An Inequality for the Predictable Projection of an Adapted Process

F. DELBAEN AND W. SCHACHERMAYER

Department of Mathematics, Vrije Universiteit Brussel

Institut für Statistik der Universität Wien

ABSTRACT. Let $(f_n)_{n=1}^N$ be a stochastic process adapted to the filtration $(\mathcal{F}_n)_{n=0}^N$. Denoting by $(g_n)_{n=1}^N$ the predictable projection of this process, i.e., $g_n = E_{n-1}(f_n)$ we show that the inequality

$$\left[E(\sum_{n=1}^N \mid g_n \mid^q)^{p/q}\right]^{1/p} \leq 2\left[E(\sum_{n=1}^N \mid f_n \mid^q)^{p/q}\right]^{1/p}$$

or, in more abstract terms

$$\|(g_n)_{n=1}^N\|_{L^p(l_N^q)} \leq 2\|(f_n)_{n=1}^N\|_{L^p(l_N^q)}$$

holds true for $1 \leq p \leq q \leq \infty$ (with the obvious interpretation in the case of $p = \infty$ or $q = \infty$).

Several similar results, pertaining also to the case $p > q$, are known in the literature. The present result may have some interest in view of the following reasons: (1) the case $p = 1$ and $2 < q \leq \infty$ seems to be new; (2) we obtain 2 as a uniform constant which is sharp in the case $p = 1$, $q = \infty$ and (3) the proof is very easy.

We denote by $(\mathcal{F}_n)_{n=0}^N$ a filtration on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let E_n be the conditional expectation with respect to $\mathcal{F}_n$. Let $(f_n)_{n=1}^N$ be a stochastic process which will be assumed in most of this note to be adapted to $(\mathcal{F}_n)_{n=1}^N$ and to satisfy the appropriate integrability conditions so that the subsequent statements make sense; we denote by $(g_n)_{n=1}^N$ the predictable projection of $(f_n)_{n=1}^N$, i.e., $g_n = E_{n-1}(f_n)$.

For $1 \leq p, q \leq \infty$ we define

$$\|(f_n)_{n=1}^N\|_{p,q} = \|(f_n)_{n=1}^N\|_{L^p(l_N^q)} = \left[E(\sum_{n=1}^N \mid f_n \mid^q)^{p/q}\right]^{1/p}.$$

We shall prove the following inequality.

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1. Lemma. *For $1 \leq p \leq q \leq \infty$ and an adapted process $(f_n)_{n=1}^N$ we have*

$$\|(g_n)_{n=1}^N\|_{L^p(l^q_N)} \leq 2\|(f_n)_{n=1}^N\|_{L^p(l^q_n)}.$$

The constant 2 is sharp for the case $p = 1, q = \infty$.

Let us first give some motivation for this inequality and relate it to known results.

Our interest in this inequality stems from an application in the field of Mathematical Finance [D-S], where the present authors needed the case $p = 2, q = \infty$. Note that the case $q = \infty$ may be rephrased in terms of the maximal functions f^* and g^* of the processes $(f_n)_{n=1}^N, (g_n)_{n=1}^N$, while the case $q = 2$ is related to the square function.

M. Yor kindly pointed out a possible connection to inequality 3 below ([L],[Y]) and S. Kwapień, P. Müller and G. Schechtman pointed out other known variants of the above inequality. Special thanks go to S. Kwapień who suggested to us — among other valuable remarks — how to adapt the proof from the case $p = 1, q = \infty$ to the general situation $1 \leq p \leq q \leq \infty$.

For the convenience of the reader we summarize the existing results. The starting point is the subsequent Stein's inequality ([S], th. 3.8).

2. The Inequality of E. Stein. *Let $1 < p < \infty$ and $1 \leq q \leq \infty$ and $(f_n)_{n=1}^N$, a (not necessarily adapted) process. There is a constant $C_p > 0$ such that*

$$\|(g_n)_{n=1}^N\|_{p,q} \leq C_p\|(f_n)_{n=1}^N\|_{p,q}.$$

In fact, Stein formulated the above result for $q = 2$ only, but his proof shows the result for all $1 \leq q \leq \infty$ (see [D] for an exposition of this more general setting).

The setting of Stein's inequality is more general than Lemma 1 in the sense that it does not require that $(f_n)_{n=1}^N$ is adapted to the filtration $(\mathcal{F}_n)_{n=1}^N$ and it also pertains to the case $p > q$. On the other hand, as shown by easy examples [S], Stein's inequality breaks down as $p \to 1$ (and $q \neq 1$) and $p \to \infty$ (and $q \neq \infty$).

Stein's inequality was also considered by W. Johnson, B. Maurey, G. Schechtman and L. Tzafriri [J-M-S-T], who extended it by replacing L^p by a rearrangement invariant space X on [0,1], for which the Boyd indices verify $0 < \beta_X \leq \alpha_X < 1$.

D. Lépingle and M. Yor noted that there is an interesting difference in the problem of the constant C_p in Stein's inequality between $p \to 1$ and $p \to \infty$. In the case $p \to \infty$ (and $q \neq \infty$) it is easy to give examples $(f_n)_{n=1}^N$ of *adapted* processes for which C_p becomes big as $p \to \infty$ (see [S] and [L]). For the convenience of the reader we sketch the typical example: Let $f_n = 2\chi_{A_n}$, where $(A_n)_{n=1}^N$, are disjoint sets with $P(A_n) = 2^{-n}$. Letting $\mathcal{F}_n = \sigma(f_1, \ldots, f_n)$ we obtain that $\|(f_n)_{n=1}^N\|_{\infty,q} = 2$, while $\|(g_n)_{n=1}^N\|_{\infty,q} = N^{\frac{1}{q}}$, which tends to infinity if $q < \infty$.

On the other hand, the situation is different for $p \to 1$ and here more can be said for *adapted* processes (see e.g.,[J], th. 1.6):

3. Inequality of D. Lépingle and M. Yor. *Let $p = 1, q = 2$ and $(f_n)_{n=1}^N$ be adapted to $(\mathcal{F}_n)_{n=1}^N$. Then*

$$\|(g_n)_{n=1}^N\|_{p,2} \leq 2\|(f_n)_{n=1}^N\|_{p,2}.$$

In a remarkable paper by J. Bourgain ([B], prop. 5) the same result was obtained with a constant $K_1 = 3$ as an auxiliary step (preceded by the phrase: "The next inequality is probably known, but we include its proof here for the sake of completeness").

It is interesting to compare the proofs: E. Stein uses Doob's maximal inequality and interpolation, Lépingle and Yor adapted the argument of C. Herz for the proof of the $H^1 - BMO$ duality (see Garsia [G], p.9) while J. Bourgain directly uses the atomic decomposition of H^1. In the two last proofs it seems crucial to restrict to the case $q = 2$. Let us also point out that in the case $1 < p < \infty$ the above inequality may also be deduced from a convexity lemma of D. Burkholder ([Bu], Lemma 16.1); it also can be derived in full generality (i.e., also in the case $p = 1$, but with a constant which is possibly worse) from results on decoupled conditionally independent tangent sequences as presented in the book of S. Kwapień and W. Woyczyński ([K-W], th. 5.2.1 and 5.6.2). See also S. Dilworth [D] for a clear exposition of the case $1 < p < \infty$.

We now pass to the proof of Lemma 1.

Proof of Lemma 1. We first prove the case $p = 1$. For $1 \leq q < \infty$ the function

$$\varphi_q(x, h) = (x^q + h^q)^{\frac{1}{q}} - h \qquad x \geq 0, h \geq 0$$

and, in the case $q = \infty$, the function

$$\varphi_\infty(x, h) = (x \vee h) - h = (x - h)_+ \qquad x \geq 0, h \geq 0$$

is, for fixed $h \geq 0$, convex in $x \in \mathbb{R}$.

If $\mathcal{G}$ is a σ-algebra contained in the σ-algebra $\mathcal{F}$, $f \in L^1(\mathcal{F}), h \in L^1(\mathcal{G})$, then by applying the conditional version of Jensen's inequality we have for each $1 \leq q \leq \infty$

$$\varphi_q(\mathbb{E}(f \mid \mathcal{G}), h) \leq \mathbb{E}(\varphi_q(f, h) \mid \mathcal{G}) \qquad \text{P.a.s.}$$

Hence, denoting by g the conditioned expectation of f with respect $\mathcal{G}$,

$$g = \mathbb{E}(f \mid \mathcal{G}),$$

we obtain, for $1 \leq q < \infty$,

$$\mathbb{E}(\varphi_q(g, h)) \leq \mathbb{E}(\varphi_q(f, h))$$

and, for $q = \infty$,

$$\mathbb{E}((g - h)_+) \leq \mathbb{E}((f - h)_+).$$

Now fix an adapted sequence $(f_n)_{n=1}^N$ and its predictable projection $(g_n)_{n=1}^N$ as in the statement of Lemma 1. We have to show that, for $1 \leq q \leq \infty$

$$\|(g_1, \ldots, g_N)\|_{L^1(l_N^q)} \leq 2\|(f_1, \ldots, f_N)\|_{L^1(l_N^q)}$$

which will readily finish the proof.

We may assume that $f_n \geq 0$. Denote, for $1 \leq n \leq N, 1 \leq q < \infty$,

$$\begin{aligned}
\hat{f}_n &= (f_1^q + \cdots + f_n^q)^{\frac{1}{q}}, \\
\hat{g}_n &= (g_1^q + \cdots + g_n^q)^{\frac{1}{q}}, \\
\hat{h}_n &= (f_1^q + g_1^q + \cdots + f_n^q + g_n^q)^{\frac{1}{q}},
\end{aligned}$$

and, in the case $q = \infty$,

$$f_n^* = f_1 \vee \cdots \vee f_n,$$
$$g_n^* = g_1 \vee \cdots \vee g_n,$$
$$h_n^* = f_n^* \vee g_n^*.$$

In order to show that

$$\mathbb{E}(\hat{g}_N) \leq 2\mathbb{E}(\hat{f}_N) \quad \text{and} \quad E(g_N^*) \leq 2E(f_N^*)$$

we shall proceed inductively showing that, in fact, for each $n = 1, \ldots, N$

$$\mathbb{E}(\hat{h}_n) \leq 2\mathbb{E}(\hat{f}_n) \quad \text{and} \quad E(h_n^*) \leq 2E(f_n^*), \qquad (*)$$

For $n = 1$ this holds true as $h_1^* \leq \hat{h}_1 \leq f_1 + g_1$ and $E(f_1) = E(g_1)$.

So suppose that $(*)$ holds true for $n-1$. We give the proof for the cases $1 \leq q < \infty$ and $q = \infty$ separately. In the case $1 \leq q < \infty$ we obtain from the argument at the beginning of the proof that

$$\mathbb{E}(\varphi_q(g_n, \hat{h}_{n-1}) \leq \mathbb{E}\varphi_q(f_n, \hat{h}_{n-1}).$$

Note that

$$\hat{f}_n = \hat{f}_{n-1} + (f_n^q + \hat{f}_{n-1}^q)^{\frac{1}{q}} - \hat{f}_{n-1}$$
$$= \hat{f}_{n-1} + \varphi_q(f_n, \hat{f}_{n-1}).$$

As, for fixed $x \geq 0$, the function $\varphi_q(x, h)$ is decreasing in h, we may estimate

$$\hat{h}_n \leq \hat{h}_{n-1} + \varphi_q(f_n, \hat{h}_{n-1}) + \varphi_q(g_n, \hat{h}_{n-1})$$

so that

$$\begin{aligned}\mathbb{E}(\hat{h}_n) &\leq \mathbb{E}(\hat{h}_{n-1}) + 2\mathbb{E}\varphi_q(f_n, \hat{h}_{n-1}) \\ &\leq 2[\mathbb{E}(\hat{f}_{n-1}) + \mathbb{E}\varphi_q(f_n, \hat{f}_{n-1})] \\ &= 2\mathbb{E}(\hat{f}_n).\end{aligned}$$

In the case $q = \infty$ we proceed similarly. From the argument of the beginning of the proof, we obtain that

$$E\left[(g_n - h_{n-1}^*)_+\right] \leq E\left[(f_n - h_{n-1}^*)_+\right].$$

Noting that

$$f_n^* = f_{n-1}^* + (f_n - f_{n-1}^*)_+$$

and

$$h_n^* \leq h_{n-1}^* + (f_n - h_{n-1}^*)_+ + (g_n - h_{n-1}^*)_+$$

we obtain that

$$\begin{aligned} E(h_n^*) &\leq E(h_{n-1}^*) + 2E\left[(f_n - h_{n-1}^*)_+\right] \\ &\leq 2\left[E(f_{n-1}^*) + E(f_n - f_{n-1}^*)_+)\right] \\ &= 2E(f_n^*), \end{aligned}$$

which finishes the proof in the case $p = 1$.

For the case $1 < p \leq q \leq \infty$ fix an adapted sequence $(f_n)_{n=1}^N, f_n \in L^p(\mathcal{F}_n)$ and $(g_n)_{n=1}^N$ its predictable projection. Letting $\bar{f}_n = f_n^p \in L^1(\mathcal{F}_n), \bar{g}_n = g_n^p \in L^1(\mathcal{F}_{n-1})$, note that by Jensen's inequality $\bar{g}_n \leq \mathbb{E}(\bar{f}_n \mid \mathcal{F}_{n-1})$. Apply the first part of the proof to $r = \frac{q}{p}$ to obtain

$$\mathbb{E}(\bar{g}_1^r + \cdots + \bar{g}_N^r)^{\frac{1}{r}} \leq 2\mathbb{E}(\bar{f}_1^r + \cdots + \bar{f}_N^r)^{\frac{1}{r}}$$

or

$$\mathbb{E}(g_1^q + \cdots + g_N^q)^{\frac{1}{q}} \leq 2^{\frac{1}{p}}\mathbb{E}(f_1^q + \cdots + f_N^q)^{\frac{1}{q}}$$

which shows Lemma 1 also for the case $1 < p \leq q \leq \infty$, with the constant 2 replaced by $2^{\frac{1}{p}}$. □

REMARK. (1) Let us show that the constant 2 is sharp for $p = 1, q = \infty$. Choose $\Omega = [0,1], \mathcal{F}$ the Lebesgue-measurable subsets and P to be Lebesgue measure. Let $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$. For $\varepsilon > 0$, let $f_1 = f_2 = \varepsilon^{-1}\chi_{[0,\varepsilon]}$, for which we get $\|(f_n)_{n=1}^2\|_{1,\infty} = 1$. The predictable projection is given by $g_1 = 1$ and $g_2 = f_2$ for which we obtain $\|(g_n)_{n=1}^2\|_{1,\infty} = 2 - \varepsilon$.

We did not succeed in determining the sharp constant in the other cases $1 \leq p \leq q \leq \infty$ (except for the trivial case $p = q$ where the sharp constant clearly is 1).

(2) One might try to prove the inequality

$$E(\sum_{n=1}^N \mid g_n \mid^q)^{p/q} \leq C_p E(\sum_{n=1}^N \mid f_n \mid^q)^{p/q}$$

by applying Jensen's inequality in an even more direct and brutal way:

$$\mid g_n \mid^q = \mid E_{n-1}(f_n) \mid^q \leq E_{n-1}(\mid f_n \mid^q)$$

whence

$$\sum_{n=1}^N \mid g_n \mid^q \leq \sum_{n=1}^N E_{n-1}(\mid f_n \mid^q)$$

and

$$E(\sum_{n=1}^N \mid g_n \mid^q) \leq E(\sum_{n=1}^N \mid f_n \mid^q).$$

But at this point this reasoning comes to an end as there is no way to derive from this last inequality the original one. Indeed, this argument already breaks down in the case $N = 1$, for $p < q$, as in this case one would need that conditional expectation is a bounded operator on L^r, where $r = p/q < 1$, which is not the case.

(3) One might want to apply interpolation to derive the inequality from the trivial cases $1 \leq p = q \leq \infty$ (where it holds with a constant 1) and the extreme case $p = 1, q = \infty$ to obtain it for the general case $1 \leq p \leq q \leq \infty$ (thus avoiding the distinction of cases in the above proof).

Indeed the interpolation theorem of Benedek-Panzone [B-P] implies that an operator T defined on $L^p(\Omega, \mathcal{F}, l_N^q)$ which is bounded for the extreme cases $(p = 1, q = 1), (p = \infty, q = \infty), (p = 1, q = \infty)$ ist bounded for all $1 \leq p \leq q \leq \infty$. But there is a difficulty: The present operator T which assigns to every sequence $(f_n)_{n=1}^N$ the sequence $g_n = \mathbb{E}(f_n \mid \mathcal{G}_n)$ is only bounded (by a constant 2) on the suspace of $L^1(\Omega, \mathcal{F}, l_N^q)$ formed by the sequences $(f_n)_{n=1}^N$ adapted to $(\mathcal{F}_n)_{n=1}^N$. It is not obvious (at least not to the authors) how to modify the proof of Benedek-Panzone to adapt it to this subspace of $L^p(\Omega, \mathcal{F}, l_N^q)$.

(4) Finally we want to point out that we formulated Lemma 1 only in the case of finite discrete time, but as noted by D. Lépingle [L], "le passage au temps continu ne présente pas de difficulté" (compare also [D-S], th. 2.3).

Let us sketch this for the case $q = \infty$ (in the case of $q < \infty$ the proper setting for the continuous time case is the concept of "processus mince", see, e.g. [J], th. 1.6.). Let $X = (X_t)_{t \in \mathbb{R}_+}$ be a positive optionel process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ and $X = (X_t)_{t \in \mathbb{R}_+}$ a positive predictable process such that Y is less then or equal to the predictable projection ${}^pX$ of X. The assertion of the lemma is

$$\mathbb{E}(Y^*) \leq 2\mathbb{E}(X^*).$$

We may write

$$Y^* = \sup(Y_{T_1} \vee Y_{T_2} \vee \cdots \vee Y_{T_n})$$

where the sup is taken over all increasing finite sequences $T_1 \leq T_2 \leq \cdots \leq T_n$ of predictable stopping limes. As it suffices to take a countable number of such sequences, there arises no problem in which sense the above supremum has to be interpreted.

From Lemma 1 we obtain

$$\begin{aligned} \mathbb{E}(Y_{T_1} \vee \cdots \vee Y_{T_n}) &\leq 2\mathbb{E}(X_{T_1} \vee \cdots \vee X_{T_n}) \\ &\leq 2\mathbb{E}(X^*), \end{aligned}$$

which readily reduces the continuous time case to the discrete time case.

P.A. Meyer kindly pointed out to us a very elegant proof of the inequality

$$\mathbb{E}(Y^*) \leq 2\mathbb{E}(X^*),$$

which directly works in the continuous time setting and which we now reproduce. We may write

$$\mathbb{E}(Y^*) = \sup \mathbb{E}(Y_S)$$

where S runs through the nonnegative random variables (again it suffices to consider a sequence $(S_n)_{n=1}^\infty$ in the definition of the above sup).

Fix S and denote by $A = (A_t)_{t \in \mathbb{R}_+}$ the dual predictable projection of the process $\chi_{[S,\infty[}$. This is an increasing process whose potential

$$\mathbb{E}(A_\infty - A_t \mid \mathcal{F}_t)$$

is bounded by one. As the jumps of A are also bounded by one we have that its left potential

$$Z_t = \mathbb{E}(A_\infty - A_{t-} \mid \mathcal{F}_t)$$

is bounded by 2.

Considering cadlag versions of the processes X^* und Y^* we then may estimate

$$\begin{aligned}
\mathbb{E}(Y_S) &= \mathbb{E}\left(\int_0^\infty Y_t dA_t\right) \\
&\leq \mathbb{E}\left(\int_0^\infty X_t dA_t\right) \\
&\leq \mathbb{E}\left(\int_0^\infty X_t^* dA_t\right) \\
&= \mathbb{E}\left(\int_0^\infty (A_\infty - A_{t-}) dX_t^*\right) \\
&= \mathbb{E}\left(\int_0^\infty Z_t dX_t^*\right) \\
&\leq 2\mathbb{E}(X_\infty^*)
\end{aligned}$$

which finishes the proof in the case $p = 1$. For the case $1 \leq p \leq \infty$ one applies Jensen's inequality as in the proof of Lemma 1 above to obtain

$$\|Y^*\|_p \leq 2^{\frac{1}{p}} \|X^*\|_p.$$

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DEPARTMENT OF MATHEMATICS — INSTITUTE OF ACTUARIAL SCIENCES VRIJE UNIVERSITEIT BRUSSEL, PLEINLAAN 2, B-1050 BRUSSEL

INSTITUT FÜR STATISTIK DER UNIVERSITÄT WIEN, BRÜNNERSTRASSE 72, A-1210 WIEN, AUSTRIA.

E-mail: FDELBAEN @ TENA2.VUB.AC.BE, wschach@stat1.bwl.univie.ac.at