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ARBITRAGE AND FREE LUNCH WITH BOUNDED RISK FOR UNBOUNDED CONTINUOUS PROCESSES

Delbaen Freddy
Institute of Actuarial Sciences
Departments of Mathematics
Vrije Universiteit Brussel
Pleinlaan 2
B-1050 Brussels
Belgium

Schachermayer Walter
Universität Wien
Institut für Mathematik
Strudlhofgasse 4
Boltzmanngasse 9
A-1090 Wien
Austria

1. Introduction and Notation.

Let $(S_t)_{0\leq t}$ be a real valued stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$. The filtration $(\mathcal{F}_t)_{0\leq t}$ is the natural filtration generated by the process S. We will also need the following concept, often used in the theory of stochastic integration. See Protter (1990) for more details.

Definition:
We say that a process $H : [0,1] \times \Omega \to \mathbf{R}$ is simple predictable if it is a finite linear combination of processes of the form. $f\, 1_{]T_1, T_2]}$ where $0 \leq T_1 \leq T_2 \leq 1$ are stopping times and f is $\mathcal{F}_{T_1}$ measurable (not necessarily bounded).

The stochastic integral $(H \bullet S)$ where $H = f\, 1_{]T_1, T_2]}$ is defined as $(H\bullet S)_t = f\,(S_{t\wedge T_2} - S_{t\wedge T_1})$. The set K is defined as $K = \{ (H\bullet S)_1 \mid H$ is simple predictable $\}$. The vector space K is a subspace of L^0, the vector space of all $\mathcal{F}_1$ measurable random variables equipped with the topology of convergence in

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measure. We identify random variables that are equal almost everywhere. The cone of nonnegative random variables of L^0 is denoted L^0_+.

If $(S_t)_{0\le t\le 1}$ models the discounted price process of a securities market, then the elements of K can be seen as the net gain or loss of the simple strategy H applied to the process $(S_t)_{0\le t\le 1}$. We say that $(S_t)_{0\le t\le 1}$ satisfies (NA) (which stands for no-arbitrage) if $K \cap L^0_+=\{0\}$. In other words if there is no simple strategy which yields a strictly positive gain with strictly positive probability, while the probability of loosing money is zero. We say that $(S_t)_{0\le t\le 1}$ satisfies (NFLBR) (which stands for no free lunch with bounded risk) if there does not exist a sequence $(f_n)_{n\ge 1}$ such that

(i) $f_n \ge -1$ **P** a.s.

(ii) $\lim_{n\to\infty} f_n = f_0$ **P** a.s.

(iii) $f_0 : \Omega \to [0,+\infty]$ and $\mathbf{P}\,[f_0 > 0] > 0$.

For the development and the use of the above notions we refer to Kreps (1981), Mc Beth (1991), Delbaen (1992) and Schachermayer (1992). Let us note that a continuous process $(S_t)_{0\le t\le 1}$ satisfies (NFLBR) if and only if there does not exist a sequence $(f_n)_{n\ge 1}$, $\| f_n \|_\infty \le 1$ and such that $E[f_n^-] \to 0$ while $E[f_n^+]$ does not, as may be easily seen from a stopping time argument. It was in this sense that (NFLBR) was used in Delbaen (1992). Let us also note that the notions (NA) and (NFLBR) depend on the choice of the integrands H we allow to be used. In the present paper we use simple predictable integrands defined above. The use of stopping times in financial modelling is perfectly allowed since it corresponds to real life situations: act on the market if certain events only depending on past information arise. For instance $H = f\, 1_{]T_1, T_2]}$ means that we buy f units at time T_1 and sell when time T_2 has come. It was shown in Delbaen (1992) that for continuous bounded processes S, the notion of (NFLBR) was equivalent to the existence of a martingale measure for S. An analogous result, without boundedness assumptions was shown in Schachermayer (1992) for processes indexed by a discrete and infinite time set. One may ask what happens for continuous processes which are unbounded. From Delbaen (1992) we may conclude that (NFLBR) still implies the existence of an equivalent local martingale measure. See Schachermayer (1992) for more details. In the present note we give two examples that clarify the situation when $(S_t)_{0\le t\le 1}$ is continuous but unbounded.

Example 1 is a $\mathbf{R}_+$ valued process $(S_t)_{0\le t\le 1}$ which has a unique equivalent local martingale measure, has moments of all orders but fails to have an equivalent martingale measure. The process $(S_t)_{0\le t\le 1}$ allows arbitrage in the sense that it fails (NA). This shows in particular that the existence of a local martingale measure does not imply (NA).

Example 2 is somewhat more subtle: the process $(R_t)_{0\le t\le 1}$ again has a unique local martingale measure but has no martingale measure. This time however the process $(R_t)_{0\le t\le 1}$ does not allow arbitrage profits and in fact it satisfies (NFLBR). This shows that for unbounded processes (NFLBR) only implies the existence of an equivalent local martingale measure.

Both examples are well known in the theory of continuous stochastic processes. See Revuz-Yor (1990) for details on these matters. We also refer to this book for the theory of Markov processes that is used in one of our proofs. For instance the measure $\mathbf{P}_x$ denotes the distribution, defined on the space of all continuous functions from $\mathbf{R}_+$ to $\mathbf{R}_+$, of a Markov process starting at the point x of $\mathbf{R}_+$.

2. The Examples.

Example 1:

Let $(B_t)_{0\le t}$ be a one dimensional Brownian motion starting at $B_0 = 1$. We stop the process B when it hits the level 0 for the first time. So let T be defined as $T= \inf\{t \mid B_t = 0\}$. It is well known that T is finite almost surely. The process $(S_t)_{0\le t\le 1}$ is now defined as

$$S_t = B_{\tan(t^*(\pi/2))\wedge T} \text{ for } t<1 \text{ and}$$
$$S_1 = B_T = 0 \text{ for } t=1.$$

The filtration $(\mathcal{F}_t)_{0\le t<1}$ is the filtration generated by the process $(S_t)_{0\le t\le 1}$. The process $(S_t)_{0\le t\le 1}$ has a unique local martingale measure. The sequence T_n defined as the first hitting time of the level n is a localizing sequence and the measure $\mathbf{P}$ is the only measure that turns $(S_t)_{0\le t\le 1}$ into a local martingale. An easy way to see this is to use the martingale representation theorem Revuz-Yor p187 theorem 3.4, then apply exercise 4.12 p 199, the reasoning is not affected by stopping the processes at time T. The variables $(S_t)_{0\le t\le 1}$ have moments of all orders but they do not form a martingale. To see that there is arbitrage take H to be equal to -1 on the set]0,1]. The outcome is the function $(H\bullet S)_1=1$ a.e.. The process $(S_t)_{0\le t\le 1}$ is only a local martingale on [0,1], but it is a martingale on [0,1[.

Related to the above example and of the same nature is the process $(S_t)_{0\le t\le\infty}$ defined as

$S_t = \exp (B_t - \frac{1}{2} t)$ for $t < \infty$

$S_\infty = 0$

The process is sometimes called the exponential of the Brownian motion. It corresponds to the process derived from a geometric random walk seen under an equivalent martingale measure. This process is often used in mathematical finance. The presence of arbitrage shows that some care has to be taken when dealing with infinite time horizon problems. It is easily seen that the process $(S_t)_{0\le t\le\infty}$ is only a local martingale on $[0,\infty]$. It is however a martingale on $[0,\infty[$.

Example 2:

The second example can most easily be introduced using a three dimensional Brownian motion $(X_t)_{0\le t}$ starting at the point (1,0,0). Proposition 2.7 p179 in Revuz-Yor (1990) shows that X will not hit the origin with certainty. Therefore the process $(R_t)_{0\le t}$ can be defined as $R_t = \frac{1}{\| X_t \|}$. (The $\| \ \|$ denotes the Euclidean norm on $\mathbf{R}^3$). The process R is a typical example of a local martingale which is not a martingale. (see Revuz-Yor (1990): exercises 2.13 and 2.14 p182). The family of random variables $(R_t)_{0\le t}$ is even uniformly integrable and each $R_t \in L^2$. (Even in L^p for every $p<3$). The process also satisfies a stochastic differential equation, namely $dR_t = -R_t^2\, d\beta_t$, where $(\beta_t)_{0\le t}$ is a one dimensional Brownian motion. The filtration $(\mathcal{F}_t)_{0\le t}$ is defined to be the natural filtration generated by the process $(R_t)_{0\le t}$. It is also equal to the natural filtration generated by the Brownian motion β. Because the process R is never zero and because of theorem 3.4 p 187 in Revuz-Yor (1990) applied to β, there is no local martingale L with $\langle R,L\rangle = 0$. It follows that the only probability measure equivalent to **P** and for which R remains a local martingale, is precisely **P**. This implies that R does not allow a non trivial equivalent martingale measure. The process R is also a

Markov process. To facilitate the notation of the proofs we will make use of the process R as defined on $[0,\infty[$. The process that will serve as a counter-example is the restriction of R to the time interval $[0,1]$ We recall that the subspace K of L^0 is defined as $K = \{ (H\bullet R)_1 \mid H \text{ is simple predictable} \}$

Proposition

R satisfies (NA):

$$K \cap L^0_+ = \{0\}$$

Proof

The process R is positive and is a local martingale. It is therefore a super martingale and for each stopping time T we have that R_T is integrable. In fact $\mathbf{E}[R_T] \le 1$. To prove the non arbitrage relation for R we first show that if $T_1 \le T_2 \le 1$ are stopping times, then the variable $R_{T_1} - R_{T_2}$ cannot be a nonnegative random variable unless it vanishes a.e..

Suppose on the contrary that $R_{T_1} - R_{T_2} \ge 0$ a.e. and $\mathbf{P}[R_{T_1} - R_{T_2} > 0] > 0$ We first take M such that $\mathbf{P}[R_{T_1} \le M, R_{T_1} > R_{T_2}] > 0$. Such a real number M clearly exists since $R_{T_1} < \infty$ a.e.. Since $R_{T_2} \le R_{T_1}$ a.e. this implies $R_{T_2} \le M$ a.e. on $\{R_{T_1} \le M\}$. Let us put $A = \{R_{T_1} \le M\} \cap \{T_1 < T_2\} \in \mathcal{F}_{T_1}$. We now define T as

$$T = \inf\{ t \mid t \ge T_1 \ R_t \ge 2M \} \wedge 1$$

We first show that $\mathbf{P}[A \cap \{T<T_2\}] = 0$. To do so we will apply the strong Markov property of the process R at the stopping time T. Since $A \in \mathcal{F}_{T_1}$ we also have that $A \cap \{T<T_2\} \in \mathcal{F}_T$. The set $\mathbf{1}_{\{T<T_2\}}$ is clearly contained in the set $\{ \exists \text{ u such that } T \le u \le T+1 \text{ and } R_u \le M\} = \{ \inf_{T\le u\le T+1} R_u \le M\}$

Now

$$\mathbf{P}[A \cap \{T<T_2\}] = \int_{A \cap \{T<T_2\}} dP\, \mathbf{1}_{\{T<T_2\}}$$

$$\le \int_{A \cap \{T<T_2\}} dP\, \mathbf{1}_{\{\inf_{T\le u\le T+1} R_u \le M\}}$$

$$\leq \int_{A \cap \{T<T_2\}} dP\, E[1_{\{\inf_{T\leq u\leq T+1} R_u \leq M\}} \mid \mathcal{F}_T]$$

$$\leq \int_{A \cap \{T<T_2\}} dP\, P_{2M}[\inf_{0\leq u\leq 1} R_u \leq M] \qquad \text{by the Markov property}$$

$$\leq P\,[A \cap \{T<T_2\}]\; P_{2M}[\inf_{0\leq u\leq 1} R_u \leq M]$$

But $P_{2M}[\inf_{0\leq u\leq 1} R_u \leq M] < 1$ since it is the probability that a 3-dimensional Brownian motion starting at $\left(\frac{1}{2M}, 0, 0\right)$ becomes bigger than $\frac{1}{M}$ in the time interval [0,1]. Therefore $P\,[A \cap \{T<T_2\}] = 0$. It follows that $1_A R_u \leq 2M$ on $[T_1, T_2]$. But this implies that the process defined as

$$\begin{aligned} Y_t(\omega) &= 0 \text{ for } t \leq T_1 \\ &= R_t(\omega) - R_{T_1}(\omega) \text{ for } T_1 \leq t \leq T_2 \text{ and } \omega \text{ in } A, \text{ zero elsewhere} \\ &= R_{T_2}(\omega) - R_{T_1}(\omega) \text{ for } t \geq T_2 \text{ and } \omega \text{ in } A, \text{ zero elsewhere} \end{aligned}$$

is a bounded local martingale and therefore a martingale.

It is therefore clear that $R_{T_2} \leq R_{T_1}$ implies that $R_{T_2} = R_{T_1}$ a.e..

We just proved that for $T_1 \leq T_2$ the variable $R_{T_1} - R_{T_2}$ cannot be in $L^0_+ \setminus \{0\}$.

Since R is a supermartingale it is obvious that also $R_{T_2} - R_{T_1}$ cannot be in $L^0_+ \setminus \{0\}$ either. From this we easily deduce that for all f, $\mathcal{F}_{T_1}$ measurable, the variable $f.(R_{T_2} - R_{T_1})$ cannot be in $L^0_+ \setminus \{0\}$.

We now show that this already implies that $K \cap L^0_+ = \{0\}$.

Indeed suppose that $K \cap L^0_+ \neq \{0\}$. By definition there is a simple predictable H of the form:

$$H = \sum_{k=1}^{n} f_k\, 1_{]T_k T_{k+1}]} \qquad f_k\ \mathcal{F}_{T_k} \text{ measurable}$$

$$0 \leq T_1 \leq T_2 \ldots \leq T_{n+1} \leq 1$$

where $g = (H\bullet R)_1 \geq 0$ and $\mathbf{P}[g > 0] > 0$. Let moreover n be the smallest natural number for which such a construction is possible. We claim that $H' = \sum_{k=1}^{n-1} f_k \mathbf{1}_{]T_k T_{k+1}]}$ already does the job, a contradiction to the choice of n.

Let $A = \{ (H\bullet S)_{T_n} < 0 \} \in \mathcal{F}_{T_n}$. Since $(H\bullet S)_{T_{n+1}} \geq 0$ a.e. we necessarily have $f_n \mathbf{1}_{]T_n T_{n+1}]} > 0$ on A. Unless $\mathbf{P}[A] = 0$ this is impossible by part 1 of the proof. Therefore $(H\bullet S)_{T_n} \geq 0$ a.e.. If $(H\bullet S)_{T_n} = 0$ a.e. then again we would have $f_n (R_{T_{n+1}} - R_{T_n}) \geq 0$ a.e. and > 0 on a set of positive measure, again a contradiction to part 1. Therefore $\mathbf{P}[(H\bullet S)_{T_n} > 0] > 0$ and $H' = \sum_{k=1}^{n-1} f_k \mathbf{1}_{]T_k, T_{k+1}]}$ did the job! q.e.d

Theorem

The process R satisfies (NFLBR):

> If (f_n) is a sequence in K, $f_n \geq -1$ a.e. and $f_n \to f_0$ in probability then $f_0 \geq 0$ implies $f_0 = 0$ a.e..

Proof

From the (NA) property it follows in the same way as in Delbaen(1992) that if H is simple predictable such that $(H\bullet R)_1 \geq -1$, then for all T stopping time $(H\bullet R)_T \geq -1$ a.e.., hence the process $(H\bullet R)$ is bounded below by -1. Since R is a local martingale $H\bullet R$ is a local martingale and since it is bounded below, it is a supermartingale. Therefore $\mathbf{E}[(H\bullet R)_1] \leq 0$. From Fatou's lemma it follows that $\mathbf{E}[f_0] \leq \underline{\lim}\, \mathbf{E}[f_n] \leq 0$ which finishes the proof. q.e.d.

The next proposition shows that when net gains are not supposed to be bounded below, the closure can become very big. This already happens when we restrict the integrands or strategies to be defined by fixed times instead of stopping times. To make the notation easier let us introduce:

H is a step function if H is a linear combination of processes of the form $f\, \mathbf{1}_{]t_1, t_2]}$ where $0 \leq t_1 \leq t_2 \leq 1$ are deterministic times and f is $\mathcal{F}_{t_1}$ measurable and bounded. These integrands were used in Stricker (1990). The set K' is defined as $K' = \{ (H \bullet R)_1 \mid H \text{ is a step function}\}$.

Proposition

i) $\overline{K \cap L^\infty} \cap L^\infty_+ = \{0\}$ The bar denotes the closure with respect to $\sigma(L^\infty, L^1)$.

ii) $\overline{K' \cap L^p} \cap L^p_+ \neq \{0\}$ The bar denotes L^p closure, p<3.

Proof:
The first statement follows from the preceding theorem. An element of $K \cap L^\infty$ has a negative expected value and therefore the same is true for elements in $\overline{K \cap L^\infty}$. The second statement is less trivial. If the intersection would only contain {0}, then we could apply theorem 3 of Stricker (1990). The conclusion would yield the existence of an equivalent martingale measure Q with density $\frac{dQ}{dP}$ in L^q $(q = \frac{p}{p-1})$. The only equivalent local martingale measure is however P itself and this is not an equivalent martingale measure. q.e.d.

3. Concluding Remarks.

Let us come back to the economic interpretation of the processes S and R of example 1 and 2. As both are $\mathbf{R}_+$ valued, they may be interpreted as the price process of a stock.

Intuitively both are loosing stocks as the processes are supermartingales but not martingales. Hence an arbitrageur would try to make arbitrage profits by going short on these stocks. (One should be careful in this kind of reasoning: for a supermartingale it is perfectly possible that there is an equivalent martingale measure.)

In the first example he or she may do so successfully. Just go short on one unit of the stock at time t=0. At time t=1 he or she closes the position by selling (=throwing away since the price =0) the stock. The net gain is one unit of money. Nevertheless the stock is locally fair since it is a local martingale. This is reflected by the fact that the arbitrageur must have very good nerves (or a good sleep). While t ranges through [0,1[the stock S may become very valuable and therefore the short position will have large negative values. Indeed it is well known that $E\left[\sup_{0\le t\le 1} S_t\right] = \infty$. Note by the way that it is only when the stock price reaches zero that the arbitrageur may be certain that his position is safe.

The second example is different: again the stock is loosing in the sense that R is a supermartingale with strictly decreasing expectation. But here an arbitrageur, even if he or she has good nerves, has no chance of winning the game. Whenever he or she goes short on the stock R, then with positive probability a loss will occur which by using simple strategies, cannot be recovered before time 1. In fact one may not even approximate a riskless profit by a sequence of simple strategies if one requires that the maximal loss is bounded by a uniform constant. The final proposition shows however that in L^p $(p<3)$, a positive function can be constructed via L^p approximations with simple strategies. It is a challenging exercise for the reader to construct such a sequence explicitly.

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