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A Proof of a Conjecture of Bobkov and Houdré

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Abstract: S. G. Bobkov and C. Houdré recently posed the following question on the internet ([1]): Let X, Y be symmetric i.i.d. random variables such that

$$P(\frac{|X+Y|}{\sqrt{2}} \geq t) \leq P(|X| \geq t),$$

for each $t > 0$. Does it follow that X has finite second moment (which then easily implies that X is Gaussian)?

In this note we give an affirmative answer to this problem and present a proof. Using a different method K. Oleszkiewicz has found another proof of this conjecture, as well as further related results.

Theorem. Let X, Y be symmetric i.i.d random variables. If, for each $t > 0$,

$$P(|X+Y| \geq \sqrt{2}t) \leq P(|X| \geq t), \tag{1}$$

then X is gaussian.

Proof. Step 1. $E|X|^p < \infty$ for $0 \leq p < 2$.

For this purpose it will suffice to show that, for $p < 2$, X has finite weak p'th moment, i.e., that there are constants C_p such that

$$P(|X| \geq t) \leq C_p t^{-p}.$$

To do so, it is enough to show that, for $\epsilon > 0, \delta > 0$, we can find t_0 such that, for $t \geq t_0$, we have

$$P(|X| \geq (\sqrt{2}+\epsilon)t) \leq \frac{1}{2-\delta} P(|X| \geq t). \tag{2}$$

Fix $\epsilon > 0$; then

$$P(|X+Y| \geq \sqrt{2}t) = 2P(X+Y \geq \sqrt{2}t) \geq$$
$$2P(X \geq (\sqrt{2}+\epsilon)t, Y \geq -\epsilon t, \text{ or } Y \geq (\sqrt{2}+\epsilon)t, X \geq -\epsilon t) =$$
$$2(2P(X \geq (\sqrt{2}+\epsilon)t)P(Y \geq -\epsilon t) - P(X \geq (\sqrt{2}+\epsilon)t)P(Y \geq (\sqrt{2}+\epsilon)t)) =$$

$$2P(|X| \geq (\sqrt{2}+\epsilon)t)(P(Y \geq -\epsilon t) - \frac{1}{2}P(X \geq (\sqrt{2}+\epsilon)t)) \geq$$
$$(2-\delta)P(|X| \geq (\sqrt{2}+\epsilon)t),$$

where $\delta > 0$ may be taken arbitrarily small for t large enough. Using (1) we obtain the inequality (2).

Step 2. Let $\alpha_1, ..., \alpha_n$ be real numbers such that $\alpha_1^2 + ... + \alpha_n^2 \leq 1$ and let $(X_i)_{i=1}^{\infty}$ be i.i.d. copies of X; then

$$E|\alpha_1 X_1 + ... + \alpha_n X_n| \leq \sqrt{2}E|X|.$$

We shall repeatedly use the following result:

Fact: Let S and T be symmetric random variables such that $P(|S| \geq t) \leq P(|T| \geq t)$, for all $t > 0$, and let the random variable X be independent of S and T. Then

$$E|S+X| \leq E|T+X|.$$

Indeed, for fixed $x \in R$, the function $h(s) = \frac{|s+x|+|s-x|}{2}$ is symmetric and non-decreasing in $s \in R_+$ and therefore

$$E|S+x| = E\frac{|S+x|+|S-x|}{2} \leq E\frac{|T+x|+|T-x|}{2} = E|T+x|.$$

Now take a sequence $\beta_1, ..., \beta_n \in \{2^{-k/2} : k \in N_0\}$, such that $\alpha_i \leq \beta_i < \sqrt{2}\alpha_i$. Then $\beta_1^2 + ... + \beta_n^2 \leq 2$ and

$$E|\alpha_1 X_1 + ... + \alpha_n X_n| \leq E|\beta_1 X_1 + ... + \beta_n X_n|.$$

If there is $i \neq j$ with $\beta_i = \beta_j$ we may replace $\beta_1, \ldots, \beta_n$ by $\gamma_1, \ldots, \gamma_{n-1}$ with $\sum_{i=1}^n \beta_i^2 = \sum_{j=1}^{n-1} \gamma_j^2$ and

$$E|\sum_{i=1}^n \beta_i X_i| \leq E|\sum_{j=1}^{n-1} \gamma_j X_j|. \tag{3}$$

Indeed, supposing without loss of generality that $i = n-1$ and $j = n$ we let $\gamma_i = \beta_i$, for $i = 1, \ldots, n-2$ and $\gamma_{n-1} = \sqrt{2}\beta_{n-1} = \sqrt{2}\beta_n$. With this definition we obtain (3) from (1) and the above mentioned fact.

Applying the above argument a finite number of times we end up with $1 \leq m \leq n$ and numbers $(\gamma_j)_{j=1}^m$ in $\{2^{-k/2} : k \in N_0\}$, $\gamma_i \neq \gamma_j$, for $i \neq j$, satisfying $\sum_{j=1}^m \gamma_j^2 \leq 2$ and

$$E|\sum_{i=1}^n \alpha_i X_i| \leq E|\sum_{j=1}^m \gamma_j X_j|.$$

To estimate this last expression it suffices to consider the extreme case $\gamma_j = 2^{-(j-1)/2}$, for $j = 1, \ldots, m$. In this case — applying again repeatedly the argument used to obtain (3):

$$E|\sum_{j=1}^{m} 2^{-(j-1)/2} X_j| \leq E|\sum_{j=1}^{m-1} 2^{-(j-1)/2} X_j + 2^{-(m-1)/2} X_m| \leq$$

$$E|\sum_{j=1}^{m-2} 2^{-(j-1)/2} X_j + 2^{-(m-2)/2} X_m| \leq \ldots \leq E|X_1 + X_2| \leq E|\sqrt{2} X_1| = \sqrt{2} E|X_1|.$$

Step 3. $EX^2 < \infty$.

We deduce from Step 2 that for a sequence $(\alpha_i)_{i=1}^{\infty}$ with $\sum_{i=1}^{\infty} \alpha_i^2 < \infty$ the series

$$\sum_{i=1}^{\infty} \alpha_i X_i$$

converges in mean and therefore almost surely. Using the notation

$$[S] = \begin{cases} S \text{ if } |S| \leq 1, \\ \operatorname{sign}(S) \text{ if } |S| \geq 1. \end{cases}$$

for a random variable S, we deduce from Kolmogorov's three series theorem that

$$\sum_{i=1}^{\infty} E([\alpha_i X_i]^2) < \infty.$$

Suppose now that $EX^2 = \infty$; this implies that, for $C > 0$, we may find $\alpha > 0$ such that

$$E([\alpha X]^2) \geq C\alpha^2.$$

From this inequality it is straightforward to construct a sequence $(\alpha_i)_{i=1}^{\infty}$ such that

$$\sum_{i=1}^{\infty} E([\alpha_i X_i]^2) = \infty, \text{ while } \sum_{i=1}^{\infty} \alpha_i^2 < \infty,$$

a contradiction proving Step 3.

Step 4. Finally, we show how $EX^2 < \infty$ implies that X is normal. We follow the argument of Bobkov and Houdré [2].

The finite second moment implies that we must have equality in the assumption of the theorem, i.e.,

$$P(|X + Y| \geq \sqrt{2}t) = P(|X| \geq t).$$

Indeed, assuming that there is strict inequality in (1) for some $t > 0$, we would obtain that the second moment of $X + Y$ is strictly smaller than the second moment of $\sqrt{2}X$, which leads to a contradiction:

$$2EX^2 > E(X+Y)^2 = EX^2 + EY^2 = 2EX^2.$$

Hence, $2^{-n/2}(X_1 + \ldots + X_{2^n})$ has the same distribution as X and we deduce from the Central Limit Theorem that X is Gaussian.

References

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