(U)) < 5\delta$. We shall show more precisely that for $\delta > 0$ and $f \in C$ with $\|f\| > 1-\delta$ there is a relative weak neighbourhood $U_{P,\varepsilon}(f)$ in the (bigger) set ball (L^1) such that $\operatorname{diam}(T(U_{P,\varepsilon}(f)) < 5\delta$.

Let f and δ be as above and find $\delta > \beta > 0$ such that $\|f\| > 1-\delta+\beta$. Let f_1 be an element of $L^\infty[0,1]$ such that $\|f_1-f\|_1 < \beta/4$ and find a partition $P = (A_1,\ldots,A_n)$ such that the set $\{f_1 > 0\}$ is in the σ-algebra generated by P and s.t.

$$\sup_{\|x^*\|\leq 1} \sum_{i=1}^{n} m(A_i).\operatorname{osc}(T^*x^*|A_i) < \beta/\|f_1\|_\infty.$$

For the relative weak neighbourhood

$$U_{P,\beta/2}(f_1) = \{g \in \text{ball }(L^1) : \sum_{i=1}^{n} |\int_{A_i} (f_1-g)\,dm| \leq \beta/2\}$$

of f_1 in ball (L^1) we shall show that

$$\operatorname{diam}\,(T(U_{P,\beta/2}(f_1))) < 5\delta.$$

This will finish the proof as

$$U_{P,\beta/4}(f) = \{g \in \operatorname{ball}\,(L^1) : \sum_{i=1}^{n} |\int_{A_i} (f-g)\,dm| \leq \beta/4\}$$

is contained in $U_{P,\beta/2}(f_1)$.

Given $g \in U_{P,\beta/2}(f_1)$ define

$$g_1 = \begin{cases} g_+ & \text{on those } A_i \text{ where } f_1 > 0 \text{ on } A_i \\ -g_- & \text{on those } A_i \text{ where } f_1 \leq 0 \text{ on } A_i. \end{cases}$$

As
$$\sum_{i=1}^{n} |\int_{A_i} g_1\,dm| \geq \sum_{i=1}^{n} |\int_{A_i} f_1\,dm| - \beta/2$$

$$= \|f_1\| - \beta/2 \geq 1-\delta+\beta/4$$

we conclude that $\|g_1\| \geq 1-\delta+\beta/4$ and therefore $\|g-g_1\|_1 \leq \delta-\beta/4$.

Now define
$$g_2 = \sum_{i=1}^{n} g_1\chi_{A_i}\cdot(\int_{A_i} f_1\,dm/\int_{A_i} g_1\,dm)$$

which is an element of the unit ball of $L^1[0,1]$ putting the same (signed) mass on each A_i as f_1 does. We may estimate

$$\|g_1-g_2\|_1 = \sum_{i=1} |\int_{A_i} (g_1-g_2)\,dm| = \sum_{i=1} |\int_{A_i} (g_1-f_1)\,dm| \leq$$

$$\leq \sum_{i=1}^{n} \left|\int_{A_i} (g-f_1)\,dm\right| + \|g-g_1\|_1 \leq$$

$$\leq \beta/2+\delta-\beta/4 \leq \delta + \beta/4$$

hence

$$\|g-g_2\| \leq \|g-g_1\| + \|g_1-g_2\| \leq 2\delta.$$

Now let g and h be two elements of $U_{P,\beta/2}(f_1)$ and g_2 and h_2 associated as above. For $i = 1,\dots,n$ the functions g_2 and h_2 have the same sign over A_i and

$$\int_{A_i} g_2\,dm = \int_{A_i} h_2\,dm = \int_{A_i} f_1\,dm$$

we conclude - similarly as in the proof of proposition 2.7 - that

$$\|T(g_2-h_2)\| = \sup_{\|x^*\|\leq 1} \langle T(g_2-h_2),x^*\rangle \leq$$

$$\leq \sup_{\|x^*\|\leq 1} \sum_{i=1}^{n} \left|\int_{A_i} (g_2-h_2).(T^*x^*)\,dm\right| \leq$$

$$\leq \sup_{\|x^*\|\leq 1} \sum_{i=1}^{n} \left|\int_{A_i} f_1\,dm\right|.\mathrm{osc}\,(T^*x^*|A_i) \leq$$

$$\leq \|f_1\|_\infty \sup_{\|x^*\|\leq 1} \sum_{i=1}^{n} m(A_i).\mathrm{osc}(T^*x^*|A_i) \leq$$

$$\leq \beta$$

hence $\|T(g-h)\| < 2\delta+2\delta+\beta < 5\delta$, which finishes the proof of the implication (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (ii') $\Rightarrow$ (ii") is evident and
(ii") $\Rightarrow$ (iv) is obvious too in view of the remark 2.6 that a convex combination of slices of F_A is relatively weakly open in F_A.

(iv) $\Rightarrow$ (i): If $M = T^*(\text{ball }(X^*))$ is not (SO) then by proposition 3.2 there is $A \subset [0,1]$, $m(A) > 0$ and $\varepsilon > 0$ such that for every $0 \leqq f \leqq \chi_A$ of positive measure and $(A_1, \ldots, A_n)$ a partition of $[0,1]$ there is $g \in M$ such that

$$\sum_{i=1}^{n} \int_{A_i} fdm.\text{osc }(g|A_i) \geqq \varepsilon.\|f\|_1.$$

As the above formula is positively homogeneous in f, it also holds true for $f \in F_A$, provided $\|f\|_\infty < \infty$.

As we assume that (iv) holds true we may find a relatively weakly open set U in F_A s.t. diam $(T(U)) < \varepsilon$.

We may find $f \in F_A$, $\|f\|_\infty < \infty$, $\beta > 0$ and a partition $P = (A_1, \ldots, A_n)$ of $[0,1]$ into sets of positive measure s.t.

$$U \supset U_{P,\beta}(f) = \{h \in F_A : \sum_{i=1}^{n} |\int_{A_i} (f-g)dm| \leqq \beta\}.$$

A glance at proposition 2.7 shows that

$$\sup_{\|x^*\| \leqq 1} \{ \sum_{i=1}^{n} \int_{A_i} fdm.\text{osc }(T^*x^*|A_i)\} \leqq \text{diam } T(U) < \varepsilon$$

a contradiction finishing the proof of (iv) $\Rightarrow$ (i) and therefore of theorem 3.4. □

4. Examples and Remarks

After establishing the general theorem 3.4 it seems appropriate to present some examples before going further in the study of classes of operators T from L^1 to X and the corresponding subsets $T^*(\text{ball}(X^*))$ of L^∞.

The operators in the next examples will act from $L^1[0,1]$ to c_o. Note that there is a one-to-one correspondence between these operators and sequences $(g_n)_{n=1}^\infty$ in $L^\infty[0,1]$ tending weak-star to 0 via the formula

$$T : L^1[0,1] \to c_o$$
$$f \to \langle f,g_n\rangle_{n=1}^\infty .$$

It is an easy and instructive exercise to verify that T is Dunford-Pettis iff $(g_n)_{n=1}^\infty$ tends to zero with respect to $\|.\|_1$. Less trivial is the observation, due to Mokobodzki (c.f. [M], [C-L]), that T is an (RNP)-operator iff f_n tends to zero pointwise almost everywhere.

<u>4.1 Remark:</u> Let us mention in this context, that it was shown in [S1] that an absolutely convex, bounded subset $M \subset L^\infty[0,1]$ is equimeasurable iff every sequence $(g_n)_{n=1}^\infty$ tending weak-star to zero converges to zero almost everywhere (this part is due to Mokobodzki) and iff every sequence $(g_n)_{n=1}^\infty$ tending to zero w.r.t. $\|.\|_1$ tends to zero almost everywhere. This result gives a direct link

between the theory of (RNP)-operators and a characterisation of integral operators due to A.V. Bukhvalov (c.f. [B5], [S1]; compare also [S3] and remark 7.4 below).

4.2 Example: An operator $T : L^1[0,1] \to c_o$ which is (SR) but not (RNP).

This example is certainly well-known (c.f. [P], 24.2.11) but we are unable to trace back who stated it first explicitely.

It is also the arch-example of a "complemented bush", a notion introduced by R.C. James and A. Ho which turnded out to be useful in studying the Krein-Milman problem (c.f. [H],[S5]).

Let $(g_j)_{j=1}^{\infty}$ be an enumeration of the indicator-functions of the dyadic intervals

$$\chi_{I_{n,k}} = \chi_{[(k-1)/2^n,k/2^n]} \qquad 1 \leq k \leq 2^n,\ n \in \mathbb{N},$$

and

$$T(f) = \langle f,g_j\rangle_{j=1}^{\infty}.$$

This sequence obviously does not tend to zero almost everywhere. On the other hand it is just as obvious that $(g_j)_{j=1}^{\infty}$ is (SO) by taking as partitions the dyadic intervals of the n-th generation. In order to be able to apply theorem 3.4 we still need the subsequent easy observation showing that the (SO)-sets have a stability property just as the equimeasurable sets.

<u>4.3 Proposition:</u> A bounded subset M of $L^\infty[0,1]$ is (SO) iff the weak-star closed, convex circled hull of M is (SO).

<u>Proof:</u> Straightforward from the definition of (SO).

<u>4.4 Example:</u> An operator $T : L^1[0,1] \to c_o$ which is Dunford-Pettis but not (SR).

This example was apparently first noticed by M. Talagrand ([T], 7-3-10).

This time let $(g_j)_{j=1}^\infty$ be an enumeration of all indicator functions of k dyadic intervals of length 2^{-k} $(k \in \mathbb{N})$. Clearly $(g_j)_{j=1}^\infty$ tends to zero w.r.t. $\|.\|_1$, hence T is Dunford-Pettis. On the other hand one easily verifies that for any partition $P = (A_1,\dots,A_n)$ of $[0,1]$ into sets of positive measure

$$\sup \{ \sum_{i=1}^{n} m(A_i).\mathrm{osc}\,(g_j|A_i) : j \in \mathbb{N} \} = 1,$$

hence T is not (SO).

<u>4.5 Remark:</u> It was noted by Talagrand ([T], 7-3-10), that the above operator is not (W^*RNP) as $(g_j)_{j=1}^\infty$ has non-measurable cluster points in the pointwise topology. The next

example is of a different nature as the operator T will be (W^*RNP). However we shall treat these aspects only in chapter 7 (theorem 7.7) below, as we need for a proper treatment some more concepts which we only introduce later.

<u>4.6 Example:</u> Another operator $T : L^1[0,1] \to c_o$ which is Dunford-Pettis but not (SR).

Let $(k_m)_{m=1}^{\infty}$ be a sequence in $\mathbb{N}$, $k_m \geq 2$, and consider the generalized Cantor-space

$$\Delta = \Delta_{(k_m)_{m=1}^{\infty}} = \prod_{m=1}^{\infty} \{1,\ldots,k_m\}$$

equipped with the canonical product-measure μ.

Let, for $m \in \mathbb{N}$ and $1 \leq j \leq k_m$,

$$h_{m,j} = \chi_{(j)} \circ \pi_m$$

where π_m is the projection onto the m-th coordinate of Δ and $\chi_{(j)}$ the indicator-function of the element $j \in \{1,\ldots,k_m\}$. Let

$$T : L^1 \to c_o$$
$$f \to ((f,h_{m,j})_{j=1}^{k_m})_{m=1}^{\infty}.$$

The set $M = ((h_{m,j})_{j=1}^{k_m})_{m=1}^{\infty}$ is not (SO). Indeed, for any partition $P = (A_1,\ldots,A_n)$ of Δ into sets of positive μ-measure

$$\sup \{ \sum_{i=1}^{n} \mu(A_i).\text{osc}\,(f|A_i) \;:\; f \in M\} = 1 \tag{1}$$

Indeed, by the Lebesgue-density-theorem, there is $m_o \in \mathbb{N}$ and atoms $B_1,\dots,B_n$ of $\mathcal{B}_{m_o}$ such that, for $1 \leq i \leq n$,

$$\mu(A_i \cap B_i) \geq \mu(B_i)(1-1/2n). \tag{2}$$

By an atom of $\mathcal{B}_{m_o}$ we mean a set B determined by a finite sequence $(p_m)_{m=1}^{m_o}$, $1 \leq p_m \leq k_m$ defined via

$$B = B_{p_1,\dots,p_{m_o}} = \{(q_m)_{m=1}^{\infty} \in \Delta \;:\; q_m = p_m \text{ for } m = 1,\dots,m_o\}.$$

Consider now $(h_{m_o+1,j})_{j=1}^{k_{m_o+1}}$ and note that each of these functions oscillates on each B_i by 1. We infer from (2) and the definition of the $h_{m,j}$, for $1 \leq i \leq n$ fixed, at least $k_{m_o+1}(1-1/2n)$ of these functions oscillate on A_i by 1. Hence there is some $1 \leq j_o \leq k_{m_o+1}$ such that h_{m_o,j_o} osciallates on each A_i by 1 which proves (1).

On the other hand T is clearly Dunford-Pettis, if we choose $(k_m)_{m=1}^{\infty}$ such that k_m tends to infinity. We shall see in 7.7 below, that M is "stable" in the sense of Talagrand [T], if we require that $(k_m^{-1})_{m=1}^{\infty}$ is summable or, more generally, if $(k_m^{-p})_{m=1}^{\infty}$ is summable for some $p < \infty$).

□

<u>4.7 Example:</u> We shall now deal with convolution operators from $L^1(\mathbb{T})$ to $C(\mathbb{T})$. We thank F. Lust-Piquard for pointing out to us the relation between the Riemann-integrability

of g and the relevant properties of the convolution operator T_g.

For simplicity we consider only the compact group $\mathbb{T} = \{e^{2\pi it} : t \in [0,1[\}$, which we identify with $[0,1[$, equipped with Haar-measure m.

Let $g \in L^\infty(\mathbb{T})$ and define

$$T_g : L^1(\mathbb{T}) \to C(\mathbb{T})$$
$$f \to f * g.$$

It is instructive to verify that the above operator is Dunford-Pettis and that it is an (RNP)-operator iff $g \in C(\mathbb{T})$. We shall show that if g is Riemann-integrable then T_g is a (SR)-operator. By theorem 3.4 we have to show that $M = T_g^*(\text{ball }(M(\mathbb{T})))$ is (SO). Noting that M is the weak-star closed convex hull of the translates $\{g_t : t \in \mathbb{T}\}$ and invoking prop. 4.4 we shall be done if we proved the subsequent assertion.

<u>Claim:</u> If $g \in L^\infty(\mathbb{T})$ is Riemann-integrable (in the sense: there is a Riemann-integrable representant of g) then $\{g_t : t \in \mathbb{T}\}$ is (SO).

<u>Proof:</u> The Riemann-integrability of g means that, for $\varepsilon > 0$, there is $n \in \mathbb{N}$ such that there are at most $\varepsilon . 2^n$ of the intervalls $I_{k,n} = [k/2^n, (k+1)/2^n]$ on which the

oscillation of g is bigger than ε. Fix $\varepsilon > 0$ and $n \in \mathbb{N}$ such that this holds true. If we consider g_t, where t is of the form $k/2^n$, note that the number of intervals $I_{k,n}$ where g_t oscillates more than ε is exactly the same as for g. If $t \in \mathbb{T}$ is arbitrary, a moment's reflection shows that the number of intervals where g_t oscillates more than ε is at most $2.\varepsilon.2^n$. Hence taking as partition $P = ([k/2^n,(k+1)/2^n[\; : \; 1 \leq k \leq 2^n)$ we obtain

$$\sup \{ \sum_{k=1}^{2^n} 2^{-n}.\mathrm{osc}\,(g_t | [k/2^n,(k+1)/2^n[) \; : \; t \in \mathbb{T}\} \leq$$

$$\leq 4\varepsilon \; \|g\|_\infty + \varepsilon,$$

thus proving the claim.

□

What we have proved is, that for a Riemann-integrable $g \in L^\infty(\mathbb{T})$ the convolution operator T_g is (SR), whence in particular (W^*RNP). On the other hand, Talagrand ([T], 15-4) has shown the much deeper fact that g is Riemann-integrable iff there is a representant g^R in the equivalence class of g such that $\{g_t^R : t \in \mathbb{T}\}$ is stable ([T], 9-1-1). Admitting for the moment the results of chapter 7 below, we see that the convolution operator T_g is (SR) iff g is Riemann-integrable iff (under the assumption of axiom F ([T], 9-1-2)) T_g is W^*RNP. Hence we have a complete characterisation of the nature of T_g in terms of the properties studied here according to whether g is continuous, not continuous but Riemann-integrable or not Riemann-integrable.

4.8 Remark: Theorem 3.4 states that an operator $T : L^1[0,1] \to X$ is (SR) iff T restricted to F is weak-to-norm continuous at every $f \in F$. Denote $\tilde{F}$ the weak-star closure of F in the double-dual $L^1[0,1]^{**}$. One may consider (and this is done extensively in [S2]) the restriction of T^{**} to $\tilde{F}$ and ask for which $\mu \in \tilde{F}$ this map is weak-star to norm continuous (see chapter 6 below).

It follows easily from the above that this is the case (for the T above) for every $f \in F$. On the other hand if $T^{**}(\mu) \in X^{**} \setminus X$ then $T^{**}|\tilde{F}$ is certainly not weak-star to norm continuous at μ. Indeed, for every relative weak-star-neighbourhood U of μ in $\tilde{F}$ we have

$$T^{**}(U) \cap X \neq \emptyset$$

hence

$$\operatorname{diam}(T^{**}(U)) \geq \operatorname{dist}(T^{**}(\mu),X) > 0.$$

The crucial point in [S2] consists in the fact that the (SR)-operator $T : L^1[0,1] \to X$ constructed there is such that $T^{**}|\tilde{F}$ is weak-star to norm continuous at $\mu \in \tilde{F}$ iff $T^{**}(\mu) \in X$. It has been observed by H.P. Rosenthal [R2] that this implies that on $\overline{T(F)}$, the closure taken in $(X,\|.\|)$, the weak and norm topologies coincide, thus relating the result of [S2] with a recent characterisation of denting points due to B. Lin, P. Lin and S. Troyanski [L-L-T].

In view of theorem 3.4 one might be tempted to conjecture that every (SR)-operator has the property that $T^{**}|\tilde{F}$ is weak-star to norm continuous at every $\mu \in \tilde{F}$ such that $T^{**}(\mu) \in X$. This, however is not true as will be shown by an example of a convolution operator T_g, where g is Riemann-integrable, in [L-S].

4.9 Remark: The property (SO) of being of small oscillation may be viewed as a kind of uniform Riemann-integrability as is indicated by the previous example 4.7:

Suppose $M \subset L^{\infty}[0,1]$ is (SO) and find, for $k \in \mathbb{N}$, a partition

$$P^k = (A_1^k, \ldots, A_{n_k}^k)$$

such that

$$\sup_{g \in M} \sum_{i=1}^{n_k} m(A_i^k) . \mathrm{osc}\,(g | A_i^k) < k^{-1}.$$

There is no loss of generality in assuming that P^{k+1} splits every member of P^k into m_k sets (so that $n_k = m_1\, m_2 \ldots m_k$), and that $((A_i^k)_{i=1}^{n_k})_{k=1}^{\infty}$ generates the Borel-σ-algebra of $[0,1]$.

Similarly as in example 4.6 above let

$$\Delta = \prod_{k=1}^{\infty} \{1,\dots,m_k\}.$$

For an atom

$$B = B_{p_1,\dots,p_{m_o}} = \{(q_m)_{m=1}^{\infty} \in \Delta : q_m = p_m \quad \text{for } 1 \leqq m \leqq m_o\}$$

define the measure $\mu(B)$ to equal the Lebesgue-measure m of the corresponding set $A_i^{m_o}$ of the partition P^{m_o} (the correspondence defined in an obvious way). Clearly μ extends to a measure on Δ which allows us to identify $L^1([0,1],m)$ with $L^1(\Delta,\mu)$. Using this identification we may view M as a subset of $L^\infty(\Delta,\mu)$. But now the partition

$$\tilde{P}^k = \{B_{p_1,\dots,p_k} : 1 \leqq p_1 \leqq m_1,\dots,1 \leqq p_k \leqq m_k\}$$

are partitions of Δ into clopen sets, such that the $g \in M$ oscillate only on few of them more than k^{-1}. This condition may be viewed as a uniform Riemann-integrability, in the sense that the (essential) lower and upper Riemann-suns taken over $\tilde{P}^k$ converge to the integrals of g uniformly in $g \in M$.

Note, however, that it is unavoidable to "change the topology" as is done above by passing from $[0,1]$ to Δ even if M is contained in $C[0,1]$ as is shown by the subsequent example: Let $1/2 > \varepsilon > 0$ and $(r_n)_{n=1}^{\infty}$ be an enumeration of $\mathbb{Q} \cap [0,1]$ and define

$$h_n : \mathbb{R} \to [0,1]$$

$$\begin{array}{lr} \text{on } r_n & = 1 \\ \text{outside } [r_n-\varepsilon/2^n,\ r_n+\varepsilon/2^n] & = 0 \\ \text{on } [r_n-\varepsilon/2^n,\ r^n] \text{ and } [r_n, r_n+\varepsilon/2^n] & \text{linear} \end{array}$$

and

$$g_n : [0,1] \to [0,1]$$
$$g_n(t) = \inf\{\sum_{i=1}^{n} g_i(t), 1\}.$$

It is easy to check that $M = (g_n)_{n=1}^{\infty}$ is (SO). However, given any increasing sequence of partitions $(P^k)_{k=1}^{\infty}$ of $[0,1]$ into intervals, the corresponding lower and upper Riemann-sums do not converge uniformly over M.

<u>4.10 Remark:</u> The characterisation of (SR)-operators in terms of weak-to-norm continuity given by theorem 3.4 regards (F,weak) as a topological space. It seems worth noting that, if we regard (F,weak) as a uniform space, we thus obtain a characterisation of (RNP)-operators analogous to 3.4 (iv).

<u>4.11 Proposition:</u> A continuous operator $T : L^1[0,1] \to X$ is compact iff the restriction of T to F is uniformly weak-to-norm continuous. Hence $T : L^1[0,1] \to X$ is (RNP) iff for $A \subset [0,1]$, $m(A) > 0$ there is $B \subset A$ such that T restricted to F_B is uniformly weak-to-norm continuous.

<u>Proof:</u> A continuous operator S between Banach spaces Y and X is compact iff the restriction of S to the unit-ball of Y is uniformly weak-to-norm continuous and $T : L^1[0,1] \to X$ is uniformly weak-to-norm continuous on the unit-ball of $L^1[0,1]$ iff it is so on F. The second assertion now follows from the first and theorem 1.15.

□

<u>4.12 Remark:</u> Let us finally note that for dual spaces X^* the situation is much easier than in general: As has been proved by J. Bourgain [B2] X does not contain ℓ^1 iff X^* is (SR) iff every $T : L^1[0,1] \to X^*$ is Dunford-Pettis (the last part follows essentially from Rosenthal's ℓ^1-theorem; see, e.g. [T], 7-3-7 and 7-3-8 and the references given there about the weak Radon-Nikodym property).

Hence, in our language, every $T : L^1[0,1] \to X^*$ is (PCP) iff every $T : L^1[0,1] \to X^*$ is Dunford-Pettis (to quote the least and most general conditions in the listing of chapter 1). Hence for dual spaces and using the quantifier "for every" we don't have much distinction between the properties of operators we are presently studying.

5. A characterisation of strongly regular spaces

Note that a Banach space X has (RNP) iff and only if every operator $T : L^1[0,1] \to X$ is (RNP). This is by no means a tautology but it relies on the - by now well-known - construction of a bounded δ-bush in a non-dentable set C (see, e.g. [D-U]).

Note also that we can not hope to have a similar result for the notion of (PCP) as theorem 3.4 tells us that the notions (PCP), (CPCP) and (SR) are equivalent for operators from $L^1[0,1]$ to X, while we know [G-M-S] that (PCP) $\not\Leftarrow$ (CPCP) (as a property for spaces).

However for the notion of (SR) we do have that X is (SR) iff every $T : L^1[0,1] \to X$ is (SR). This results from theorem 5.2 below, which we formulate in the more general "local" setting. Theorem 5.2 will result from the subsequent crucial lemma 5.1. It might be helpful to look at example 4.6 which is the arch-example of the situation described by lemma 5.1 and theorem 5.2.

5.1 Lemma: Let $\alpha > 0$ and $C \subset X$ be convex and bounded such that every convex combination of slices of C has diameter greater than 2α. Let x_0 and $x_1,\dots,x_k$ in C be such that

$$x_0 = k^{-1} \sum_{i=1}^{k} x_i .$$

Given $\varepsilon > 0$ there is $m \in \mathbb{N}$ and elements $((x_{i,j})_{j=1}^{m})_{i=1}^{k}$ in C such that

(i) $\quad \|x_i - m^{-1} \sum_{j=1}^{m} x_{i,j}\| < \varepsilon \quad$ for $i = 1,\dots,k$

(ii) $\quad \|x_o - k^{-1} \sum_{i=1}^{k} x_{i,j}\| > \alpha \quad$ for $j = 1,\dots,m$

<u>Proof:</u> First note that it follows from the assumption and Bourgain's lemma 1.2a that every convex combination of relatively weakly open subsets of C has diameter greater than 2α.

Consider the $(k+1)$-fold product space X^{k+1} - equipped with the norm $\|(x_i)_{i=0}^{k}\| = \sup\{\|x_i\| : i = 0,\dots,k\}$ - and let D be the subset

$$D = \{(\xi_o,\dots,\xi_k) \in C^{k+1} : \xi_o = k^{-1} \sum_{i=1}^{k} \xi_i\},$$

which is bounded and convex. Let

$$V_\alpha = \{\vec{\xi} = (\xi_o,\dots,\xi_k) \in D : \|x_o - \xi_o\| < \alpha\}.$$

We shall prove the crucial formula, which is a kind of multidimensional non-dentability

$$\overline{\mathrm{co}}(D \setminus V_\alpha) \supset D, \tag{1}$$

where $\overline{\mathrm{co}}$ denotes the closed, convex hull.

Indeed, if (1) fails we may apply the separation theorem to the pairs of Banach spaces $(X^{k+1},(X^*)^{k+1})$ to find $\delta > 0$,

$$\vec{\eta} = (\eta_o,\ldots,\eta_k) \in V_\alpha$$

and

$$\vec{f} = (f_o,\ldots,f_k) \in (X^*)^{k+1}$$

such that for $\vec{\xi} \in D$ with

$$\langle\vec{\xi}-\vec{\eta},\vec{f}\rangle = \sum_{i=0}^{k} \langle\xi_i-\eta_i,f_i\rangle > -\delta$$

we have $\vec{\xi} \in V_\alpha$, i.e. $\|\xi_o-x_o\| < \alpha$.

Consider, for $i = 1,\ldots,k$, the relatively weakly open subsets U_i of C

$$U_i = \{\xi_i \in C : |\langle\xi_i-\eta_i,f_i\rangle| < \delta/2n \text{ and}$$

$$|\langle\xi_i-\eta_i,f_o\rangle| < \delta/2$$

and let

$$W = k^{-1}\sum_{i=1}^{k} U_i.$$

If $\xi_o = k^{-1}\sum_{i=1}^{k}\xi_i$ is an element of W (i.e., $\xi_i \in U_i$) then $\vec{\xi} = (\xi_o,\xi_1,\ldots,\xi_k)$ is in D and

$$\langle\vec{\xi}-\vec{\eta},\vec{f}\rangle = \sum_{i=0}^{k}\langle\xi_i-\eta_i,f_i\rangle > -\delta/2 - \delta/2 = -\delta$$

hence $\|\xi_o-x_o\| < \alpha$. But this implies that W has diameter $\leq 2\alpha$, a contradiction proving (1).

Applying (1) to $\vec{x} = (x_o,x_1,\ldots,x_k)$ we may find vectors $\vec{x}_1 = (k^{-1}\sum_{i=1}^{k}x_{i,1},x_{1,1},\ldots,x_{k,1}),\ldots,\vec{x}_m = (k^{-1}\sum_{i=1}^{k}x_{i,m},x_{1,m},\ldots$ $\ldots,x_{k,m})$ in $D \setminus V_\alpha$ such that

$$\|\vec{x} - m^{-1} \sum_{j=1}^{m} \vec{x}_j\| = \sup_{i=0,\ldots,k} \|x_i - m^{-1} \sum_{j=1}^{m} x_{i,j}\| < \varepsilon .$$

Hence the $((x_{i,j})_{j=1}^{m})_{i=1}^{k} \in C$ above satisfy (i) and (ii) thus proving lemma 5.1.

□

<u>5.2 Theorem:</u> Let C be closed, convex, bounded in X and $\alpha > 0$ such that every convex combination of slices of C has diameter greater than 2α. Then there is an operator $T : L^1[0,1] \to X$ such that $T(F) \subset C$ and T is not (SR).

<u>Proof:</u> Let $(\varepsilon_n)_{n=1}^{\infty}$ be a sequence of strictly positive reals such that $\Sigma\varepsilon_n < \alpha/4$. Let $x_o \in C$. We proceed by induction on n. For $n = 1$ find $k_1 \in \mathbb{N}$ and $x_1,\ldots,x_{k_1} \in C$ such that

$$\|x_o - k_1^{-1} \sum_{i_1=1}^{k_1} x_{i_1}\| < \varepsilon_1$$

and

$$\|x_o - x_{i_1}\| \geq \alpha \qquad\qquad i_1 = 1,\ldots,k_1$$

(For this first step we need only the assumption that every slice has diameter $> 2\alpha$.)

Let x_o^1 be the exact average of the $x_1,\ldots,x_{k_1}$, i.e.

$$x_o^1 = k_1^{-1} \sum_{i_1=1}^{k_1} x_{i_1}$$

so that $\|x_o - x_o^1\| < \varepsilon_1$.

For $n = 2$ apply lemma 5.1 to $(x_o^1, x_1, \ldots, x_{k_1})$ to find k_2 and $\{x_{i_1,i_2} : 1 \leq i_1 \leq k_1, 1 \leq i_2 \leq k_2$ such that

(i) $\quad \|x_{i_1} - k_2^{-1} \sum_{i_2=1}^{k_2} x_{i_1,i_2}\| < \varepsilon_2 \qquad 1 \leq i_1 \leq k_1$

(ii) $\quad \|x_o^1 - k_1^{-1} \sum_{i_1=1}^{k_1} x_{i_1,i_2}\| > \alpha \qquad 1 \leq i_2 \leq k_2$

Let x_o^2 be the exact average of the elements of the second generation, i.e.

$$x_o^2 = (k_1 k_2)^{-1} \sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} x_{i_1,i_2}$$

so that $\|x_o^1 - x_o^2\| < \varepsilon_2$.

For the n'th step suppose we have found natural numbers $k_1, \ldots, k_n$ and elements $\{x_{i_1,\ldots,i_n} : 1 \leq i_1 \leq k_1, \ldots, 1 \leq$ $\leq i_n \leq k_n\}$ such that

(i) $\quad \|x_{i_1,\ldots,i_{n-1}} - k_n^{-1} \sum_{i_n=1}^{k_n} x_{i_1,\ldots,i_n}\| < \varepsilon_n$

for $1 \leq i_1 \leq k_1, \ldots, 1 \leq i_{n-1} \leq k_{n-1}$

(ii) $\quad \|x_o^{n-1} - (k_1 \ldots k_{n-1})^{-1} \sum_{i_1=1}^{k_1} \ldots \sum_{i_{n-1}=1}^{k_{n-1}} x_{1_i,\ldots,i_n}\| > \alpha$

for $1 \leq i_n \leq k_n$.

Let

$$x_o^n = (k_1 \ldots k_n)^{-1} \sum_{i_1=1}^{k_1} \ldots \sum_{i_n=1}^{k_n} x_{i_1,\ldots,i_n}$$

so that $\|x_o^{n-1} - x_o^n\| < \varepsilon_n$.

Apply lemma 5.1 to $\{x_o^n, x_{i_1,\ldots,i_n} : 1 \leq i_1 \leq k_1, \ldots$ $\ldots, 1 \leq i_n \leq k_n\}$ to find k_{n+1} and $\{x_{i_1,\ldots,i_{n+1}} :$ $: 1 \leq i_1 \leq k_1, \ldots, 1 \leq i_{n+1} \leq k_{n+1}\}$ such that

(i) $$\|x_{i_1,\ldots,i_n} - k_{n+1}^{-1} \sum_{i_{n+1}=1}^{k_{n+1}} x_{i_1,\ldots,i_{n+1}}\| < \varepsilon_{n+1}$$

$$\text{for } 1 \leq i_1 \leq k_1, \ldots, 1 \leq i_n \leq k_n$$

(ii) $$\|x_o^n - (k_1 \ldots k_n)^{-1} \sum_{i_1=1}^{k_1} \ldots \sum_{i_n=1}^{k_n} x_{i_1,\ldots,i_{n+1}}\| > \alpha$$

$$\text{for } 1 \leq i_{n+1} \leq k_{n+1}.$$

This finishes the induction step. Let y_o equal the limit of $(x_o^n)_{n=1}^{\infty}$ and, for $1 \leq i_1 \leq k_1, \ldots, 1 \leq i_n \leq k_n$

$$y_{i_1,\ldots,i_n} = \lim_{p\to\infty} (k_{n+1} \ldots k_p)^{-1} \sum_{i_{n+1}=1}^{k_{n+1}} \ldots \sum_{i_p=1}^{k_p} x_{i_1,\ldots,i_n,\ldots,i_p}.$$

Clearly the above limits exist and the $y_{i_1,\ldots,i_n}$ form a "bush". It will be (notationally) convenient to assume $y_o = 0$ which may be done without loss of generality by passing from C to $C - y_o$.

Similarly as in example 4.6 we define a generalized Cantor-set

$$\Delta = \prod_{n=1}^{\infty} \{1, \ldots, k_n\}$$

equipped with the canonical product measure μ. Clearly $L^1(\Delta,\mu)$ may be identified with $L^1([0,1],m)$. Define the operator $T : L^1(\mu) \to X$ on the atoms

$$B_{i_1,\ldots,i_n} = \{(j_k)_{k=1}^{\infty} \in \Delta : j_1 = i_1,\ldots,j_n = i_n\}$$

by

$$T(\chi_{B_{i_1,\ldots,i_n}}/\mu(B_{i_1,\ldots,i_n})) = y_{i_1,\ldots,i_n}.$$

Clearly T extends by linearity and continuity to all of $L^1(\mu)$ and $T(F) \subset C$. We shall show that $M = T^*(\text{ball }(X^*))$ is not of small oscillation. Indeed, suppose we can find a partition $P = (A_1,\ldots,A_m)$, such that

$$\sup_{g\in M} \{\sum_{j=1}^{m} \mu(A_j).\text{osc }(g|A_j)\} < \alpha/8 \tag{2}$$

Note that it follows from the assumption $y_o = T(1) = 0$ that, for $x^* \in X^*$

$$\int T^*x^*d\mu = \langle T(1),x^*\rangle = 0$$

hence (2) implies

$$\sup_{g\in M} \{\sum_{j=1}^{m} \mu(A_j) \sup (g|A_j)\} < \alpha/8. \tag{3}$$

For $n \in \mathbb{N}$ and $1 \leq i_n \leq k_n$ we have

$$\|y_o-(k_1\ldots k_{n-1})^{-1} \sum_{i_1=1}^{k_1}\cdots \sum_{i_{n-1}=1}^{k_{n-1}} y_{i_1,\ldots,i_n}\| > \alpha-2\eta_n > \alpha/2$$

where $\eta_n = \sum_{i=n}^{\infty} \varepsilon_i$, hence there is $x^*_{i_n}$ in the unit-ball of X^* such that

$$(k_1 \ldots k_{n-1})^{-1} \sum_{i_1=1}^{k_1} \ldots \sum_{i_{n-1}=1}^{k_{n-1}} \langle y_{i_1,\ldots,i_n}, x^*_{i_n} \rangle > \alpha/2.$$

Writing

$$g_{i_n} = T^*(x^*_{i_n})$$

and

$$R_{i_n} = \cup\{B_{i_1,\ldots,i_n} : 1 \leqq i_1 \leqq k_1, \ldots, 1 \leqq i_{n-1} \leqq k_{n-1}\}$$

we obtain

$$\int_{R_{i_n}} g_{i_n} d\mu > \alpha/2k_n = (\alpha/2)\mu(R_{i_n}). \tag{4}$$

Formula (4) means that the average of g_{i_n} over R_{i_n} is greater than $\alpha/2$. If Q_{i_n} is a subset of R_{i_n} such that

$$\mu(Q_{i_n}) > (1-\alpha / 4\|T\|).\mu(R_{i_n})$$

we may conclude (in view of $\|g_{i_n}\|_\infty \leqq \|T\|$) that

$$\int_{Q_{i_n}} g_{i_n} d\mu > (\alpha/4)\mu(R_{i_n}). \tag{5}$$

Let us turn our attention to the partition $P = (A_1,\ldots,A_m)$. For $n \in \mathbb{N}$ and $1 \leqq j \leqq m$ let

$$I_n^j = \{(i_1,\ldots,i_n) : \mu(B_{i_1,\ldots,i_n} \cap A_j) > (1/2)\mu(B_{i_1,\ldots,i_n})\}$$

and for $1 \leqq i_{n+1} \leqq k_{n+1}$

$$J^j_{n,i_{n+1}} = \{(i_1,\ldots,i_n) : \mu(B_{i_1,\ldots,i_n,i_{n+1}} \cap A_j) >$$
$$> (1-\alpha/8\|T\|)\mu(B_{i_1,\ldots,i_n,i_{n+1}})\}.$$

For $(i_1,..,i_n) \in I_n^j$ or J_{n-1,i_n}^j let $E_{i_1,..,i_n} =$
$= B_{i_1,..,i_n} \cap A_j$.

It follows from classical measure theory that

$$\lim_{n\to\infty} \left(\sum_{j=1}^{m} \# I_n^j\right)/(k_1...k_n) = 1$$

and

$$\lim_{n\to\infty} \left(\sum_{j=1}^{m} \sum_{i_{n+1}=1}^{k_{n+1}} \# J_{n,i_{n+1}}^j\right)/(k_1...k_{n+1}) = 1.$$

Hence we may choose $n \in \mathbb{N}$ and an appropriate $i_{n+1} \in \{1,...,k_{n+1}\}$ such that

$$\sum_{j=1}^{m} \# I_n^j/(k_1...k_n) > 1-\alpha/16\|T\|$$

and

$$\sum_{j=1}^{m} \# J_{n,i_{n+1}}^j/(k_1...k_n) > 1-\alpha/16\|T\|$$

whence

$$\sum_{j=1}^{m} \# (I_n^j \cap J_{n,i_{n+1}}^j)/(k_1...k_n) > 1-\alpha/8\|T\| \tag{6}$$

Note that $\{E_{i_1,..,i_n} : (i_1,..,i_n) \in I_n^j,\ 1 \leq j \leq m\}$ is a collection of disjoint subsets of Δ of positive measure refining the partition $P = (A_1,...,A_m)$, hence we infer from (3) that

$$\sup_{g\in M} \left\{ \sum_{j=1}^{m} \sum_{(i_1,..,i_n)\in I_n^j} \mu(E_{i_1,..,i_n}) \sup (g|E_{i_1,..,i_n})\right\} < \alpha/8. \tag{7}$$

On the other hand we may estimate for $g_{i_{n+1}} \in M$

$$\sum_{j=1}^{m} \sum_{(i_1,\ldots,i_n)\in I_n^j} \mu(E_{i_1,\ldots,i_n}) \sup (g_{i_{n+1}} | E_{i_1,\ldots,i_n}) \geq$$

$$\geq \sum_{j=1}^{m} \sum_{(i_1,\ldots,i_n)\in I_n^j \cap J^j_{n,i_{n+1}}} \mu(E_{i_1,\ldots,i_n}) \sup (g_{i_{n+1}} | E_{i_1,\ldots,i_{n+1}}) \geq$$

$$\geq \inf \{\mu(E_{i_1,\ldots,i_n})/\mu(E_{i_1,\ldots,i_{n+1}}) : (i_1,\ldots,i_n) \in I_n^j \cap J^j_{n,i_{n+1}}\}.$$

$$\cdot \sum_{j=1}^{m} \sum_{(i_1,\ldots,i_n)\in I_n^j \cap J^j_{n,i_{n+1}}} \mu(E_{i_1,\ldots,i_{n+1}}) \sup (g_{i_{n+1}} | E_{i_1,\ldots,i_{n+1}}) \geq$$

$$\geq (k_{n+1}/2) \int_{Q_{i_{n+1}}} g_{i_n} \, d\mu > (k_{n+1}/2)(\alpha/4k_{n+1}) = \alpha/8.$$

Here $Q_{i_{n+1}}$ denotes the set

$$Q_{i_{n+1}} = \bigcup_{j=1}^{m} \bigcup_{(i_1,\ldots,i_n)\in I_n^j \cap J^j_{n,i_{n+1}}} (E_{i_1,\ldots,i_n,i_{n+1}})$$

for which in view of (6) and the definition of $J^j_{n,i_{n+1}}$ we have

$$\mu(Q_{i_{n+1}}) > (1-\alpha/4\,\|T\|)\,k_{n+1}^{-1}$$

hence the last line of the above inequalities follows from (5). So we arrive at a contradiction to (7) thus proving theorem 5.2.

□

<u>5.3 Remark:</u> An analysis of the proof above shows that we get in fact (with the above notation) for every partition $P = (A_1,\dots,A_m)$ of Δ

$$\sup_{g\in M} \{ \sum_{j=1}^{m} \mu(A_j).\mathrm{osc}\,(g|A_j)\} \geq \alpha.$$

6. Regular operators and sets of regular oscillation

In this section we give a characterisation of regular operators in terms of sets of regular oscillation. It will turn out, that the results are analoguous to the characterisation of strongly regular operators in terms of sets of small oscillation given in section 3. We want to thank N. Ghoussoub for suggesting the definition of regular operators, which was the motivation for developing the results of this section.

6.1 Definition: A bounded subset M of $L^\infty[0,1]$ is of regular osciallation (RO), if, for z^{**} in the weak-star-closure $\widetilde{M}$ of M in the bidual $L^\infty[0,1]^{**}$ and $\varepsilon > 0$, there is $A \subset [0,1]$, $m(A) > 1-\varepsilon$ such that

$$z^{**}|_{L^\infty(A)^*} = g|_{L^\infty(A)^*}$$

for a (unique) $g \in L^\infty(A, m|_A)$.

6.2 Remark: It is not obvious (but true) that (SO) $\Rightarrow$ (RO). This will follow from the subsequent theorem 6.5. We shall also prove this explicitely in the next section using a notion and an argument due to J. Bourgain.

6.3 Proposition: M is (RO) iff for $A \subset [0,1]$, $m(A) > 0$ there is $B \subset A$, $m(B) > 0$ such that M restricted to B is (RO).

<u>Proof:</u> Suppose B_1 and B_2 are two disjoint sets such that M restricted to B_1 and B_2 is (RO). We shall show that the restriction of M to $B_1 \cup B_2$ is (RO). Indeed, let $z^{**} \in \widetilde{M}$ and find subsets $C_1 \subset B_1$, $C_2 \subset B_2$ of large measure such that the restriction of z^{**} to C_1 and to C_2 is induced by an element of L^∞, hence the restriction of z^{**} to $C_1 \cup C_2$ - which is a large subset of $B_1 \cup B_2$ - is induced by an element of L^∞, which had to be shown.

Applying an exhaustion argument we finish the proof of the proposition.

□

In view of the definition of (RO) it is not surprising that we shall have to investigate the higher duals of $L^1[0,1]$. We have to introduce some additional notation: $\widetilde{F}$ (resp. $\widetilde{F}_A$) will denote the weak-star-closure of F (resp. F_A) in $L^1[0,1]^{**}$. For $\mu \in \widetilde{F}$, $P = (A_1,\dots,A_n)$ a partition of $[0,1]$ into (equivalence classes of) sets of strictly positive Lebesgue-measure and $\varepsilon \geqq 0$ define

$$\widetilde{V}_{P,\varepsilon}(\mu) = \{\nu \in \widetilde{F} : \sum_{i=1}^{n} |\langle\mu-\nu,\chi_{A_i}\rangle| \leqq \varepsilon\}$$

and

$$\widetilde{V}^{\circ}_{P,\varepsilon}(\mu) = \{\nu \in \widetilde{F} : \sum_{i=1}^{n} |\langle\mu-\nu,\chi_{A_i}\rangle| < \varepsilon\}.$$

This notation is consistent with the above in the sense that, for $f \in F$, $\widetilde{V}_{P,\varepsilon}(f)$ is the weak-star-closure of $V_{P,\varepsilon}(f)$ in $L^1[0,1]^{**}$.

Again, one easily checks that $\tilde{V}_{P,\varepsilon}(\mu)$ defines a weak-star neighbourhood-basis of μ in $\tilde{\mathcal{F}}$ as P runs through the finite partitions of $[0,1]$ into (equivalence classes of) sets of positive measure and ε through $]0,1]$.

For $A \subset [0,1]$, $m(A) > 0$ and $\beta > 0$ denote $\tilde{S}_{A,\beta}$ to be the weak-star-slice of $\tilde{\mathcal{F}}$

$$\tilde{S}_{A,\beta} = \{\nu \in \tilde{\mathcal{F}} : \langle \nu, \chi_A \rangle > 1-\beta\}.$$

Similarly as in 2.3 one verifies for $\mu \in \tilde{\mathcal{F}}$

$$\tilde{V}^{o}_{P,\varepsilon}(\mu) = \sum_{i=1}^{n} \langle \mu, \chi_{A_i} \rangle \, \tilde{S}_{A_i,\varepsilon/2}$$

and

$$\tilde{V}_{P,o}(\mu) = \sum_{i=1}^{n} \langle \mu, \chi_{A_i} \rangle \, \tilde{\mathcal{F}}_{A_i}.$$

For $z^{**} \in L^{\infty}[0,1]^{**}$ and (an equivalence class of) a set $A \subset [0,1]$, $m(A) > 0$, define

$$\begin{aligned} \operatorname{osc}(z^{**} \mid A) &= \operatorname{osc}(z^{**} \mid \tilde{\mathcal{F}}_A) = \\ &= \sup \{\langle z^{*}, \mu-\nu \rangle : \mu, \nu \in \tilde{\mathcal{F}}_A\}. \end{aligned}$$

As $\tilde{\mathcal{F}}$ is a norming set for $L^{\infty}[0,1]^{**}$ one verifies

$$\operatorname{osc}(z^{**} \mid A) = \inf \{\|\chi_A . z^{**} - \gamma . \chi_A\|_{L^{\infty}[0,1]^{**}} : \gamma \in \mathbb{R}\}$$

Indeed, let

$$\gamma_1 = \sup \{\langle z^{*}, \mu \rangle : \mu \in \tilde{\mathcal{F}}_A\} \quad \text{and} \quad \gamma_2 = \inf \{\langle z^{*}, \nu \rangle : \nu \in \tilde{\mathcal{F}}_A\}$$

then $\gamma = (\gamma_1 - \gamma_2)/2$ gives the appropriate value in the above infimum.

For an operator $T : L^1[0,1] \to X$ and $x^{***} \in X^{***}$ we therefore get

$$\begin{aligned} \operatorname{osc}\,(x^{***} \mid T^{**}(\tilde{\mathcal{F}}_A)) &= \operatorname{osc}\,(T^{***}(x^{***}) \mid \tilde{\mathcal{F}}_A) \\ &= \inf\,\{ \|\chi_A \cdot T^{***}(x^{***}) - \gamma \cdot \chi_A\|_{L^\infty[0,1]^{**}} : \gamma \in \mathbb{R} \} \end{aligned}$$

For a partition $P = (A_1, \ldots, A_n)$ and $\mu \in \tilde{\mathcal{F}}$ we get

$$\begin{aligned} \operatorname{osc}\,(x^{***} \mid T^{**}(\tilde{V}_{P,0}(\mu))) &= \operatorname{osc}\,(x^{***} \mid T^{**}(\sum_{i=1}^{n} \langle \mu, \chi_{A_i} \rangle \tilde{\mathcal{F}}_{A_i})) \\ &= \sum_{i=1}^{n} \langle \mu, \chi_{A_i} \rangle \inf\,\{ \|\chi_{A_i} T^{***} x^{***} - \gamma_i \chi_{A_i}\| : \gamma_i \in \mathbb{R} \} \end{aligned}$$

and similarly as in 2.7 for $\varepsilon > 0$

$$\begin{aligned} &\sum_{i=1}^{n} \langle \mu, \chi_{A_i} \rangle \inf\,\{ \|\chi_{A_i} T^{***} x^{***} - \gamma_i \chi_{A_i}\| : \gamma_i \in \mathbb{R} \} \leq \\ &\quad \leq \operatorname{osc}\,(x^{***} \mid T^{**}(\tilde{V}_{P,\varepsilon}(\mu))) \leq \\ &\quad \leq \sum_{i=1}^{n} \langle \mu, \chi_{A_i} \rangle \inf\,\{ \|\chi_{A_i} T^{***} x^{***} - \gamma_i \chi_{A_i}\| : \gamma_i \in \mathbb{R} \} + \\ &\qquad + 2\varepsilon \|T\|. \end{aligned}$$

We have assembled the tools to prove a result analoguous to 3.3.

<u>6.4 Proposition:</u> Let $T : L^1[0,1] \to X$ be an operator, $f \in F$ and suppose that the restriction of T^{**} to $\tilde{F}$ is weak-star to weak continuous at f. Let $B = f > 0$. Then the restriction of $M = T^*(\text{ball } X^*)$ to B is a set of regular oscillation.

<u>Proof:</u> In view of 6.3 it will suffice to show that, for $\alpha > 0$, M restricted to $B_\alpha = \{f > \alpha\}$ is (RO). Let $z^{**} \in \tilde{M}$. As $\tilde{M} = T^{***}(\text{ball } (X^{***}))$ there is $x^{***} \in \text{ball } (X^{***})$ with $T^{***}(x^{***}) = z^{**}$.

By assumption, for $\delta > 0$, there is a parition $P = (A_1, \dots, A_n)$ and $0 < \varepsilon < \delta/2\|T\|$ such that

$$\operatorname{osc}\,(x^{***} \,|\, T^{**}(\tilde{V}_{P,\varepsilon}(f))) < \delta.$$

There is no loss of generality in assuming that B_α belongs to the σ-algebra generated by P, i.e. there is $I \subset \{1, \dots, n\}$ such that

$$B_\alpha = \bigcup_{i \in I} A_i.$$

By the above remarks we obtain

$$\sum_{i \in I} m(A_i) . \inf \{ \|\chi_{A_i} z^{**} - \gamma_i \chi_{A_i}\| : \gamma_i \in \mathbb{R} \} \leq$$

$$\leq \alpha \Big(\sum_{i \in I} \big(\int_{A_i} f dm \big) . \inf \{ \|\chi_{A_i} z^{**} - \gamma_i \chi_{A_i}\| : \gamma_i \in \mathbb{R} \} \Big) \leq$$

$$\leq \alpha(\delta + 2\varepsilon \|T\|) \leq 2\alpha\delta$$

hence, for $\eta > 0$, there is a subset $J \subset I$ such that

$$\sum_{i\in J} m(A_i) > m(B_\alpha)-\eta$$

and

$$\text{dist}\,(\chi_{A_i} z^{**}, L^\infty[0,1]) \leqq \inf\{\|\chi_{A_i} z^{**} - \gamma_i \chi_{A_i}\| : \gamma_i \in \mathbb{R}\} \leqq$$

$$\leqq 2\alpha\delta\eta^{-1} \qquad \text{for } i \in J.$$

As the choice of $\delta > 0$ is still free, we see that, for $\kappa > 0$ and $j \in \mathbb{N}$, there are finitely many disjoint subsets $A_1^j, \dots, A_{m_j}^j$ of B_α such that for

$$B^j = \bigcup_{i=1}^{m_j} A_i^j$$

we get $m(B^j) > m(B_\alpha) - \kappa . 2^{-j}$ and

$$\text{dist}\,(\chi_{B_j} z^{**}, L^\infty[0,1]) =$$

$$= \sup_{1\leqq i\leqq m_j} \text{dist}\,(\chi_{A_i^j} z^{**}, L^\infty[0,1]) < j^{-1}.$$

The set

$$B^o = \bigcap_{j=1}^{\infty} B^j$$

satisfies $m(B^o) > m(B_\alpha) - \kappa$

and

$$\text{dist}\,(\chi_{B^o} z^{**}, L^\infty[0,1]) = 0$$

i.e. z^{**} restricted to B^o is induced by an element of $L^\infty[0,1]$.

□

We now can state the general theorem analoguous to 3.4:

<u>6.5 Theorem:</u> Let $T : L^1[0,1] \to X$ be an operator. T.f.a.e.

(i) $T^*(\text{ball } (X^*))$ is of regular oscillation (RO);

(ii) T is a regular operator (R);

(iii) T^{**} restricted to $\tilde{F}$ is weak-star to weak continuous at every $f \in F$;

(iii') T^{**} restricted to $\tilde{F}$ is weak-star to weak continuous at $f = 1$.

(iii") For $\varepsilon > 0$ there are $f_1,\dots,f_n \in F$ such that T restricted to $\tilde{F}$ is weak-star to weak continuous at each f_i and $m(\bigcup_{i=1}^{n} \{f_i > 0\}) > 1-\varepsilon$;

(iv) For $A \subset [0,1]$, $m(A) > 0$, $x^{***} \in X^{***}$ and $\varepsilon > 0$ there is a relatively weak-star open subset U of $\tilde{F}_A$ such that $\operatorname{osc}(x^{***} | T^{**}(U)) < \varepsilon$.

<u>Proof:</u>

(i) $\Rightarrow$ (iii): Let $x^{***} \in X^{***}$, $\|x^{***}\| \leq 1$. We have to show that, for $\delta > 0$ there is a partition $P = (A_1,\dots,A_n)$ and $\varepsilon > 0$ such that

$$\operatorname{osc}(x^{***} | T^{**}(\tilde{V}_{P,\varepsilon}(f))) < \delta.$$

In view of the formula preceding proposition 3.4 this will hold true if $\varepsilon > \delta/4\|T\|$ and the partition $P = (A_1,\ldots,A_n)$ satisfies

$$\sum_{i=1}^{n} \left(\int_{A_i} f\,dm\right) \inf\{\|\chi_{A_i} T^{***}x^{***} - \gamma_i\chi_{A_i}\| : \gamma_i \in \mathbb{R}\} < \delta/2.$$

Find $\gamma > 0$ such that the integral of f over a set of measure less than γ is less than $\delta/4\|T\|$ and find - by assumption - a set $A \subset [0,1]$, $m(A) > 1-\gamma$ such that $T^{***}(x^{***})$ restricted to A is induced by some $g \in L^\infty[0,1]$. Let $A_1 = [0,1] \setminus A$ and $(A_2,\ldots,A_n)$ be a partition of A such that for $2 \leqq i \leqq n$, g oscillates less than $\delta/4$ on A_i. We may estimate the above expression by

$$\left(\int_{A_1} f\,dm\right)\|T\| + \sum_{i=2}^{n} \left(\int_{A_i} f\,dm\right).\mathrm{osc}\,(g|A_i) < \delta/4 + \delta/4 = \delta/2$$

thus showing (iii).

(iii) $\Rightarrow$ (iii') $\Rightarrow$ (iii") is obvious and
(iii") $\Rightarrow$ (i) follows from the two preceding propositions.

(i) $\Rightarrow$ (ii): Let C be a convex, bounded subset of $L^1[0,1]$ with $\|C\| = 1$ and $x^{***} \in X^{***}$, $\|x^{***}\| \leqq 1$. We shall show that, for $\varepsilon > 0$, there is a slice S of C such that on the weak-star-closure $\tilde{S}$ of S in $L^1[0,1]^{**}$, the element $z^{**} = T^{***}(x^{***}) \in L^\infty[0,1]^{**}$ oscillates less than ε.

As has been observed in [G-G-M] it follows quickly from Bourgain's lemma 1.2a that it is equivalent to show that there is a relatively weakly open subset U of C such that z^{**} oscillates on $\widetilde{U}$ les than ε.

As we may find sets A of arbitrarily large measure such that z^{**} restricted to A is induced by an element $g \in L^{\infty}[0,1]$ we may find such an $A \subset [0,1]$ and $f \in C$ such that $\|f.\chi_A\|_1 > 1-\varepsilon/4\|T\|$.

We shall show that there is a relative weak neighbourhood W of f in the (bigger) set ball $(L^1[0,1])$ such that z^{**} oscillates on $\widetilde{W}$ less than ε. Indeed let $A_1 = [0,1] \setminus A$ and $(A_2,\dots,A_n)$ a partition of A such that, for $2 \leq i \leq n$, g oscillates less than $\varepsilon/4$ on A_i and either $A_i \subset \{f \geq 0\}$ or $A_i \subset \{f < 0\}$.

$$W = \{h \in \text{ball } (L^1[0,1]) : \sum_{i=1}^{n} |\int_{A_i} (f-h)\,dm| < \varepsilon/4\|T\|\}.$$

Then we may estimate (similarly as in the proof of 3.4).

$$\text{osc } (z^{**}|\widetilde{W}) \leq \sup \{\|h.\chi_{A_1}\|.\|z^{**}\| : h \in W\} +$$

$$+ \sum_{i=2}^{n} (\int_{A_i} f\,dm).\text{osc } (g|A_i) + (\varepsilon/4\|T\|)\|z^{**}\| <$$

$$< \varepsilon/2 + \varepsilon/4 + \varepsilon/4 = \varepsilon,$$

which shows (ii).

(ii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i): In view of the above remark (second paragraph of (i) $\Rightarrow$ (ii)) and proposition 2.2 the assumption implies that there is a (compact) $K \subset A$, $m(K) > 0$ and $\beta > 0$ such that $\operatorname{osc}(x^{***} | T^{**}(\tilde{S}_{K,\beta})) < \varepsilon$; in particular the oscillation of $z^{**} = T^{***}(x^{***})$ on $\tilde{\mathcal{F}}_K$ is less than ε. Repeating the argument of the proof of proposition 6.4 combined with a standard exhaustion argument gives a set $B \subset [0,1]$, $m(B) > 1-\varepsilon$, and such that z^{**} restricted to B is induced by an element of $L^\infty[0,1]$.

$\square$

7. W^*RNP-operators, stable subsets of L^∞ and related notions

In the preceding sections we have given characterisations of the properties (RNP), (PCP), (CPCP), (SR) and (R) of an operator $T : L^1[0,1] \to X$ in terms of the set $M = T^*(\text{ball}\ (X^*))$. As regards the question as to when T is a Dunford-Pettis-operator the situation becomes almost dull: This is easily seen to be equivalent to the $\|.\|_1$-compactness of M. Hence we are left with (W^*RNP) as the last property to investigate: This was done in ([T], 7-4-1) and the condition on M is given by the subsequent definition.

7.1. Definition: A subset M of $L^\infty[0,1]$ is called "pointwise relatively compact under a lifting" (RCL) if

(i) there is a lifting $\rho : L^\infty[0,1] \to \mathcal{L}^\infty[0,1]$ such that $\rho(M)$ is pointwise relatively compact in $\mathcal{L}^\infty[0,1]$.

Under axiom (F), a weakening of Martin's axiom ([T], 9-1-2) this condition is equivalent to any one of the following:

(ii) for every lifting $\rho : L^\infty[0,1] \to \mathcal{L}^\infty[0,1]$, $\rho(M)$ is pointwise relatively compact in $\mathcal{L}^\infty[0,1]$.

(iii) there is a lifting $\rho : L^\infty[0,1] \to \mathcal{L}^\infty[0,1]$ such that $\rho(M)$ is stable ([T], 9-1-1).

(iv) for every lifting $\rho : L^\infty[0,1] \to \mathcal{L}^\infty[0,1]$ the set $\rho(M)$ is stable.

<u>7.2 Remark:</u> The implication (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) hold true without any set-theoretical assumptions (c.f. [T]). The validity of (i) $\Rightarrow$ (iii) follows from axiom (F) and apparently it is unknown whether this implication holds true absolutely (c.f. [T], 9-1-2). Hence the implication (i) $\Rightarrow$ (ii) certainly holds true under axiom F; but it is not clear (to the author) whether (i) $\Rightarrow$ (ii) holds true absolutely.

In the sequel we shall not deal with results depending on special axioms. For example the equivalence of (iii) and (iv) above holds absolutely and we feel that the notion of stability makes perfect sense for equivalence classes of functions, i.e. for subsets of $L^\infty(m)$ (instead of sets of functions, i.e. subsets of $\mathcal{L}^\infty(m)$ (this is implicitly used in [T], 7-4-1 d), for example). The next proposition will clarify the situation: The first two conditions use directly the equivalence classes of functions without the help of a lifting. They emphasize the intuitive idea, that $M \subset L^\infty[0,1]$ is not stable if for almost all (u_q,v_q), $1 \leq q \leq p$ in some set A there is $g \in M$ which oscillates on each of these pairs of points.

We need some notation: For $g \in \mathcal{L}^\infty[0,1]$ and $p \in \mathbb{N}$ define the function $g^{(p)}$ on $[0,1]^{2p}$ by

$$g^{(p)}(u_1,\dots,u_p,v_1,\dots,v_p) = [\min_{1\leq q\leq p} (g(v_q)-g(u_q))]_+$$

Clearly $g^{(p)}$ is an element of $\mathcal{L}^\infty([0,1]^{2p},m^{2p})$ and the equivalence class of $g^{(p)}$ depends only on the equivalence class of g. Hence, for $g \in L^\infty[0,1]$ we may define $g^{(p)} \in L^\infty([0,1]^{2p})$.

<u>7.3 Proposition:</u> For a bounded subset M of $L^\infty[0,1]$ t.f.a.e.

(i) There is (an equivalence class of) a set $A \subset [0,1]$, $m(A) > 0$, and $\alpha > 0$ such that, for $p \in \mathbb{N}$,

$\sup \{g^{(p)} : g \in M\} \geq \alpha.\chi_A 2p,$

the sup taken in the sense of the complete lattice $L^\infty([0,1]^{2p})$.

(ii) There is (an equivalence class of) a set $A \subset [0,1]$, $m(A) > 0$, and $\alpha > 0$ such that, for $p \in \mathbb{N}$ and $\varepsilon > 0$, there are $g_1,\dots,g_n \in M$ with

$$\|\alpha.\chi_A 2p - (\sup_{1\leq i\leq n} g_i^{(p)}) \wedge \alpha.\chi_A 2p\|_{L^1([0,1]^{2p})} < \varepsilon.$$

(iii) For every lifting $\rho : L^\infty[0,1] \to \mathcal{L}^\infty[0,1]$, $\rho(M)$ is not stable.

(iv) There is a lifting $\rho : L^\infty[0,1] \to \mathcal{L}^\infty[0,1]$ such that $\rho(M)$ is not stable.

If M fails the above (equivalent) conditions we call it "<u>stable under a lifting</u>" (SL) or - if there is not danger of confusion - just stable.

<u>Proof:</u> We shall use the following fact: If $(f_\alpha)_{\alpha \in I}$ is an upwards directed bounded system in $L^\infty([0,1]^{2p})$ and f is the supremum, then f_α tends to f with respect to $\|.\|_1$. If $\rho : L^\infty[0,1] \to \mathcal{L}^\infty[0,1]$ is a lifting then also $\rho(f_\alpha)$ (which is well-defined) tends to $\rho(f)$ with respect $\|.\|_1$.

(i) $\Rightarrow$ (ii) follows from this remark by taking $(f_\alpha)_{\alpha \in I}$ to be the finite suprema of $g^{(p)}$.

(ii) $\Rightarrow$ (iii) follows from ([T], 11-1-1). Note however that the "brute force" ([T], p. 212) of this result applied to the present situation is a kind of shooting with canons on pigeons. The alert reader will find a more direct (but longer to write) argument.

(iii) $\Rightarrow$ (iv) is obvious and

(iv) $\Rightarrow$ (i) also follows from the above remark. Indeed, the definition ([T], 9-1-1) of stability implies that - assuming (iv) - there is a set $A \subset [0,1]$, $m(A) > 0$ and $\alpha > 0$ such that every $p \in \mathbb{N}$

$$\sup\,(\rho g^{(p)} : g \in M) \geqq \alpha . \chi_A p$$

holds true on a subset of $[0,1]^{2p}$ of outer measure 1 (the sup taken pointwise).

From the above remark we conclude that $\sup (g^{(p)} : g \in M)$ - this time the sup taken in the lattice $L^\infty([0,1]^{2p})$ - is bigger than $\alpha.\chi_A 2p$.

$\square$

7.4 Remark: Let us announce that it is proved in [S3] that a circled, convex, bounded set $M \subset L^\infty[0,1]$ is not stable iff there is $A \subset [0,1]$, $m(A) > 0$ and $\alpha > 0$ such that for every $\varepsilon > 0$

(i') $\sup \{g^p : g \in M, \|g\|_1 < \varepsilon\} \geqq \alpha.\chi_A 2p.$

This result is parallel to the characterisation of equimeasurable sets mentioned in 4.1 above and its proof uses similar arguments as [S1].

We now introduce the last concept, due to Bourgain (in the unpublished note [B3]; see [R-S] and [T], 9-5-4 for a definition). This concept too was introduced for sets of functions but again we believe that this notion makes perfect sense for sets of equivalence classes of functions.

7.5 Definition: A bounded subset M of $L^\infty[0,1]$ is said to satisfy "Bourgain's condition under a lifting" (BL) if, for (on equivalence class of) a set $A \subset [0,1]$, $m(A) > 0$ and $\varepsilon > 0$, there is a partition $P = (A_1,\ldots,A_n)$ of A into (equivalence classes of) subsets of A, $m(A_i) > 0$, such that for each $g \in M$ there is some $1 \leqq i \leqq n$ such that

$$\text{essential osc } (g|A_i) < \varepsilon.$$

7.6 Remark: It is obvious that M satisfies (BL) iff, for some (equivalently for every) lifting $\rho : L^\infty[0,1] \to \mathcal{L}^\infty[0,1]$, ρM satisfies Bourgain's condition ([T], 9-5-4). If there is no danger of confusion we shall then just say $M \subset L^\infty[0,1]$ satisfies Bourgain's condition (B).

We can now state the theorem clarifying (partially) the relations of the above properties.

7.7 Theorem: For a bounded subset M of $L^\infty[0,1]$ we have the following diagram of implications

$$
\begin{tikzcd}
 & & & \text{stable (SL)} \arrow[lld, Rightarrow, shift left, "/" marking] \arrow[d, Rightarrow, shift right] \\
\text{small oscillation (SO)} \arrow[r, Rightarrow] & \text{Bourgain's condition (BL)} \arrow[rru, Rightarrow, shift left] \arrow[r, Rightarrow] & \text{regular oscillation (RO)} \arrow[r, Rightarrow] & \text{pointwise relatively compact under a lifting (RCL)} \arrow[u, Rightarrow, shift right, "\text{Axiom (F)}"']
\end{tikzcd}
$$

7.8 Remark: We do not know whether (BL) $\Rightarrow$ (SO), (RO) $\Rightarrow$ (BL) and (SL) $\Rightarrow$ (RO), although we know that one of the two latter implications must fail in view of (SL) $\not\Rightarrow$ (BL). We also do not know whether (RO) $\Rightarrow$ (SL) absolutely.

In ([T], 9-5-4) an example is given of a set which is stable but fails Bourgain's condition (B). Note however that this example consists of functions on $[0,1]$ which are different from zero only on finitely many points. Hence it does not fit into our situation where we are dealing with sets of equivalence classes of functions. Hence we believe that our example showing (SL) $\not\Rightarrow$ (BL) does bring new information. Let us point out that we do not know whether our subsequent example $M = (h_{m,j})_{m=1}^{\infty}$ is a set of regular oscillation (RO) or - equivalently - whether the operator of example 4.6 is a regular operator.

Before embarking the proof let us deduce that regular operators $T : L^1[0,1] \to X$ are W^*RNP-operators. The according result for sets was proved - under a technical separability assumption - by Ghoussoub, Godefroy and Maurey ([G-G-M], th. IV.7).

<u>7.9 Theorem:</u> A regular operator $T : L^1[0,1] \to X$ is (W^*RNP). Hence if $D \subset X$ is a convex, bounded, regular set in X such that $T(F) \subset D$ then T is regular and therefore (W^*RNP).

<u>Proof:</u> T is regular iff $T^*(\text{ball }(X^*))$ is (RO) (theorem 6.5) and T is W^*RNP iff $T^*(\text{ball }(X^*))$ is (RCL) ([T], 7-4-1). Hence the first part follows from theorem 7.7 and the second part is obvious in view of theorem 6.5.

□

<u>Proof of theorem 7.7:</u> (SO) $\Rightarrow$ (BL) is obvious.

(BL) $\Rightarrow$ (RO): This argument is due to J. Bourgain and given in [B3]: Let $\varepsilon > 0$ and $(g_\alpha)_{\alpha \in I}$ be an ultrafilter in M converging weak-star to $z^{**} \in \widetilde{M}$. By an exhaustion argument and the property of an ultrafilter we find, for $p \in \mathbb{N}$, finitely many $A_1^p,\dots,A_{n_p}^p$ with

$$m\Big(\bigcup_{i=1}^{n_p} A_i^p\Big) > 1 - \varepsilon/2^p$$

and such that for $\alpha \geqq \alpha_p$ each g_α oscillates on A_i^p less than p^{-1}, for $1 \leqq i \leqq n_p$. Defining

$$A = \bigcap_{p=1}^{\infty} \bigcup_{i=1}^{n_p} A_i^p$$

we obtain a set of measure bigger than $1-\varepsilon$ and such that z^{**} restricted to A is induced by an element of $L^\infty[0,1]$ (compare the proof of 6.4), which shows that M is of regular oscillation (RO).

(RO) $\Rightarrow$ (RCL): Let $\rho : L^\infty[0,1] \to \mathcal{L}^\infty[0,1]$ be a lifting and $(g_\alpha)_{\alpha\in I}$ an ultrafilter in M; we have to show that the pointwise limit g of $(\rho(g_\alpha))_{\alpha\in I}$ is an element of $\mathcal{L}^\infty[0,1]$.

There is a weak-star limit z^{**} of $(g_\alpha)_{\alpha\in I}$ in $\widetilde{M}$. By assumption there is a sequence of (equivalence classes of) sets $A_n \subset [0,1]$, $m(A_n) \to 1$ and elements $g_n \in L^\infty(A_n, m|_{A_n})$ - which we identify with $g_n = g_n\chi_{A_n} \in L^\infty[0,1]$ - such that the restriction of z^{**} to $L^\infty(A_n)^*$ equals g_n. Hence $(\chi_{A_n}\cdot g_\alpha)_{\alpha\in I}$ converges weakly to g_n in $L^\infty[0,1]$ and as ρ is a continuous linear map (in fact an isometric embedding) $\rho(\chi_{A_n}\cdot g_\alpha) = \chi_{\rho(A_n)}\cdot\rho(g_\alpha)$ converges weakly to $\rho(g_n)$ in the Banach space $\mathcal{L}^\infty[0,1]$, hence in particular pointwise. Hence the function g equals on the set $\rho(A_n)$ the Lebesgue-measurable function $\rho(g_n)$ which readily shows that $g \in L^\infty[0,1]$.

(BL) $\Rightarrow$ (SL) is not hard to see (compare [T], 9-5-4) and

(SL) $\Rightarrow$ (RCL) and (RCL) $\Rightarrow$ (SL) under axiom (F) is extensively treated in ([T], chapter 9).

We still have to show that (SL) $\not\Rightarrow$ (BL). This will be done by the sequence $((h_{m,j})_{j=1}^{k_m})_{m=1}^{\infty}$ given in example 4.6. An inspection of the argument given there shows that $M = ((h_{m,j})_{j=1}^{k_m})_{m=1}^{\infty}$ not only fails to be of small oscillation (SO) but fails in fact Bourgain's condition (B).

However, we shall presently see that if there is $p \in \mathbb{N}$ such that

$$\sum_{m=1}^{\infty} k_m^{p-1} < \infty$$

the set M is stable (under a lifting). Let $B_{m,j} = \{h_{m,j}=1\}$ and consider $h_{m,j}^{(p)}$ - with the notation of 7.2 above:

$$h_{m,j}^{(p)}(u_1,\ldots,u_p,v_1,\ldots,v_p) = \begin{cases} 1 & \text{if each } v_p \in B_{m,j} \text{ and} \\ & \text{no } u_q \in B_{m,j} \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\|h_{m,j}^{(p)}\|_{L^1([0,1]^{2p})} \leq k_m^{-p}$$

and

$$\| \sup_{1 \leq j \leq k_m} h_{m,j}^{(p)} \|_1 \leq \sum_{j=1}^{k_m} \|h_{m,j}^{(p)}\|_1 \leq k_m^{-p+1}.$$

Suppose now $M = ((h_{m,j})_{m=1}^{k_m})_{m=1}^{\infty}$ is not stable. It follows easily from the result announced in 7.4 that there is $A \subset [0,1]$, $m(A) > 0$ and $\alpha > 0$ such that for every $m_o \in \mathbb{N}$ and $p \in \mathbb{N}$

$$\sup \{h_{m,j}^{(p)} : 1 \leq j \leq m, \ m \geq m_o\} \geq \alpha . \chi_A 2p,$$

the sup taken in the lattice $L^{\infty}[0,1]$.
If k_m^{-p+1} is summable choose m_o such that

$$\sum_{m=m_o}^{\infty} k_m^{-p+1} < \alpha . m(A)\, 2p.$$

Then we may estimate

$$\|\sup (h_{m,j}^{(p)} : 1 \leq j \leq k_m, \ m \geq m_o)\|_1 \leq \sum_{m=m_o}^{\infty} \sum_{j=1}^{k_m} \|h_{m,j}^{(p)}\|_1 \leq$$

$$\leq \sum_{m=m_o}^{\infty} k_m^{-p+1} < \|\alpha . \chi_A 2p\|_1 .$$

This contradiction shows that M is indeed stable.

We shall give yet another proof that M is stable which is technically a little more involved but does not make use of the result announced in 7.4: Suppose again M is not stable and find $A \subset [0,1]$, $m(A) > 0$ and $\alpha > 0$ such that, for $p \in \mathbb{N}$

$$\sup \{h_{m,j}^{(p)} : 1 \leq j \leq k_m,\ m \in \mathbb{N}\} \geq \alpha.\chi_A 2p.$$

Fix $p \in \mathbb{N}$ such that k_m^{-p+1} is summable. We may find $m_o \in \mathbb{N}$ and an

$$B = B_{i_1,\dots,i_{m_o}} = \{(j_m)_{m=1}^{\infty} \in \Delta : j_m = i_m \text{ for } 1 \leq m \leq m_o\},$$

where $1 \leq i_1 \leq k_1,\dots,1 \leq i_{m_o} \leq k_{m_o}$ such that

$$m(B \cap A) > m(B)/2.$$

We may choose m_o big enough such that

$$\sum_{m=m_o}^{\infty} k_m^{-p-1} < \alpha/2^{2p}.$$

Let A_1 be the set $A \cap B$ and note that we get for A_1 the estimate

$$\sup \{h_{m,j}^{(p)}.\chi_B 2p : 1 \leq j \leq k_m, m \in \mathbb{N}\} \geq \alpha.\chi_{A_1} 2p.$$

Note that for $m \leq m_o$ the $h_{m,j}$ do not oscillate on B hence $h_{m,j}^{(p)}.\chi_B 2p$ equals zero. For $m > m_o$ $h_{m,j}.\chi_B 2p(u_1,\dots,u_m,v_1,\dots,v_m) = 1$ implies that each $v_q \in B \cap B_{m,j}$. Having in mind that

$$m(B \cap B_{m,j}) = m(B)/k_m \quad \text{for } m > m_o \text{ and } 1 \leq j \leq k_{m_o}.$$

we may estimate

$$\| h_{m,j}^p.\chi_B 2p \|_1 \leq m(B)^{2p}/k_m^p.$$

Hence

$$\|\sup (h^{(p)}_{m,j} \cdot \chi_B 2p : 1 \leqq j \leqq k_m, m \in \mathbb{N})\|_1 \leqq$$

$$\leqq \sum_{m=m_o+1}^{\infty} \sum_{j=1}^{k_m} m(B)^{2p}/k_m^p =$$

$$= m(B)^{2p} . \sum_{m=m_o+1}^{\infty} k_m^{-p+1} <$$

$$< \alpha . (m(B)/2)^{2p} \leqq \|\alpha . \chi_A 2p\|_1 .$$

This contradiction shows that M is stable. It also finishes the proof of 7.7 and - last not least - finishes this paper.

□

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