A COUNTER-EXAMPLE TO SEVERAL PROBLEMS IN THE THEORY OF ASSET PRICING

W. Schachermayer

Institut für Mathematik der Universität Wien,
Strudlhofgasse 4, A-1090 Wien, Austria.

September 92

Abstract. We construct a continuous bounded stochastic process $(S_t)_{t\in[0,1]}$ which admits an equivalent martingale measure but such that the minimal martingale measure in the sense of Föllmer and Schweizer does not exist. This example also answers (negatively) a problem posed by I. Karatzas, J. P. Lehozcky and S. E. Shreve as well as a problem posed C. Stricker.

1. Introduction

The problem of deciding wether a stochastic process $(S_t)_{t\in[0,1]}$ (or $(S_t)_{t\in\mathbb{R}_+}$) admits an equivalent martingale measure has attracted a lot of interest, especially for applications in mathematical finance (see. e.g. Harrison-Kreps (79), Harrison-Pliska (81), Kreps (81), Duffie-Huang (86), Stricker (90), Foellmer-Schweizer (90), Delbaen (92), Ansel-Stricker (92), Schachermayer (92). Essentially there have been two approaches to this problem: The first one consists in trying to establish the equivalence of some kind of "no arbitrage" or "no free lunch" condition to the existence of an equivalent martingale measure P^*. The theorems obtained in this direction are mere existence theorems and they all rely in some way on the Hahn–Banach theorem. Starting from the case of a finite probability space, which was solved in Harrison-Kreps (79) and Harrison-Pliska (81), this approach was developed in Duffie-Huang (86), Stricker (90), Dalang-Morton-Willinger (91) and culminated – for the case of continuous, bounded processes – in Delbaen's work (92), who gave for this case a necessary and sufficient condition for the existence of an equivalent martingale measure, namely the non-existence of a "free lunch with bounded risk" (see Schachermayer (92) for the definition and economic motivation).

The other approach to the present problem consists of directly constructing an equivalent martingale by using (a variant of) the Girsanov theorem. This approach started with the classical setting of the Black-Scholes formula (which is of course at the root of all this research): in the Black-Scholes economy the price process $(S_t)_{t\in[0,T]}$ is a (geometric) Brownian motion with a drift. The Girsanov theorem allows one to explicitly calculate the density of an equivalent martingale measure $\hat{P}$ which, in the case of the Black-Scholes economy, turns out to be unique.

Föllmer and Schweizer [F-S 90] extended this approach to the case of continuous processes $(S_t)_{t\in[0,1]}$ for which the equivalent martingale measure is not necessarily unique. They characterised the measure $\hat{P}$ which one obtains via the Girsanov-type formula ($t = 1$ denoting the time horizon and $\mathcal{E}$ denoting the Doleans-Dade exponential of a stochastic process; for further details, see below)

$$\hat{G}_t = \mathcal{E}(-\alpha \cdot M)_t$$

Key words and phrases. Equivalent Martingale Measure, Föllmer-Schweizer Decomposition, Girsanov Transformation.
Acknowledgement: We are indebted to F. Delbaen for much help and advice on this paper; we also thank C. Stricker for a joyful discussion in Oberwolfach

Typeset by $\mathcal{AMS}$-TEX

and

$$d\hat{P} = \hat{G}_1 dP$$

as the measure that should be used to minimize the quadratic risk of hedging. (This characterisation holds true provided the above defined $\hat{P}$ exists and is equivalent to P). Foellmer and Schweizer also gave a characterisation of $\hat{P}$ (if it exists) as the equivalent martingale measure that minimizes a certain functional envolving the relative entropy of $\hat{P}$ with respect to P.

The measure $\hat{P}$ was also implicitly introduced in the work of Karatzas, Lehoczky, Shreve and Xu (91) and found several applications since (see Ansel-Stricker (92a) and the references given there).

This approach was continued by Ansel and Stricker (92). They extended the work of Föllmer-Schweizer also to the case of processes with jumps and showed that in this case the approach of constructing the desired measure $\hat{P}$ via the Girsanov theorem fails in general: An example where Ω consists of three points shows that the exponential process $\mathcal{E}(-\alpha.M)_t$, may take negative values and therefore there is no hope that the above formula defines a probability measure $\hat{P}$.

In view of this example from Ansel-Stricker (92) it became clear that one can only hope for a good theorem on the existence of an equivalent martingale measure via the Girsanov theorem if one restricts attention to the case of continuous processes. Ansel and Stricker conjectured that for a continuous process $(S_t)_{t\in[0,1]}$ the existence of some equivalent martingale measure P^* implies the existence of the minimal measure $d\hat{P} = \mathcal{E}(-\alpha \cdot M)_1 dP$ defined above, a conjecture which is backed – at least morally speaking – by a strong orthogonality result which was established in (Ansel-Stricker (92), th. 5).

Unfortunately – and to the surprise of the author – this conjecture turns out to be wrong. We construct a counterexample which shows that even the existence of an equivalent martingale measure P^* with uniformly bounded density $\frac{dP^*}{dP}$ does not imply the existence of the minimal martingale measure.

The example also gives a negative solution to a question considered by I. Karatzas, J.P. Lehoczky and S.E. Shreve which arose from a problem in mathematical finance related to the Föllmer-Schweizer decomposition. Obviously this question is also of natural interest in the general theory of stochastic processes: suppose that G and N are non-negative local martingales satisfying $G_0 \equiv N_0 \equiv 1$, such that their pointwise product GN is a uniformly integrable martingale. In other words, we suppose that G and N are strongly orthogonal (Protter (90), ch.IV.3). Does this imply that G and N are necessarily true martingales?

In the case of discrete processes counterexamples were obtained by I. Karatzas, J.P. Lehoczky and S.E. Shreve as well as by D. Lépingle (see Lépingle (92) and the references given there). The question of what happens for continuous processes remained open, and we shall see below that the answer is no in the continuous case too.

Finally our construction also furnishes a counterexample to a question pertaining to the characterisation of attainable claims, which was raised by C. Stricker during the Oberwolfach meeting on mathematical finance in August 92 and which arose from the work of S. Jacka (92) and J.-P. Ansel and C. Stricker (92b) on this topic.

Let us now summarize our example (for precise definitions see below):

1.1 Theorem. *There exists a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$ and a uniformly bounded adapted continuous semimartingale $(S_t)_{t\in[0,1]}$ with the following properties:*

(i) Each martingale adapted to the filtration $(\mathcal{F}_t)_{t\in[0,1]}$ is continuous.

(ii) There exists an equivalent martingale measure P^* for S such that the density dP^*/dP is uniformly bounded.

(iii) There does not exist the minimal martingale measure $\hat{P}$.

(iv) There exists an equivalent measure Q such that for $N_t = E(dQ/dP|\mathcal{F}_t)$ and the Girsanov type local martingale $\hat{G}$ we have that the product $\hat{G}N$ is a true martingale while $\hat{G}$ is not.

The answer to Stricker's question is summarized in the next statement:

1.2 Proposition. *There exists a martingale M defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, Q)$ and a contingent claim (i.e., a random variable) $f \in L^1_+(\Omega, \mathcal{F}, Q)$ such that*

(i) f is integrable with respect to each equivalent martingale measure P and

(ii) $E_Q(f) \geq E_P(f)$ for each equivalent martingale measure P while

(iii) $E_Q(f) > E_P(f)$ for some equivalent martingale measure P.
The term martingale measure refers to a measure under which M is a martingale.

Let us now turn to the organisation of the paper. We first construct in section 2 a discrete example $(S_t)_{t\in\mathbb{N}_0}$ displaying the same features as the announced continuous counterexample $(S_t)_{t\in[0,1]}$. The reason is that the discrete example is technically simpler and everything can be calculated explicitly by elementary algebra while at the same time it contains the entire idea of the construction. The final continuous example will just be an adaptation of the discrete example and will be presented in section 3.

2. The Discrete Example

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, P)$ will denote a filtered probability space, where I equals $[0,1], \mathbb{R}_+$ or $\mathbb{N}_0$. We shall consider adapted processes $(S_t)_{t\in I}$ which are special semimartingales so that there is a canonical Doob-Meyer decomposition

$$S_t = M_t + A_t,$$

where $(M_t)_{t\in I}$ is a local martingale under P and $(A_t)_{t\in I}$ is predictable and of finite variation on compact subsets of I. We shall always have the case that the angle bracket process $\langle M, M\rangle$ is defined and, in fact, in both of the subsequent examples the angle bracket process $\langle M, M\rangle$ will coincide with the bracket process $[M, M]$. We shall also have the case that the process $(A_t)_{t\in I}$ is absolutely continuous with respect to $\langle M, M\rangle_t$, i.e.

$$dA_t = \alpha_t d\langle M, M\rangle_t = \alpha_t d[M, M]_t$$

for a predictable process $(\alpha_t)_{t\in I}$. Furthermore we shall also always assume that the process $(\alpha^2 \cdot [M, M])_t = (\alpha^2 \cdot \langle M, M\rangle)_t$ is defined.

Let us consider the Girsanov-type stochastic process

$$\begin{aligned}\hat{G}_t &= \mathcal{E}(-\alpha \cdot M)_t \\ &= \exp(-(\alpha \cdot M)_t - \frac{1}{2}(\alpha^2 \cdot [M, M])_t)\Pi_{0\leq s\leq t}(1 - \alpha_s \Delta M_s)\exp(\alpha_s \Delta M_s + \frac{1}{2}\alpha_s^2(\Delta M_s)^2),\end{aligned}$$

where $\mathcal{E}$ denotes the Doleans-Dade exponential (see e.g. Protter (90), ch. II.8). If $(S_t)_{t\in[0,1]}$ is a continuous process this reduces to the formula

$$\hat{G}_t = \exp(-\int_0^t \alpha_s dM_s - \frac{1}{2}\int_0^t \alpha_s^2 d[M, M]_s).$$

In this case $(\hat{G}_t)_{t\in[0,1]}$ is a continuous $\mathbb{R}_+$ valued local martingale and therefore a supermartingale. The importance of the process $(\hat{G}_t)_{t\in I}$ stems from the fact that $(\hat{G}_t S_t)_{t\in I}$ is a local martingale under P (see Föllmer-Schweizer (90) and Ansel-Stricker (92a).

2.1 Definition. (compare Foellmer-Schweizer (90) and Ansel-Stricker (92a))

$(S_t)_{t\in[0,1]}$ admits a minimal local martingale measure *if the process $(\hat{G}_t)_{t\in[0,1]}$ defined by $\hat{G}_t = \mathcal{E}(-\alpha\cdot M)_t$ is a martingale and $\hat{G}_1$ is strictly positive P-almost surely.*

In this case the measure $\hat{P}$ with density $\frac{d\hat{P}}{dP}$, turns $(S_t)_{t\in[0,1]}$ into a local martingale. If $\hat{P}$ is in fact a martingale measure for the process S, we call $\hat{P}$ the minimal martingale measure *associated to the process S.*

In the case $I = \mathbb{R}_+$ or $I = \mathbb{N}_0$ a similar definition may be given by requiring that $(\hat{G}_t)_{t\in I}$ is a uniformly integrable martingale and that $\hat{G}_\infty = \lim_{t\to\infty}\hat{G}_t$ is strictly positive almost surely. Note that, in the case when $(S_t)_{t\in I}$ is bounded, the existence of a minimal local martingale measure is equivalent to the existence of a minimal martingale measure.

We now start with the construction of the discrete example.

Let $\Omega_1 = \Omega_2 = \{-1,+1\}^{\mathbb{N}}$ equipped with the respective Borel-σ-algebras $\mathcal{F}^1$ and $\mathcal{F}^2$. Ω will denote $\Omega = \Omega_1\times\Omega_2$ equipped with the σ-algebra $\mathcal{F}^1\otimes\mathcal{F}^2$. We shall denote the elements of Ω by $\omega = (\omega_1,\omega_2) = ((\eta_n)_{n=1}^\infty, (\theta_n)_{n=1}^\infty)$.

Let P_1 denote the Haar measure on $\mathcal{F}^1$, i.e.,

$$P_1 = \bigotimes_{n=1}^{\infty}(\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}),$$

where δ denotes the Dirac measure.

On $\mathcal{F}^2$ we shall consider two measures P_2 and Q_2 given by

$$P_2 = \bigotimes_{n=1}^{\infty}((1-2^{-n})\delta_1 + 2^{-n}\delta_{-1})$$

and

$$Q_2 = \bigotimes_{n=1}^{\infty}(\frac{1}{4}\delta_1 + \frac{3}{4}\delta_{-1}).$$

Finally define the measures P and Q on $\mathcal{F}^1\otimes\mathcal{F}^2$ by $P = P_1\otimes P_2$, $Q = P_1\otimes Q_2$.

Let $(M_t)_{t\in\mathbb{N}_0}$ be the process

$$M_t(\omega) = \sum_{n=1}^{t}\eta_n.$$

Obviously M is a martingale with respect to its natural filtration under P as well as under Q. Define $(A_t)_{t\in\mathbb{N}_0}$ as the deterministic (and therefore predictable) process

$$A_t = -\sum_{n=1}^{t}(1-2^{-n}).$$

Note that $\langle M,M\rangle_t = [M,M]_t = t$ and that the process A_t may be written as $A_t = (\alpha\cdot\langle M,M\rangle)_t$ with $\alpha_t = -(1-2^{-t})$.

Let T denote the function on Ω given by

$$T(\omega) = \min\{t \in \mathbb{N}_0 : \eta_t = -1 \quad \text{or} \quad \theta_{t+1} = -1\}$$

with $T(\omega) = \infty$ if all the coordinates of ω equal 1.

Denote by $(S_t)_{t\in\mathbb{N}_0}$ the process $M + A$ stopped at T, i.e.,

$$S_t = M_{t\wedge T} + A_{t\wedge T},$$

and let $(\mathcal{F}_t)_{t\in\mathbb{N}_0}$ be the filtration generated by $(S_t)_{t\in\mathbb{N}_0}$ and $\mathcal{F}$ the σ-algebra generated by $(\mathcal{F}_t)_{t\in\mathbb{N}_0}$. Note that T is a stopping time with respect to $(\mathcal{F}_t)_{t\in\mathbb{N}_0}$.

The above equation gives the Doob–Meyer decomposition of S_t into its martingale and into its predictable part (with respect to Q as well as to P).

Note that each $\mathcal{F}_t$ is generated by a partition of Ω into finitely many atoms. We denote by B_t the atom of $\mathcal{F}_t$ given by

$$B_t = \{\omega : \eta_1 = \cdots = \eta_t = \theta_1 = \cdots = \theta_t = 1\}.$$

Note that, for $t \geq 0$ we obtain $\mathcal{F}_{t+1}$ from $\mathcal{F}_t$ by leaving all the atoms of $\mathcal{F}_t$ unchanged except for splitting B_t into three atoms, i.e.,

$$B_t = B_{t+1} \cup C_{t+1} \cup D_{t+1}$$

where

$$C_{t+1} = B_t \cap \{\theta_{t+1} = -1\}$$

and

$$D_{t+1} = B_t \cap \{\eta_{t+1} = -1, \theta_{t+1} = 1\}.$$

Let us calculate the measures of these sets with respect to P and Q:

$$\begin{array}{ll} P(B_t) = 2^{-t}\prod_{n=1}^{t}(1-2^{-n}), & Q(B_t) = 8^{-t}, \\ P(C_t) = 2^{-2t+1}\prod_{n=1}^{t-1}(1-2^{-n}), & Q(C_t) = 6\cdot 8^{-t}, \\ P(D_t) = 2^{-t}\prod_{n=1}^{t}(1-2^{-n}), & Q(D_t) = 8^{-t}. \end{array}$$

As $\mathcal{F}$ is generated by the countable partition of Ω into the atoms $\{(C_t)_{t=1}^{\infty}, (D_t)_{t=1}^{\infty}\}$ and since P and Q both are strictly positive on each of them, we readily conclude that the measures P and Q are equivalent on $\mathcal{F}$.

Now we calculate the Girsanov type process $(\hat{G}_t)_{t\in\mathbb{N}_0}$ for S which is the same for P as well as for Q because the Doob-Meyer decomposition $S_t = M_{t\wedge T} + A_{t\wedge T}$ does not depend on the choice of P or Q. The exponential formula simplifies in the present case to

$$\hat{G}_t = \mathcal{E}(-\alpha \cdot M)_{t\wedge T} = \prod_{s=1}^{t\wedge T}(1 - \alpha_s \bigtriangleup M_s).$$

Noting that $\alpha_s = -(1-2^{-s})$, $\{T \geq s\} = B_s \cup D_s$ and $\triangle M_s$ equals 1 on B_s and -1 on D_s, one obtains inductively

$$\hat{G}_t(\omega) = \begin{cases} \prod_{n=1}^{t}(2-2^{-n}), & \omega \in B_t \\ 2^{-t}\prod_{n=1}^{t-1}(2-2^{-n}), & \omega \in D_t \\ \hat{G}_{t-1}(\omega), & \text{elsewhere.} \end{cases}$$

In particular $\hat{G}_\infty = \lim_{t\to\infty}\hat{G}_t$ exists almost surely (with respect to P as well as to Q) and is given by

$$\hat{G}_\infty(\omega) = \begin{cases} \prod_{n=1}^{t-1}(2-2^{-n}), & \omega \in C_t \\ 2^{-t}\prod_{n=1}^{t-1}(2-2^{-n}), & \omega \in D_t. \end{cases}$$

After all these calculations we arrive at the crucial observation: $\hat{G}_\infty$ closes the martingale $(\hat{G}_t)_{t\in\mathbb{N}_0}$ with respect to Q while it does not so with respect to P. Indeed, an elementary (and boring) calculation reveals that $\lim_{t\to\infty} E_Q(\hat{G}_\infty - \hat{G}_t) = 0$. On the other hand, to see that $\lim_{t\to\infty} E_P(\hat{G}_\infty - \hat{G}_t) > 0$, the easiest way is to verify that $(\hat{G}_t)_{t\in\mathbb{N}}$ is not uniformly integrable in $L^1(P)$. Indeed

$$E_P(\hat{G}_t \cdot \chi_{B_t}) = 2^{-t}\cdot\prod_{n=1}^{t}(1-2^{-n})\prod_{n=1}^{t}(2-2^{-n}) = (2-2^{-t})\prod_{n=1}^{t}(1-2^{-n})^2,$$

which tends to a strictly positive number for $t\to\infty$ while $P(B_t)$ tends to zero.

Looking at definition 2.1 of a minimal martingale measure we may conclude the following: Considering the process $(S_t)_{t\in\mathbb{N}_0}$ based on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}_0}, P)$ there does not exist a minimal local martingale measure, while if we consider the process $(S_t)_{t\in\mathbb{N}_0}$ based on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}_0}, Q)$ there does exist a minimal local martingale measure with density equal to $\hat{G}_\infty$ (relative to Q). Denote this measure on $\mathcal{F}$ by P^*, i.e.

$$\frac{dP^*}{dQ} = \hat{G}_\infty.$$

One easily checks that $(S_t)_{t\in\mathbb{N}_0}$ is a uniformly integrable martingale under P^*, whence in particular P^* is a martingale measure for the process S.

As P^* is equivalent to Q and Q is equivalent to P this implies in particular that $(S_t)_{t\in\mathbb{N}_0}$ does admit a martingale measure equivalent to P but, as we just saw, there is no minimal (even local) martingale measure with respect to P.

Finally we shall show that the density $\frac{dP^*}{dP}$ is uniformly bounded. Let us calculate the value of

$$\frac{dP^*}{dP} = \frac{dP^*}{dQ}\frac{dQ}{dP} = \hat{G}_\infty\frac{dQ}{dP}$$

on each C_t and D_t. For $\omega \in C_t$ we get

$$\frac{dP^*}{dP}(\omega) = \hat{G}_\infty(\omega)\cdot\frac{Q(C_t)}{P(C_t)} = \prod_{n=1}^{t-1}(2-2^{-n})\frac{6\cdot 8^{-t}}{2^{-2t+1}\cdot\prod_{n=1}^{t-1}(1-2^{-n})} = 3(1-2^{-t})$$

and for $\omega \in D_t$ we get

$$\frac{dP^*}{dP}(\omega) = \hat{G}_\infty(\omega)\frac{Q(D_t)}{P(D_t)} = 2^{-t}\prod_{n=1}^{t-1}(2-2^{-n})\cdot\frac{8^{-t}}{2^{-t}\prod_{n=1}^{t}(1-2^{-n})} = 4^{-t}$$

and these numbers are, of course, uniformly bounded.

The construction of Example 2.2 thus is complete.

q.e.d.

Remarks. (a) The process $(S_t)_{t\in\mathbb{N}_0}$ above is not uniformly bounded, but this is only a superficial feature. Passing to the process $(\tilde{S}_t)_{t\in\mathbb{N}_0}$ with differences

$$\tilde{S}_t - \tilde{S}_{t-1} = 2^{-t}(S_t - S_{t-1})$$

we obtain a uniformly bounded process with all the above properties unchanged. We just did not want to obscure the above calculation by one more factor of 2^{-t}, which is essentially irrelevant.

(b) Let us give an intuitive interpretation of the above process S. At each time t two coins are flipped independently; one, described by the random variable η_t, being fair while the other, described by θ_t, is very likely to take the value $\theta_t = 1$ (with respect to the measure P). If for all s smaller than t you always had the outcome $\eta_s = \theta_s = 1$ then you are still in business at time t: You first observe the outcome of the coin θ_t. If $\theta_t = -1$ you stop the game; otherwise you play a game at time t whose outcome is determined by the (fair) coin η_t. Nevertheless the game is very unfair as you win only the sum 2^{-t} if $\eta_t = 1$ appears while you loose $2 - 2^{-t}$ if the result is $\eta_t = -1$.

With respect to P the flipping of the coin θ_t is essentially irrelevant as θ_t is very likely to equal 1. Hence with respect to P the above game is similar to playing only the unfair game governed by η_t and it is an easy calculation that for this latter game there exists no equivalent martingale measure at all. This is the reason behind the above verified absence of the minimal martingale measure with respect to P.

The situation is different if we consider Q. In this case the coin θ_t is quite likely to equal -1 (namely with probability $3/4$ for each t; we have chosen $3/4$ instead of $1/2$ for technical reasons, namely to force dP^*/dP to be in L^∞). Thus a premature end of the game caused by θ is quite likely to take place and the unfairness of the bets on the random variable η_t does not have the same impact as under the measure P.

This is – very roughly – the intuitive idea behind the above construction and this idea may be carried over to the continuous time example constructed in section 3 below: We only replace the flipping of the coins η_t and θ_t by running Brownian motions with apropriate drifts and stopping these processes conveniently.

(c) We could have constructed the above example on a denumerable probability space Ω by shrinking the atoms $(C_t)_{t=1}^\infty$ and $(D_t)_{t=1}^\infty$ to one-point-sets. The example given by Karatzas, Lehoczky and Shreve as well as the one given by Lépingle (92) also use denumerable probability spaces Ω. But we felt that the use of $\Omega = \{-1,+1\}^{\mathbb{N}} \times \{-1,+1\}^{\mathbb{N}}$ is more transparent and should also facilitate the passage from the discrete example to the continuous one.

3. The continuous example

We start with an easy lemma which allows to fabricate the counterparts to the random variables η_t in the discrete example.

3.1 Lemma. *For $1 > \alpha > 0$ there is a uniformly bounded diffusion process $(X_t)_{t\in[0,1]}$ on $(\Omega, \mathcal{F}, P)$ adapted to its natural filtration $(\mathcal{F}_t)_{t\in[0,1]}$ with the following property: for X there is a unique equivalent martingale measure $\hat{P}$ on $\mathcal{F}$ such that the density $d\hat{P}/dP$ assumes the values α and $2-\alpha$ with probability $\frac{1}{2}$.*

Proof. Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion on $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in\mathbb{R}_+}, P)$, starting at $B_0 \equiv 0$, where $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$ is the natural filtration of B_t. Let Y be the process

$$Y_t = B_t - t$$

which is of the form $Y_t = B_t + \alpha_t.\langle B, B\rangle_t$ with $\alpha_t \equiv -1$.

Form the stochastic exponential

$$\hat{G}_t = \mathcal{E}(-\alpha_t.B_t) = \mathcal{E}(B_t) = \exp(B_t - \frac{t}{2})$$

and let T be the stopping time when $\hat{G}_t$ becomes for the first time either α or $2-\alpha$. Clearly T is almost surely finite and $P(\hat{G}_T = \alpha) = P(\hat{G}_T = 2-\alpha) = \frac{1}{2}$ (by the martingale property of $(\hat{G}_t)_{t\le T}$).

By the martingale representation property of Brownian motion with respect to its natural filtration we conclude that the measure $\hat{P}$ on $\mathcal{G}_T$ with density $d\hat{P}/dP = \hat{G}_T$ is the unique equivalent martingale measure on $\mathcal{G}_T$ for the process $Z_t = Y_{t\wedge T}$.

To replace the index set $\mathbb{R}_+$ by $[0,1]$ define the process $(W_t)_{t\in[0,1]}$ by

$$W_t = \begin{cases} Z_{t/(1-t)}, & \text{for} \quad t \in [0,1[, \\ Z_\infty = Z_T, & \text{for} \quad t = 1. \end{cases}$$

Finally, to make the process uniformly bounded, define the stopping times T_n by letting $T_0 = 0$ and, for $n \in \mathbb{N}$,

$$T_n = \inf\{t \in [0,1] : \quad | W_t | \ge n\}$$

with $T_n = 1$ if the above set is empty; define X_t inductively on the stochastic intervals $[\![T_{n-1}, T_n]\!]$ by

$$X_t - X_{T_{n-1}} = 2^{-n}(W_t - W_{T_{n-1}}) \qquad \text{for} \quad t \in [\![T_{n-1}, T_n]\!].$$

The process $(X_t)_{t\in[0,1]}$ satisfies the requirements.

q.e.d.

We now start the construction of the continuous example:

Let $(\Omega^1, \mathcal{F}^1, P^1)$ be the product $(\prod_{n=1}^\infty \Xi_n, \bigotimes_{n=1}^\infty \mathcal{G}_n, \bigotimes_{n=1}^\infty P_n^1)$, where $(\Xi_n, \mathcal{G}_n, P_n^1)_{n=1}^\infty$ is a sequence of copies of the probability spaces appearing in lemma 3.1 for $\alpha = 2^{-n}$. Denote by $(X_t^n)_{t\in[0,1]}$ the corresponding processes given by lemma 3.1 which we assume to be uniformly bounded by 2^{-n} in absolute value.

Denote by $\hat{P}_n$ the measures on $\mathcal{G}_n$ given by lemma 3.1 and define the following elements of $\mathcal{G}_n$:

$$G_n^+ = \{\frac{d\hat{P}_n}{dP_n^1} = 2 - 2^{-n}\} \qquad \text{and} \qquad G_n^- = \Xi_n \setminus G_n^+ = \{\frac{d\hat{P}_n}{dP_n^1} = 2^{-n}\}.$$

As regards the second factor of our probability space let $(\Omega^2, \mathcal{F}^2, P^2) = (\prod_{n=1}^\infty \Psi_n, \bigotimes_{n=1}^\infty \mathcal{H}_n, \bigotimes_{n=1}^\infty P_n^2)$ where $(\Psi_n, \mathcal{H}_n, P_n^2)_{n=1}^\infty$ are copies of probability spaces on which there are defined standard Brownian motions $(W_t^n)_{t\in[0,1]}$ with $W_0^n \equiv 0$, for each $n \in \mathbb{N}$.

Fix for each $n \in \mathbb{N}$ a set $H_n^+ \in \mathcal{H}_n$ which satisfies $P_n^2(H_n^+) = 1 - 2^{-n}$. Letting $H_n^- = \Psi_n \setminus H_n^+$ define the measure Q_n on $\mathcal{H}_n$ by defining the density function

$$\frac{dQ_n}{dP_n^2}(\psi_n) = \begin{cases} (4(1-2^{-n}))^{-1} & \text{for} \quad \psi_n \in H_n^+, \\ 3 \cdot 2^{n-2} & \text{for} \quad \psi_n \in H_n^-. \end{cases}$$

so that $Q_n(H_n^+) = 1/4$ and $Q_n(H_n^-) = 3/4$. Let the measure Q^2 on $\mathcal{F}^2$ be given by the product measure $\bigotimes_{n=1}^\infty Q_n$.

Define again $(\Omega, \mathcal{F})$ as $(\Omega^1 \times \Omega^2, \mathcal{F}^1 \otimes \mathcal{F}^2)$ equipped with the measures $P = P^1 \otimes P^2$ and $Q = P^1 \otimes Q^2$.

The elements ω of Ω will be denoted by $\omega = ((\chi_n)_{n=1}^\infty, (\psi_n)_{n=1}^\infty)$ with $\chi_n \in \Xi_n$ and $\psi_n \in \Psi_n$.

In analogy with the discrete example we define the sets $B_0 = \Omega$, and, for $k \in \mathbb{N}$,

$$\begin{aligned} B_k =& \{\omega : \chi_n \in G_n^+ \quad \text{and} \quad \psi_n \in H_n^+ \quad \text{for} \quad n = 1, \dots, k\} \\ C_k =& B_{k-1} \cap \{\psi_k \in H_k^-\} \\ D_k =& B_{k-1} \cap \{\chi_k \in G_k^- \quad \text{and} \quad \psi_k \in H_k^+\}. \end{aligned}$$

As in the discrete example $\{\{C^k\}_{k=1}^\infty, \{D^k\}_{k=1}^\infty\}$ forms an almost sure partition of Ω and the P and Q measures of B_k, C_k and D_k equal the corresponding measures in the discrete example.

Now we define the random variable T by

$$T(\omega) = \begin{cases} 1 - \frac{1}{2k} & \text{if} \quad \psi_k \in H_k^- \quad \text{and} \quad \chi_j \in G_j^+, \psi_j \in H_j^+ \quad \text{for} \quad j < k, \\ 1 - \frac{1}{2k+1} & \text{if} \quad \chi_k \in G_k^- \quad \text{and} \quad \chi_j \in G_j^+ \, \text{for}\, j < k, \, \psi_j \in H_j^+ \, \text{for}\, j \leq k. \end{cases}$$

so that T equals $1 - \frac{1}{2k}$ on C_k and $1 - \frac{1}{2k+1}$ on D_k.

Now we define the process $(S_t)_{t \in [0,1]}$ inductively on the intervals $[1 - \frac{1}{2n-1}, 1 - \frac{1}{2n}]_{n=1}^\infty$ and $[1 - \frac{1}{2n}, 1 - \frac{1}{2n+1}]_{n=1}^\infty$. Let $S_0 \equiv 0$ and, for $n \in \mathbb{N}$ and $t \in [1 - \frac{1}{2n-1}, 1 - \frac{1}{2n}]$, let

$$S_t - S_{1-\frac{1}{2n-1}} = 0$$

while, for $n \in \mathbb{N}$ and $t \in [1 - \frac{1}{2n}, 1 - \frac{1}{2n+1}]$

$$S_t - S_{1-\frac{1}{2n}} = \begin{cases} X^n_{(t-(1-\frac{1}{2n}))(2n)(2n+1)} & \text{if} \quad T > 1 - \frac{1}{2n}, \\ 0 \quad \text{otherwise.} \end{cases}$$

This defines S_t for $t \in [0, 1[$ and, for $t = 1$, let $S_1 = S_T$.

We also define a "dummy process" $(U_t)_{t \in [0,1]}$ inductively on the intervals $[1 - \frac{1}{2n-1}, 1 - \frac{1}{2n}]$ and $[1 - \frac{1}{2n}, 1 - \frac{1}{2n+1}]$. Let $U_0 \equiv 0$ and, for $n \in \mathbb{N}$ and $t \in [1 - \frac{1}{2n-1}, 1 - \frac{1}{2n}]$

$$U_t - U_{1-\frac{1}{2n-1}} = \begin{cases} W^n_{(t-(1-\frac{1}{2n-1}))(2n-1)(2n)} & \text{if} \quad T > 1 - \frac{1}{2n-1}, \\ 0 \quad \text{otherwise.} \end{cases}$$

while, for $n \in \mathbb{N}$ and $t \in [1 - \frac{1}{2n}, 1 - \frac{1}{2n+1}]$

$$U_t - U_{1-\frac{1}{2n}} = 0.$$

Again let $U_1 = U_T$. Let $(\mathcal{F}_t)_{t\in[0,1]}$ be the filtration on Ω generated by the process $(S_t + U_t)_{t\in[0,1]}$ and $\mathcal{F} = \mathcal{F}_1$. Note that this filtration is finer than the one generated by $(S_t)_{t\in[0,1]}$ due to the information revealed by the "dummy process" $(U_t)_{t\in[0,1]}$. Also note that T is a stopping time with respect to $(\mathcal{F}_t)_{t\in[0,1]}$.

We shall see that the process $(S_t)_{t\in[0,1]}$ behaves in a similar way as the discrete process $(S_t)_{t\in\mathbb{N}_0}$ constructed above. Let us first give the intuitive interpretation similarly as in remark b of section 2: Again you are playing games. During the intervals $[1-\frac{1}{2n-1}, 1-\frac{1}{2n}]$ you only observe the process $(U_t)_{t\in[0,1]}$ and do not make any bets; at time $1-\frac{1}{2n}$ you observe whether your state of the world ω is in H_n^+ or not. If this is the case (and if you have not stopped playing already before) you are betting on the process S: You make a bet on its behaviour during the time interval $[1-\frac{1}{2n}, 1-\frac{1}{2n+1}]$; note that during this intervall the process S is – essentially – the process X_n, i.e., a Brownian motion with an unfavourable drift.

At time $1-\frac{1}{2n+1}$ you observe whether your state of the world ω belongs to G_n^-; if this is the case you stop playing and otherwise you continue.

This should explain the similarities with the discrete example and motivate the subsequent calculations.

Note that the measures P and Q are equivalent on $\mathcal{F}$. This follows from the fact that they are equivalent on each $\mathcal{F}_{1-\frac{1}{n}}$ and that $\mathcal{F}$ and $\mathcal{F}_{1-\frac{1}{2n+1}}$ coincide outside of B_n.

Let us now calculate the Girsanov type process

$$\hat{G}_t = \mathcal{E}(-\alpha \cdot M)_t$$

for $(S_t)_{t\in[0,1]}$, where $S = M + A = M + \alpha \cdot \langle M, M\rangle$ is the Doob-Meyer decomposition of S. We remark again that this decomposition and therefore the process $\hat{G}$ does not depend on the choice of P or Q.

It quickly follows from lemma 3.1 that $\hat{G}_{1-\frac{1}{2n+1}}$ is almost surely constant on each of the sets $B_n, (C_j)_{j\le n}$ and $(D_j)_{j\le n}$ and that on each of these sets it assumes there the same values as the function $\hat{G}_n$ in the discrete example.

We now show that the process S satisfies the assertions of theorem 1.1.

Proof of theorem 1.1. (i) To verify that each martingale adapted to $(\mathcal{F}_t)_{t\in[0,1]}$ is continuous note that the process $S + U$ is just built from Brownian motions (with some drift and stopped at stopping times which use only the information revealed by the preceding Brownian motion). Hence the assertion quickly follows from the well-known fact that each local martingale adapted to the natural filtration of a Brownian motion is necessarily continuous (see, e.g., Protter (90), th. IV.3.42).

(ii) and (iii) We have just seen that the function $\hat{G}_1 = \lim_{t\to 1}\hat{G}_t$ takes the same values as the function $\hat{G}_\infty$ from the discrete example on the sets $(C_k)_{k=1}^\infty, (D_k)_{k=1}^\infty$. Hence the proof of the discrete example carries over verbatim to the present case: for $(S_t)_{t\in[0,1]}$ there does not exist a minimal martingale measure $\hat{P}$ with respect to P as $E_P(\hat{G}_1) < 1$. On the other hand, there does exist an equivalent martingale measure P^*, even with $\frac{dP^*}{dP} \in L^\infty$; for example, consider the minimal measure P^* with respect to Q given by the formula $\frac{dP^*}{dQ} = \hat{G}_1$.

(iv) Note that $N_t = E(dQ/dP|\mathcal{F}_t)$ is a martingale (with respect to P) and the Girsanov type process $\hat{G}_t$ is a local martingale but not a martingale. The product $\hat{G}N$ is a true martingale (with respect to P) as $\hat{G}$ is a true martingale with respect to Q.

q.e.d.

Remark. The theorem is somewhat asymetric in its statement (iv): Only $\hat{G}$ fails to be a martingale while the second factor N is a true martingale. But it is not hard to modify the construction so that both factors fail to be martingales: it suffices to take two independent copies of the above examples and to define a process indexed by $I = [0,2]$ by repeating the above construction on $[0,1]$ and reversing the roles of $\hat{G}$ and N on $[1,2]$. We leave the details to the reader.

Summing up we have shown that the product of two non-negative local martingales, starting both at 1, may be a true martingale while both factors — or precisely one of the factors — may fail to be a true martingale.

Proof of proposition 1.2. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, Q)$, P and $(S_t)_{t\in[0,1]}$ be as in the proof of theorem 1.1 and let M be the martingale part of the Doob-Meyer decomposition of $S = M + A$. It is easy to verify that M is a true martingale under Q as well as under P.

The nonnegative random variable f will be $\hat{G}_1$. We have seen that

$$E_Q(f) = 1 \qquad \text{and} \qquad E_P(f) < 1.$$

If $\tilde{P}$ is any probability measure equivalent to Q such that M is a martingale under $\tilde{P}$, then the process $(\hat{G}_t)_{t\in[0,1]} = \mathcal{E}(-\alpha.M)$ is a local martingale and therefore a supermartingale with respect to $\tilde{P}$. Therefore $E_{\tilde{P}}(f) \leq 1$ which proves proposition 1.2.

q.e.d.

References

J. P. Ansel, C. Stricker (1992), *Lois de Martingale, Densités at Décomposition de Föllmer Schweizer*, preprint, 18 p.

J. P. Ansel, C. Stricker (1992a), *Unicité et existence de la loi minimale*, to appear in Séminaire de Probabilités, 8p.

J.P. Ansel, C. Stricker (1992b), *Couverture des actifs contingents*, preprint, 11 p.

K. Back, S. R. Pliska (1990), *On the Fundamental Theorem of Asset Pricing with an infinite State Space*, Journal of Mathematical Economics **20**, 1 – 18.

R. C. Dalang, A. Morton, W. Willinger (1990), *Equivalent Martingale measures and no–arbitrage in stochastic securities market models*, Stochastics and Stochastic Reports **29**, 185-201.

F. Delbaen (1992), *Representing Martingale Measures when Asset Prices are continuous and bounded*, preprint (to appear in Math. Finance), preprint, 44 p.

D. Duffie, C. F. Huang (1986), *Multiperiod security markets with differential information*, J. Math. Econ. **15**, 283-303.

H. Föllmer, M. Schweizer (1990), *Hedging of Contingent Claims under Incomplete Information*, appeared in M. H. A. Davis and R. J. Elliott (eds.): Applied Stochastic Analysis, Stochastic Monographs, Gordon and Breach, London/New York **5** (1991), 389 - 414.

M. J. Harrison, D. M. Kreps (1979), *Martingales and arbitrage in multiperiod securities markets*, J. Econ. Theory **20**, 381-408.

M. J. Harrison, S. R. Pliska (1981), *Martingales and stochastic integrals in the theory of continuous trading*, Stoch. Proc. Appl. **11**, 215-260.

N. Hofmann, E. Platen, M. Schweizer (1992), *Option Pricing under Incompleteness and Stochastic Volatility*, Math. Finance 2, 153 – 187.

S.D. Jacka (1992), *A Martingale Representation Result and an Application to Incomplete Financial Markets*, preprint, to appear in Math. Finance (1992).

I. Karatzas, J.P. Lehoczky, S.E. Shreve, G.L. Xu (1991), *Martingale and duality methods for utility maximisation in an incomplete market*, SIAM J. of Control and Optimisation **29**, 702-730.

D. M. Kreps (1981), *Arbitrage and Equilibrium in Economics with infinitely many Commodities*, J. of Math. Econ. **8**, 15-35.

D. Lépingle (1992), *Orthogonalité et Intégrabilité uniforme de martingales discrètes*, to appear in Séminaire de Probabilités, 3p.

P.Protter (1990), *Stochastic Integration and Differential Equations*, Applications of Math. Vol.21, Springer.

W. Schachermayer (1992), *Martingale Measures for discrete time processes with infinite horizon*, preprint (submitted to Mathematical Finance), 30 p.

M. Schweizer (1992a), *Mean-Variance Hedging for General Claims*, Annals of Applied Probability 2, 171 –179.

M. Schweizer (1992b), *Martingale Densities for General Asset Prices*, J. Math. Economics 21, 363 – 378.

C. Stricker (1990), *Arbitrage et lois de martingale*, Ann. Inst. H. Poincaré **26**, 451-460.