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THE EXISTENCE OF ABSOLUTELY CONTINUOUS LOCAL MARTINGALE MEASURES

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Abstract. We investigate the existence of an absolutely continuous martingale measure. For continuous processes we show that the absence of arbitrage for general admissible integrands implies the existence of an absolutely continuous (not necessarily equivalent) local martingale measure. We also rephrase Radon-Nikodym theorems for predictable processes.

1.Introduction.

In Delbaen and Schachermayer (1994a) we showed that for locally bounded finite dimensional stochastic price processes S, the existence of an equivalent (local) martingale measure – sometimes called risk neutral measure – is equivalent to a property called No Free Lunch with Vanishing Risk (NFLVR). We also proved that if the set of (local) martingale measures contains more than one element, then necessarily, there are non equivalent absolutely continuous local martingale measures for the process S. We also gave an example, see Delbaen and Schachermayer (1994a) Example 7.7, of a process that does not admit an equivalent (local) martingale measure but for which there is a martingale measure that is absolutely continuous. The example moreover satisfies the weaker property of No Arbitrage with respect to general admissible integrands. We were therefore lead to the investigation of the relation between the two properties, the existence of an absolutely continuous martingale measure (ACMM) and the absence of arbitrage for general admissible integrands (NA).

From an economic viewpoint a local martingale measure $\mathbb{Q}$, that gives zero measure to a non negligible event, say F, poses some problems. The price of the contingent claim that pays one unit of currency subject to the occurrence of the event F, is given by the probability $\mathbb{Q}[F]$. Since F is negligible for this probability, the price of the commodity becomes zero! In mosteconomic

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models preference relations are supposed to be strictly monotone and hence there would be an infinite demand for this commodity. At first sight the property (ACMM) therefore seems meaningless in the study of general equilibrium models. But as the present paper shows, for continuous processes it is a consequence of the absence of arbitrage (NA). We therefore think that the (ACMM) property has some interest also from the economic viewpoint.

Throughout the paper all variables and processes are defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. The space of all measurable functions, equipped with the topology of convergence in probability is denoted by $L^0(\Omega,\mathcal{F},\mathbb{P})$ or simply $L^0(\Omega)$ or L^0. If $F \in \mathcal{F}$ has non zero measure then the closed subspace of functions, vanishing on the complement F^c of F is denoted by $L^0(F)$. The conditional probability with respect to a non negligible event F is denoted by $\mathbb{P}_F$ and is defined as $\mathbb{P}_F[B] = \frac{\mathbb{P}[F\cap B]}{\mathbb{P}[F]}$. To simplify terminology we say that a probability $\mathbb{Q}$ that is absolutely continuous with respect to $\mathbb{P}$ is supported by the set F if $\mathbb{Q}$ is equivalent to $\mathbb{P}_F$, in particular we then have $\mathbb{Q}[F] = 1$. Indicator functions of sets F, etc. are denoted by $\mathbf{1}_F$ etc.. The probability space Ω is equipped with a filtration $(\mathcal{F}_t)_{0\le t<\infty}$. We use the time set $[0,\infty[$ as this is the most general case. Discrete time sets and bounded time sets can easily be imbedded in this framework. We will mainly study continuous processes and in this case the discrete time set makes no sense at all. However section 2 contains some results that remain valid for processes with jumps.

We assume that the filtration $(\mathcal{F}_t)_{0\le t<\infty}$ satisfies the usual assumptions, i.e. it is right continuous and saturated for $\mathbb{P}$-null sets. Stopping times are with respect to this filtration. We draw the attention of the reader to the problem that when $\mathbb{P}$ is replaced by an absolutely continuous measure $\mathbb{Q}$, these usual hypotheses will no longer hold. In particular we will have to saturate the filtration with the $\mathbb{Q}$-null sets.

The process S, sometimes denoted as $(S_t)_{0\le t<\infty}$, is a fixed cadlag, locally bounded process that is a semimartingale with respect to $(\Omega,(\mathcal{F}_t)_{0\le t<\infty},\mathbb{P})$. The process S is supposed to take values in the d-dimensional space $\mathbb{R}^{\mathrm{d}}$ and may be interpreted as the (discounted) price process of d stocks. If T_1 and T_2 are two stopping times such that $T_1 \le T_2$ then $]\!]T_1,T_2]\!]$ is the stochastic interval $\{(t,\omega) \mid t < \infty,\ T_1(\omega) < t \le T_2(\omega)\} \subset [0,\infty[\times\Omega$. Other intervals are denoted in a similar way.

If H is a predictable process we say that H is simple if it is a linear combination of elements of the form $f\,\mathbf{1}_{]T_1,T_2]}$ where $T_1 \le T_2$ are stopping times and f is $\mathcal{F}_{T_1}$ measurable. For the theory of stochastic integration we refer to Protter's book (1990) and for vector stochastic integration we refer to Jacod's book (1979). The reader who is not familiar with vector stochastic integration can think of S as being one dimensional, i.e. $d = 1$. If H is a d-dimensional predictable process that is S integrable, then the process obtained by stochastic integration is denoted $H \cdot S$, its value at time t is $(H \cdot S)_t$.

A strategy is a predictable process that is integrable with respect to the semimartingale S and that satisfies $H_0 = 0$. As in Delbaen and Schachermayer (1994a) we will need the concept of admissible strategy.

1.1 Definition. *An S-integrable predictable strategy H is k-admissible, for $k \in \mathbb{R}_+$, if the process $H \cdot S$ is always bigger than $-k$ and if the limit $\lim_{t\to\infty}(H \cdot S)_t$ exists almost surely. In particular if H is 1-admissible then $H \cdot S \ge -1$.*

For a discussion of this topic and its origin in mathematical finance we refer to Harrison and

Pliska (1981).

We also refer to Harrison and Pliska (1981) for a discussion of the fact that – by considering the discounted values of the stock price S – there is no loss of generality in assuming that the "riskless interest rate r" is assumed to be zero, as we shall assume throughout the paper to alleviate notation. The outcome $(H \cdot S)_\infty$ can be seen as the net profit (or loss) by following the strategy H. If the time set is bounded, then of course, the condition on the existence of the limit at ∞ becomes vacuous. As shown in our paper Delbaen and Schachermayer (1994a), theorem 3.3, the existence of the limit at infinity follows from arbitrage properties.

Fundamental in the proof of the existence of an equivalent local martingale measures are the sets

$$\mathcal{K}_1 = \{ (H \cdot S)_\infty \mid H \text{ 1-admissible strategy } \} \text{ and}$$
$$\mathcal{K} = \{ (H \cdot S)_\infty \mid H \text{ admissible } \}.$$

From Delbaen and Schachermayer (1994a), corollary 3.7, we recall

1.2 Definition. *We say that the semimartingale S satisfies the No Arbitrage condition with respect to general admissible integrands or(abbreviated (NA)) if*

$$\mathcal{K} \cap L^0_+(\Omega) = \{0\}.$$

We say that the semimartingale S satisfies the No Free Lunch Property with respect to general admissible integrands (abbreviated (NFLVR)) if for a sequence of S-integrable strategies $(H_n)_{n\geq 1}$ such that each H_n is a δ_n-admissible strategy and where δ_n tends to zero, we have that $(H \cdot S)_\infty$ tends to zero in probability $\mathbb{P}$.

The following theorem describes the relation between the $NFLVR$ property and the existence of a local martingale measure. The equivalence of these two properties (A resp. D below) is the subject of Delbaen and Schachermayer (1994a), corollary 3.8 and Theorem 1.1. The equivalence with properties B and C below was proved in Delbaen and Schachermayer (1994c), theorem 4, see also Delbaen and Schachermayer (1994d).

1.3 Theorem. *For a locally bounded semimartingale S the following properties are equivalent:*

(A) S satisfies (NFLVR).

(B)

(1) *S satisfies the property (NA) and*

(2) *$\mathcal{K}_1$ is bounded in L^0.*

(C)

(1) *S satisfies the property (NA) and*

(2) *there is a strictly positive local martingale L such that at infinity $L_\infty > 0$, $\mathbb{P}$ a.s. and such that LS is a local martingale.*

(D) S admits an equivalent local martingale measure $\mathbb{Q}$.

In the present paper we will enlarge the scope of the preceding theorem by giving conditions for the existence of an absolutely continuous local martingale measure. In particular we shall prove in section 4 the following central result of the paper.

1.4 Main Theorem. *If the continuous semimartingale S satisfies the No Arbitrage property with respect to general admissible integrands, then there is an absolutely continuous local martingale measure for the process S.*

The paper is organised as follows. Section 2 contains some well known material on the existence of predictable Radon-Nikodym derivatives. The results are mainly due to C. Doléans and are scattered in the "Séminaires". We need a more detailed version for finite dimensional processes. More precisely we treat the case of a predictable measure taking values in the set of positive operators on the space $\mathbb{R}^d$, and we investigate under what conditions a vector measure has a Radon-Nikodym derivative with respect to this operator-valued measure. In this context we say that an operator is positive when it is positive definite. (If we would be aiming for a coordinate free approach, we would rather interprete such an operator valued measure as taking values in the set of semi-positive bilinear forms on $\mathbb{R}^d$). This Radon-Nikodym problem, even for deterministic processes is not treated in the literature (to the best of our knowledge). The proofs are straightforward generalisations of the one-dimensional case. For completeness we give full details.

We need these techniques to prove in section 3 the fact that if the continuous semimartingale $S = M + A$ does not allow arbitrage (with respect to general admissible integrands) then dA may be written as $dA = d\langle M, M\rangle h$ for some predictable $\mathbb{R}^d$-valued process h. This result seems wellknown to people working in Mathematical Finance, but to the best of our knowledge at least the d-dimensional version of this theorem has not been presented in the literature. In section 3 we then investigate the no arbitrage properties and we introduce the concept of immediate arbitrage. We also give an example that illustrates this phenomenon.

In section 4 we prove the main theorem stated above.

The results of this paper were presented at the seminar in the Tokyo University in summer 1993. We thank Professor Kotani and Professor Kusuoka for the invitation and for discussions on this topic. We also thank W. Brannath for discussions on the current proof of theorem 3.7, and an anonymous referee for valuable suggestions.

After finishing this paper we were informed of the paper of Levental and Skorohod (1994), which has a very significant overlap with our results here. In particular, although our framework is more general, the content and the probabilistic approach we give here to proving Theorem 3.7 are essentially identical to that of Lemma 2 of Levental and Skorohod (1994). Their proof appears to have been constructed earlier than ours, although this theorem based on a rather more complicated analytic proof was already presented by the present authors during the SPA conference in Amsterdam in June 93 (1993) and in the seminar of Tokyo University in September 93. Also, theorem 1 of Levental and Skorohod (1994) corresponds to our Main Theorem 1.4 under the additional assumption that the local martingale part M of the continuous semimartingale S is of the form $M = \Sigma \cdot W$, where W is a d-dimensional Brownian motion defined on its (saturated) natural filtration and $\Sigma = (\Sigma_t)_{0\le t\le 1}$ an adapted matrix valued process such that each Σ_t is invertible.

2. The Predictable Radon-Nikodym Derivative

In this section we will prove Radon-Nikodym theorems for stochastic measures. We first deal with the case of one dimensional processes. A stochastic measure on $\mathbb{R}_+$ is described by a stochastic process of finite variation. In our setting it is convenient to require that the measure has no mass at zero, i.e. the initial value of the process is 0. If we have two predictable stochastic measures defined by the finite variation processes A and B respectively, we can for almost every ω in Ω decompose the A-measure in a part absolutely continuous with respect to the B-measure and a component that is singular to it. We are interested in the problem whether such a decomposition can be done in a measurable or even predictable way. Similar problems can be stated for the optional and for the measurable case. For applications in section 3, we only need the case of continuous processes. However the more general case is almost the same and therefore we treat, at little extra cost, processes with jumps.

2.1 Theorem.

(1) *If $A: \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is a predictable, cadlag process of finite variation on finite intervals, then the process V defined as V_t equal to the variation of A on the interval $[0,t]$, is cadlag and predictable.*

(2) *If $A: \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is a predictable, cadlag process of finite variation on finite intervals, if V is defined as in 1, then there is a decomposition of $\mathbb{R}_+ \times \Omega$ into two disjoint, predictable subsets, D^+ and D^-, such that*

$$A_t = \int_0^t (\mathbf{1}_{D^+} - \mathbf{1}_{D^-})\, dV$$

(3) *If $A: \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is a predictable, cadlag process of finite variation on finite intervals, if V is cadlag, predictable and increasing, then there are $\varphi: \mathbb{R}_+ \times \Omega \to \mathbb{R}$ predictable and N a predictable subset of $\mathbb{R}_+ \times \Omega$ such that*

$$A_t = \int_{[0,t]} \varphi_u \, dV_u + \int_{[0,t]} \mathbf{1}_N(u)\, dA_u \quad \textit{and} \quad \int_{\mathbb{R}_+} \mathbf{1}_N \, dV_u = 0$$

Proof. (1) We give the proofs only in the case $A_0 = V_0 = 0$. For the proof we need some results from the general theory of stochastic processes. (see Dellacherie and Meyer (1980)) One of these results says that there is a sequence of predictable stopping times $(T_n)_{n\geq 1}$ that exhausts all the jumps of A. Fix n and let $(\tau_k)_{0\leq k\leq N_n}$ be the finite ordered sequence of stopping times obtained from the set $\{0, 1/2^n, \dots n/2^n, T_1, \dots T_n\}$.

Put $V^n = \sum_{k=0}^{N_n-1} |A_{\tau_{k+1}} - A_{\tau_k}|\, \mathbf{1}_{[\![\tau_{k+1},\infty[\![}$

Because A is predictable, the variables A_{τ_k} are $\mathcal{F}_{\tau_k -}$ measurable and hence the processes V^n are predictable. Because V^n tends pointwise to V, this process is also predictable.

(2) The second part is proved using a constructive proof of the Hahn-Jordan decomposition theorem. It could be left as an exercise but we promised to give details. Let $V = var(A)$ as obtained in the first part. Being predictable and cadlag, the process is locally bounded (Dellacherie and Meyer (1980)) and hence there is an increasing sequence $(T_n)_{n\geq 1}$ of stopping times such that $T_n \uparrow \infty$ and $V_{T_n} \leq n$. Define now

$$\mathcal{H} = \{\varphi \mid \varphi \text{ predictable and } \mathbf{E}[\int_{\mathbb{R}_+} \varphi^2\, dV_u] < \infty\}$$

With the obvious inner product $\langle\varphi,\psi\rangle = \mathbf{E}[\int \varphi_u\psi_u\, dV_u]$, the space $\mathcal{H}$ divided by the obvious subspace $\{\varphi \mid \mathbf{E}[\int \varphi^2\, dV_u] = 0\}$, is a Hilbert space. For each n we define the linear functional L^n on $\mathcal{H}$ as

$$L^n(\varphi) = \mathbf{E}[\int_{[0,T_n]} \varphi_u\, dA_u].$$

Since

$$|\int_{[0,T_n]} \varphi_u\, dA_u| \le \int_{[0,T_n]} |\varphi_u|\, dV_u \le \sqrt{n}(\int_{[0,T_n]} \varphi_u^2\, dV_u)^{1/2},$$

the functional L^n is well defined. Therefore there is ψ^n such that

$$L^n(\varphi) = \mathbf{E}[\int_{[0,T_n]} \varphi_u\psi^n_u\, dV_u].$$

Clearly the elements ψ^n and ψ^{n+1} agree for functions φ supported on $[\![0,T_n]\!]$. Hence (with the convention that $T_0 = 0$) we have that $\psi = \sum_{n\ge1}\psi^n\mathbf{1}_{]\!]T_{n-1},T_n]\!]}$ is predictable and satisfies for all n:

$$L^n(\varphi) = \mathbf{E}[\int_{[\![0,T_n]\!]} \varphi\psi\, dV].$$

Let now $C_t = A_t - \int_0^t \psi_u\, dV_u$. We will show that $C = 0$. First we show that C is continuous. Let τ be a predictable stopping time. Define $\varphi = \Delta C_\tau \mathbf{1}_{[\![\tau]\!]}$. By definition of C and by the property of ψ we have for all n that $\mathbf{E}[(\Delta C)^2_{\tau\wedge T_n}] = 0$. This shows that C is continuous. Next we put $\varphi = C\mathbf{1}_{[\![0,T_n\wedge t]\!]}$ and we find that $\mathbf{E}[C^2_{T_n\wedge t}] = 0$. From this it follows that for all t we have that $C_t = 0$. Because C is cadlag, this implies that the process C vanishes identically.

Until now we proved that in a predictable way $dA = \psi\, dV$. Let now $D^+ = \{\psi = 1\}$ and let $D^- = \mathbb{R}_+ \times \Omega \setminus D^+$. Both sets are predictable and from ordinary measure theory we deduce that $A_t = \int_0^t (\mathbf{1}_{D^+} - \mathbf{1}_{D^-})\, dV$. This gives us the desired Hahn-Jordan decomposition.

(3) The third part is again standard, a constructive proof of Lebesgue's decomposition theorem. Let A and V be given. As in ordinary measure theory we decompose A into its positive and its negative part. Part 2 shows that this can be done in a predictable way. It is therefore sufficient to prove the claim for A increasing. We define $B = A + V$. We now repeat the proof of the second part and we find ψ predictable $0 \le \psi \le 1$ and $dA = \psi dB$. Let $N = \{\psi = 1\}$, a predictable set. We find $dA = \psi\, dA + \psi\, dV$. As in the classical proof we deduce from this equality that $dA = \mathbf{1}_N\, dA + \varphi\, dV$ where $\int \mathbf{1}_N\, dV = 0$ and where φ is predictable. $\square$

2.2 Corollary. *If A and V are as in part 3 of theorem 2.1, if $dA \ll dV$ with respect to the predictable sigma-algebra, i.e. for each predictable set N the property $\int \mathbf{1}_N\, dV = 0$ implies that also $\int \mathbf{1}_N\, dA = 0$, then for almost all ω the measure $dA(\omega)$ is absolutely continuous with respect to $dV(\omega)$ on $\mathbb{R}_+$.*

For applications in finance we need a vector measure generalisation of the preceding results. The theory was developped by Jacod (1979). We need two kinds of vector measures. The first kind is an ordinary vector measure taking values in $\mathbb{R}^d$. The second kind is an operator valued measure that takes values in the set of all operators on $\mathbb{R}^d$. In daily language, in the space of all $d \times d$ matrices. Positive measures on $\mathbb{R}_+$ are generalised as measures that take values in the cone $Pos(\mathbb{R}^d)$ of all positive semi-definite operators on $\mathbb{R}^d$. In this setting the variation process V becomes a predictable, cadlag, increasing process $V: \mathbb{R}_+ \times \Omega \to Pos(\mathbb{R}^d)$. On the set of all operators we put the nuclear norm; for positive operators this simply means the trace of the operator. Let now $\lambda_t = trace(V_t)$. The process λ is predictable, cadlag and increasing. Again we assume $V_0 = 0$, which results in $\lambda_0 = 0$. We have that $dV \ll d\lambda$ in the sense that all elements of the matrix function define measures that are absolutely continuous with respect to λ. If we calculate the Radon-Nikodym derivative using dyadic approximations we see that $dV = \sigma\, d\lambda$, where σ is a predictable process taking values in $Pos(\mathbb{R}^d)$.

For a positive operator a we have that the range $R(a)$ is invariant under a and that on $R(a)$ the operator a is invertible. If we define P_a as the orthogonal projection on $R(a)$ we see that $a^{-1} = a^{-1} \circ P_a$ is a generalised inverse of a. More precisely we have $a \circ a^{-1} := a^{-1} \circ a = P_a$. The correspondence between a, a^{-1} and P_a can be described in a Borel measurable way. This is an easy exercise but we promised to give details.

First note that, for each strictly positive operator b_0, the map $b \to b^{-1}$ is continuous at b_0.

To calculate P_a we simply take the limit

$$\lim_{\varepsilon \downarrow 0} a \circ (a + \varepsilon\, id)^{-1}.$$

This constructive definition shows that the mapping $a \to P_a$ is a Borel measurable mapping. The same trick is used to obtain the generalised inverse

$$a^{-1} = \lim_{\varepsilon \downarrow 0} a \circ (a + \varepsilon\, id)^{-2}.$$

The processes σ^{-1} and P_σ are therefore still predictable since they are the composition of a predictable and a Borel measurable mapping.

We will now describe a kind of absolute continuity of a vector measure with respect to an operator valued measure. Let ν be a measure defined on the σ-ring of relatively compact Borel sets of $\mathbb{R}_+$ and taking values in $\mathbb{R}^d$. Let μ be a measure defined on the same σ-ring and taking values in $Pos(\mathbb{R}^d)$. We say that $\nu \ll \mu$, if whenever $f: \mathbb{R}_+ \to \mathbb{R}^d$ is a Borel function such that either $f(t) = 0$ or $\|f(t)\| = 1$, the expression $d\mu\, f = 0$ (as a vector measure) implies $f' d\nu = 0$ (as a scalar measure). (f' is the transpose of f). One can show that in this case the measure ν has a Radon-Nikodym derivative with respect to μ. Again we will need a predictable version of this theorem, so we give details.

Suppose that $A: \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ is predictable, cadlag and of finite variation on finite intervals. Suppose that $A_0 = 0$. Let V be as above, predictable, cadlag taking values in $Pos(\mathbb{R}^d)$ and increasing. Suppose that for every predictable process $f: \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$, such that $\|f(t,\omega)\|$ is either 0 or 1, the relation $dV\, f = 0$ implies that $f' dA = 0$. This means that $dA \ll dV$ in a predictable way. Let $\lambda = trace(V)$ and let N be a predictable null set for λ, i.e. $\mathbf{1}_N\, d\lambda = 0$. For such a predictable set N and for each predictable k we have $\mathbf{1}_N\, dV\, k = 0$. The hypothesis on A then implies that $\mathbf{1}_N\, k' dA = 0$. This shows that $dA \ll d\lambda$ and the predictable Radon-Nikodym theorem (applied for each coordinate) shows the existence of a predictable $\mathbb{R}^d$-valued process g such that $dA = g\, d\lambda$. Now $(id - \sigma \circ \sigma^{-1})\, dV = dV\, (id - \sigma \circ \sigma^{-1}) = (id - \sigma \circ \sigma^{-1})\sigma\, d\lambda = 0$ and by the assumption on A we have $(id - \sigma \circ \sigma^{-1})\, dA = 0$. This implies that $(id - \sigma \circ \sigma^{-1})\, g\, d\lambda = 0$ and that up to null sets for λ, we have $g \in R(\sigma)$. Let now $h = \sigma^{-1}(g)$. Then obviously $\sigma(h) = g$

(because $g \in R(\sigma)$!!), $h \in R(\sigma)$ and $dA = \sigma\, h\, d\lambda = dV\, h$. The range $R(\sigma)$ could have been called the infinitesimal range $R(dV)$ of the measure V. It is easy to show that it does not depend on the control measure. We completed the proof of the following theorem.

2.3 Theorem. *If V is an increasing predictable cadlag process, taking values in the cone of the positive semi-definite operators on $\mathbb{R}^d$, then the vector measure defined by the predictable $\mathbb{R}^d$-valued cadlag process A of finite variation is of the form $dA = dV\, h$, for some predictable $\mathbb{R}^d$-valued process h, if and only if for each predictable $\mathbb{R}^d$-process f, such that $\|f(t,\omega)\|$ is either 0 or 1, the relation $dV\, f = 0$ implies $f' dA = 0$.*

2.4 Remark. If S is a semimartingale with values in $\mathbb{R}^d$, then the bracket $[S,S]$ and — if it exists — also the bracket $\langle S,S\rangle$ define increasing processes with values in $Pos(\mathbb{R}^d)$. The fact that values are taken in $Pos(\mathbb{R}^d)$ is a reformulation of the Kunita-Watanabe inequalities:

$$|d[S^i,S^j]| \le \sqrt{d[S^i,S^i]\, d[S^j,S^j]}$$
$$|d\langle S^i,S^j\rangle| \le \sqrt{d\langle S^i,S^i\rangle\, d\langle S^j,S^j\rangle}$$

3. The No Arbitrage Property and Immediate Arbitrage.

We now turn to the main theme of the paper, a detailed analysis of the notion of "no arbitrage". We start with an easy lemma, which turns out to be very useful. It shows that the general case of an arbitrage may be reduced to two special kinds of arbitrage.

3.1 Lemma. *If the cadlag semimartingale S does not satisfy the No Arbitrage property with respect to general admissible integrands then at least one of the two following statements holds*

(1) *There is an S-integrable strategy H and a stopping time T, $\mathbb{P}[T<\infty]>0$ such that H is supported by $[\![T, T+1[\![$, $H\cdot S \ge 0$ and $(H\cdot S)_t > 0$, for $t > T$.*
(2) *There is an S-integrable 1-admissible strategy K, $\varepsilon > 0$ and two stopping times $T_1 \le T_2$ such that $T_2 < \infty$ on $\{T_1 < \infty\}$, $\mathbb{P}[T_2<\infty] > 0$, $K = K\mathbf{1}_{]\!]T_1,T_2]\!]}$ and $(K\cdot S)_{T_2} \ge \varepsilon$ on the set $\{T_2 < \infty\}$.*

Proof. Let S allow arbitrage and let H be a 1-admissible strategy that produces arbitrage, i.e., $(H\cdot S)_\infty \ge 0$ with strict inequality on a set of strictly positive probability. We now distinguish two cases. Either the process $H\cdot S$ is never negative or the process $H\cdot S$ becomes negative with positive probability. In the first case let $T = \inf\{t : (H\cdot S)_t > 0\}$.

Let $(\theta_n)_{n=1}^\infty$ be a dense in $]0,1[$ and let $\tilde H = \sum_{n=1}^\infty 2^{-n} H\mathbf{1}_{[\![T,T+\theta_n[\![}$. Then $\tilde H$ satisfies (1). We thank an anonymous referee for correcting a slip in a previous version of this paper at this point.

In the second case we first look for $\varepsilon > 0$ such that $\mathbb{P}[\inf_t (H\cdot S)_t < -2\varepsilon] > 0$. We then define T_1 as the first time the process $H\cdot S$ goes under -2ε, i.e.

$$T_1 = \inf\{t \mid (H\cdot S)_t < -2\varepsilon\}.$$

On the set $\{T_1 < \infty\}$ we certainly have that the process $H\cdot S$ has to gain at least 2ε. Indeed at the end the process $H\cdot S$ is positive and therefore the time

$$T_2 = \inf\{t \mid t > T_1, (H\cdot S)_t \ge -\varepsilon\}$$

is finite on the set $\{T_1 < \infty\}$. We now put $K = H\mathbf{1}_{]\!]T_1,T_2]\!]}$. The process K is 1-admissible since $(K\cdot S)_t \ge -1 + 2\varepsilon$ on the set $\{T_1 < \infty\}$. Also $(K\cdot S)_{T_2} \ge \varepsilon$ on the set $\{T_1 < \infty\}$. □

3.2 Definition. *We say that the semimartingale S admits immediate arbitrage at the stopping time T, where we suppose that $\mathbb{P}[T < \infty] > 0$, if there is an S-integrable strategy H such that $H = H\mathbf{1}_{]\!]T,\infty]\!]}$, and $(H \cdot S)_t > 0$ for $t > T$.*

3.3 Remark. (a) Let us explain why we use the term immediate arbitrage. Suppose S admits immediate arbitrage at T and that H is the strategy that realises this arbitrage opportunity. Clearly $H \cdot S \geq 0$ and $(H \cdot S)_{T+t} > 0$ for all $t > 0$ almost surely on $\{T < \infty\}$. Hence we can make an arbitrage almost surely *immediately* after the stopping time T has occured.

(b) Lemma 3.1 shows that either we have an immediate arbitrage oppurtunity or we have a more conventional form of arbitrage. In the second alternative the strategy to follow is also quite easy. We wait until time T_1 and then we start our strategy K. If the strategy starts at all (i.e. if $T_1 < \infty$) then we are sure to collect at least the amount ε in a finite time. It is clear that such a form of arbitrage is precisely what one wants to avoid in economic models. The immediate arbitrage seems, at first sight, to be some mathematical pathology that can never occur. However the concept of immediate arbitrage can occur as the following example shows. In model building one therefore cannot neglect the phenomenon.

3.4 Example. Take the one-dimensional Brownian motion $W = (W_t)_{t\in[0,1]}$ with its usual filtration. For the price process S we take $S_t = M_t + A_t = W_t + \sqrt{t}$ which satisfies the differential equation $dS_t = dW_t + \frac{dt}{2\sqrt{t}}$. We will show that such a situation leads to "immediate" arbitrage at time $T = 0$. Take $H_t = \frac{1}{\sqrt{t}(\ln t)^2}$. With this choice the integral on the drift-term $\int_0^t H_u \frac{du}{2\sqrt{u}} = \frac{1}{2}(\ln(t^{-1}))^{-1}$ is convergent.

As for the martingale part, the random variable $\int_0^t H_u\, dW_u$ has variance $\int_0^t \frac{1}{u(\ln u)^4}\, du$ which is of the order $\ln(t^{-1})^{-3}$. The iterated logarithm law implies that for $t = t(\omega)$ small enough $|(H \cdot W)_t(\omega)| \leq C\sqrt{(\ln(t^{-1}))^{-3} \ln\ln((\ln(t^{-1}))^3)} \leq C'(\ln(t^{-1}))^{-5/4}$. It follows that, for t small enough, we necessarily have that $(H \cdot S)_t(\omega) > 0$. We now define the stopping time T as $T = \inf\{t > 0 \mid (H \cdot S)_t = 0\}$ and, for $n > 0$, $T_n = T \wedge n^{-1}$. Clearly $(H \cdot S)^T \geq 0$ and $\mathbb{P}[(H \cdot S)_{T_n} > 0]$ tends to 1 as n tends to infinity. By considering the integrand $L = \sum_{n=1}^{\infty} \alpha_n H^{T_n}$ for a sequence $\alpha_n > 0$ tending to zero sufficiently fast, we can even obtain that $(L \cdot S)_t$ is almost surely strictly positive for each $t > 0$.

We now give some more motivation why such a form of arbitrage is called immediate arbitrage. In the preceding example, for each stopping time $T > 0$ the process $S - S^T$ admits an equivalent martingale measure $\mathbb{Q}(T)$ given by the density $f_T = \exp(-\frac{1}{2}\int_T^1 \frac{1}{\sqrt{u}}\, dW_u - 1/8 \int_T^1 \frac{1}{u}\, du)$. We can check this by means of the Girsanow-Maruyama formula or we can checkit even more directly via Itô's rule. This statement shows that if one wants to make an arbitrage profit one has to be very quick since a profit has to be the result of an action taken before time T.

Let us also note that the process S also satisfies the (NA) property for simple integrands. As is well known it suffices to consider integrands of the form $f\mathbf{1}_{]\!]T_0,T_1]\!]}$ where f is $\mathcal{F}_{T_0}$-measurable (see Delbaen and Schachermayer (1994e)). Let us show that such an integrand does not allow an arbitrage. Take $T_0 \leq T_1$ two stopping times. We distinguish between $\mathbb{P}[T_0 > 0] = 1$ and $T_0 = 0$. (The $0-1$ law for $\mathcal{F}_0$ (Blumenthal's theorem) shows that one of the two holds).

If $T_0 > 0$, $\mathbb{P}$ a.s. then the result follows immediately from the existence of the martingale measure $\mathbb{Q}(T_0)$ for the process $S - S^{T_0}$.

If $T_0 = 0$ we have to prove that $S_{T_1} \geq 0$ (or $S_{T_1} \leq 0$) implies that $S_{T_1} = 0$ a.s..

We concentrate on the first case and assume to the contrary that $S_{T_1} \geq 0$ and $\mathbb{P}\{S_{T_1} > 0\} > 0$. Note that it follows from the law of the iterated logarithm that $\inf\{t | S_t < 0\} = 0$ almost surely, hence the stopping times

$$T_\varepsilon = \inf\{t > \varepsilon | S_t < -\varepsilon\}$$

tend to zero a.s. as ε tends to zero. Let $\varepsilon > 0$ be small enough such that $\{T_\varepsilon < T_1\}$ has positive measure to arrive at a contradiction:

$$0 > \mathbf{E}_{\mathbb{Q}(T_\varepsilon)}[S_{T_\varepsilon} \mathbf{1}_{(T_\varepsilon < T_1)}] = \mathbf{E}_{\mathbb{Q}(T_\varepsilon)}[S_{T_1} \mathbf{1}_{(T_\varepsilon < T_1)}] \geq 0.$$

The following theorem, which is based on the material developped in section 2, is well known and has been around for some time. At least in dimension $d = 1$ the result should be known for a long time. For dimension $d > 1$, the presentation below is, we guess, new.

3.5 Theorem. *If the d-dimensional, locally bounded semimartingale S satisfies the (NA) property for general admissible integrands, then the Doob-Meyer decomposition $S = M + A$ satisfies $dA = d\langle M,M\rangle\, h$ where h is a d-dimensional predictable process and where $d\langle M,M\rangle$ denotes the operator valued measure defined by the $d \times d$ matrix process $(\langle M,M\rangle)_{i,j\leq d}$. The process h may be chosen to take its values in the infinitesimal range $R(d\langle M,M\rangle)$.*

Proof. We apply the criterion of section 2. Take f a d-dimensional predictable process such that the measure $d\langle M,M\rangle\, f$ is zero and such that either f has norm one or norm zero. It is obvious that the stochastic integral $f' \cdot M$ exists and results in the zero process. If the process $f' \cdot A$ is not zero then we replace f by the sign function coming from the Jordan-Hahn decomposition of $f' \cdot A$. This sign function ϕ is a predictable process equal to $+1$ or -1. The predictable integrand $g = \phi\, f$ still satisfies $g \cdot M = 0$ but the component $g' \cdot A$ now results in an arbitrage profit. This contradiction shows that the criterion of section 2 is fulfilled and hence the existence of the process h is proved. If we write $d\langle M,M\rangle$ as $\sigma\, d\lambda$ for some control measure λ and an operator valued predictable process σ, then we may, by the results of section 2, suppose that h_t is in the range of the operator σ_t. $\square$

The following theorem is the basic theorem in dealing with the (NA) property in the case of continuous price processes.

3.6 Theorem. *If the continuous semimartingale S with Doob-Meyer decomposition $S = M + A$ satisfies the (NA) property for general admissible integrands, then we have $dS = dM + d\langle M,M\rangle\, h$ where the predictable process h satisfies:*

(1)
$$T = \inf\left\{ t \mid \int_0^t h'_u\, d\langle M,M\rangle_u\, h_u = \infty \right\} > 0 \quad a.s..$$

(2) *The $[0,\infty]$-valued increasing process $\int_0^t h'_u\, d\langle M,M\rangle_u\, h_u$ is continuous; in particular it does not jump to ∞.*

Proof. The existence of the process h follows from the preceding theorem. The stopping time T is well defined. The first claim on the stopping time T follows from the second, so we limit the proof to the second statement. We will prove that the set

$$F = \{T < \infty\} \cap \{\int_T^{T+\varepsilon} h_t'\, d\langle M, M\rangle_t\, h_t = \infty\ \forall \varepsilon > 0\}$$

has zero measure. Clearly F is, by right continuity of the filtration, an element of the σ-algebra $\mathcal{F}_T$. As the process $\langle M, M\rangle_t$ is continuous, assertion (2) will follow from the fact that $\mathbb{P}[F] = 0$. Suppose now to the contrary that F has strictly positive measure. We then look at the process $\mathbf{1}_F(S - S^T)$, adapted to the filtration $(\mathcal{F}_{T+t})_{t\geq 0}$ and we replace the probability $\mathbb{P}$ by $\mathbb{P}_F$. With this notation the theorem is reduced to the case $T = 0$. This case is treated in the following theorem. It is clear that this will end the proof. ☐

3.7 Immediate Arbitrage Theorem. *Suppose the d-dimensional continuous semi- martingale S has a Doob-Meyer decomposition given by*

$$dS_t = dM_t + d\langle M, M\rangle_t\, h_t$$

where h is a d-dimensional predictable process. Suppose that a.s.

$$\int_0^\varepsilon h_t'\, d\langle M, M\rangle_t\, h_t = \infty \quad \forall\ \varepsilon > 0. \tag{1}$$

Then for all $\varepsilon > 0$, there is an S-integrable strategy H such that $H = H\mathbf{1}_{[\![0,\varepsilon]\!]}$, $H \cdot S \geq 0$ and $\mathbb{P}[(H \cdot S)_t > 0] = 1$, for each $t > 0$. In other words S admits immediate ¿arbitrage at time $T = 0$.

Proof. The proof of the theorem is based on the following lemma

3.8 Lemma. *If (1) holds almost surely then for any $a, \varepsilon, \eta > 0$ we can find $0 < \delta < \varepsilon/2$ and an a-admissible integrand H with*

$H = H\,\mathbf{1}_{]\!]\delta,\varepsilon]\!]}$
$\int_\delta^\varepsilon |H_s'\, dA|_s + \int_\delta^\varepsilon H_s'\, d\langle M, M\rangle_s\, H_s < 2 + a$
$P[(H \cdot S)_\varepsilon \geq 1] \geq 1 - \eta.$

Proof of the Lemma. Fix $a, \varepsilon, \eta > 0$ and let $R \geq \max\{\frac{8}{\eta}\left(\frac{1+a}{a}\right)^2, (1+a)^2\}$. Since (1) is satisfied almost surely, we have that

$$\lim_{\delta\downarrow 0, K\uparrow\infty} \mathbb{P}\left[\int_\delta^\varepsilon \mathbf{1}_{\{|h|\leq K\}}\, h_t'\, d\langle M, M\rangle_t\, h_t \geq R\right] = 1.$$

Hence we can find a $K > 0$ and a $0 < \delta < \varepsilon/2$ such that

$$\infty > \int_\delta^\varepsilon \mathbf{1}_{\{|h|\leq K\}}\, h_t'\, d\langle M, M\rangle_t\, h_t \geq R$$

on a $\mathcal{F}_\varepsilon$-measurable set Λ with $P[\Lambda] \geq 1 - \frac{\eta}{2}$. Let

$$T = \inf\{t > 0 \mid \int_\delta^t \mathbf{1}_{\{|h| \leq K\}}\, h_t'\, d\langle M, M\rangle_t\, h_t \geq R\} \wedge \varepsilon$$

and let $H = \frac{1+a}{R}\, h \mathbf{1}_{]\delta, T]}\, \mathbf{1}_{\{|h|\leq K\}}$. Then

$$\int_0^\varepsilon H_s'\, d\langle M, M\rangle_s\, H_s \leq \frac{(1+a)^2}{R}$$

and

$$\int_0^\varepsilon |H_s\, dA|_s \leq (1+a) \text{ a.s..}$$

Therefore H is S-integrable. Moreover, $(H \cdot A)_\varepsilon = 1 + a$ on Λ.

Since $\|H \cdot M\|_2^2 = E[\int_0^\varepsilon H_s' d\langle M, M\rangle_s\, H_s] \leq \frac{(1+a)^2}{R}$ we obtain from Doob's inequality together with Tchebychew's inequality (both in their L^2–version)

$$\mathbb{P}\left[(H \cdot M)^* \geq a\right] \leq 4 \left(\frac{1+a}{a}\right)^2 \frac{1}{R} \leq \frac{\eta}{2} \tag{2}$$

We now localize H to be a-admissible. Let

$$T_2 = \inf\{t > 0 \mid (H \cdot M)_t < -a\} \wedge T.$$

Then $T_2 = T$ on $\{(H \cdot M)^* \leq a\}$ and from (2) we obtain

$$\begin{aligned} \mathbb{P}\left[(H\, \mathbf{1}_{[0, T_2]} \cdot S)_\varepsilon \geq 1\right] &\geq \mathbb{P}\left[\{(H \cdot A)_\varepsilon \geq 1 + a\} \cap \{(H \cdot M)^* < a\}\right] \\ &\geq \mathbb{P}[\Lambda] - \mathbb{P}\left[(H \cdot M)^* \geq a\right] \geq 1 - \eta \end{aligned}$$

which proves the lemma. $\square$

Proof of the Immediate Arbitrage Theorem. Assume that (1) is valid for almost every $\omega \in \Omega$. We will now construct an integrand which realizes immediate arbitrage. Let $\varepsilon_0 > 0$ be such that $\varepsilon_0 \leq \min(\varepsilon, \frac{1}{2})$. By lemma 3.8 we can find a stricly decreasing sequence of positive numbers $(\varepsilon_n)_{n\geq 0}$ with $\lim_{n\to\infty} \varepsilon_n \to 0$ and integrands $H_n = H_n \mathbf{1}_{]\varepsilon_{n+1}, \varepsilon_n]}$ such that H_n is 4^{-n}–admissible, $\int_{\varepsilon_{n+1}}^{\varepsilon_n} |(H_n)_s'\, dA_s| + \int_{\varepsilon_{n+1}}^{\varepsilon_n} (H_n)_s'\, d\langle M, M\rangle_s\, (H_n)_s < 3/2^n$ and $\mathbb{P}\left[(H_n \cdot S)_{\varepsilon_n} \geq 2^{-n}\right] \geq 1 - 2^{-n}$. Let $\hat{H} = \sum_{n=1}^\infty H_n$. Then $\hat{H}$ is S-integrable. Define

$$T(\omega) = \inf\left\{t > 0 \mid (\hat{H} \cdot S)_t = 0\right\}.$$

We claim that $T(\omega) > 0$ for almost every $\omega \in \Omega$. Since $\mathbb{P}\left[(H_n \cdot S)_{\varepsilon_n} < 2^{-n}\right] \leq 2^{-n}$, we obtain from the Borel–Cantelli Lemma that for almost every $\omega \in \Omega$ there is a $N(\omega) \in \mathbb{N}$ with $(H_n \cdot S)_{\varepsilon_n}(\omega) > 2^{-n}$ for all $n > N(\omega)$. If $n > N(\omega)$ and $\varepsilon_{n+1} < t \leq \varepsilon_n$ then

$$(\hat{H} \cdot S)_t(\omega) = \underbrace{\sum_{k>n}^\infty (H_k \cdot S)_{\varepsilon_k}(\omega)}_{\geq 2^{-n}} + \underbrace{(H_n \cdot S)_t(\omega)}_{\geq -2^{-(n+1)}} \geq \frac{1}{2^{n+1}}$$

and we have verified the claim. Hence

$$\lim_{t\to 0} \mathbb{P}\left[\left(\hat{H}\,\mathbf{1}_{[\![0,T]\!]}\cdot S\right)_t > 0\right] = 1\ .$$

Finally let

$$H = \sum_{n=1}^{\infty} 2^{-n} \hat{H} \mathbf{1}_{]\!]0,T\wedge\varepsilon_n[\![}$$

to find an S-integrable predictable process supported by $[0,\varepsilon]$ such that $(H\cdot S)_t > 0$ for each $t > 0$. $\square$

4. The existence of an absolutely continuous martingale measure.

We start this section with the investigation of the support of an absolutely continuous risk neutral measure. The theory is based on the analysis of the density given by a Girsanow–Maruyama transformation. If $dS_t = dM_t + d\langle M, M\rangle_t\, h_t$ defines the Doob-Meyer decomposition of a continuous semimartingale, where h is a d-dimensional predictable process and where M is a d-dimensional continuous local martingale, then the Girsanov–Maruyama transformation is, at least formally, given by the local martingale $L_t = \exp(\int_0^t -h'_u\, dM_u - 1/2 \int_0^t h'_u\, d\langle M, M\rangle_u\, h_u)$, $L_0 = 1$. Formally one can verify that LS is a local martingale. However, things are not so easy. First of all, there is no guarantee that the process h is M-integrable, so L need not be defined. Second, even if L is defined, it may only be a local martingale and not a uniformly integrable martingale. The examples in Schachermayaer (1993) and in Delbaen and Schachermayer (1994b) show that even when an equivalent risk neutral measure exists, the local martingale L need not to be uniformly integrable. In other words a risk neutral measure need not be given by L. Third, in case the two previous points are fulfilled, the density L_∞ need not be different from zero a.s..

What can we save in our setting? In any case, theorem 3.6 shows that in the case when S satisfies the No Arbitrage property for general admissible integrands, the process h satisfies the properties

(1)
$$T = \inf\{t \mid \int_0^t h' d\langle M,M\rangle\, h = \infty\} > 0 \ \ a.s..$$

(2) The $[0,\infty]$-valued proces $\int_0^t h'_u\, d\langle M,M\rangle_u\, h_u$ is continuous; in particular it does not jump to ∞.

In this case the stochastic integrals $h\cdot M$ and $h\cdot S$ can be defined on the interval $[\![0,T[\![$ and at time T we have that L_T can be defined as the left limit. The theory of continuous martingales (Revuz and Yor (1991)) shows that

$$\{L_T = 0\} = \left\{\int_0^T h'_t\, d\langle M, M\rangle_t\, h_t = \infty\right\}.$$

If after time T, i.e. for $t > T$, we put $L_t = 0$ the process L is well defined, it is a continuous local martingale, satisfies $dL_t = -L_t\, h'_t\, dM_t$ and LS is a local martingale. The process $X = \frac{1}{L} - 1$

is also defined on the interval $[0,T[$ and on the set $\{L_T = 0\}$ its left limit equals infinity. The crucial observation is now that on the interval $[0,T[$, we have that $dX_t = \frac{1}{L_t} h'_t \, dS_t$.

This follows simply by plugging in Itô's formula (compare Delbean and Schachermayer (1994c)).

For each $\varepsilon > 0$ let τ^ε be the stopping time defined by $\tau^\varepsilon = \inf\{t \mid L_t \le \varepsilon\}$. Because the process X is always larger than -1, the stopped processes X^{τ^ε} are outcomes of admissible integrands. If $\mathbb{Q}$ is an absolutely continuous probability measure such that S becomes a local martingale than, by theorem 1.3 we have that the set $\mathcal{H} = \left\{X_\infty^{\tau^\varepsilon} \mid \varepsilon > 0\right\}$ is bounded in $L^0(\{\frac{d\mathbb{Q}}{d\mathbb{P}} > 0\})$. But it is clear that on the set $\{L_T = 0\}$, the set $\mathcal{H}$ is unbounded.

As a consequence we obtain the following

4.1 Lemma. *If the continuous semimartingale S satisfies the No Arbitrage condition with respect to general admissible integrands and if $\mathbb{Q}$ is an absolutely continuous local martingale measure for S, then $\{\frac{d\mathbb{Q}}{d\mathbb{P}} > 0\} \subset \{L_T > 0\}$.*

In order to prove the existence of an absolutely continuous local martingale measure $\mathbb{Q}$ we therefore should restrict ourselves to measures supported by

$$F = \{L_T > 0\}.$$

Note that the No Arbitrage condition implies that $\mathbb{P}[F] > 0$. Indeed, suppose that $\mathbb{P}[F] = 0$ and let

$$U = \inf\{t : L_t \le \frac{1}{2}\}.$$

We than have that $\mathbb{P}[U < \infty] = 1, L_U \equiv \frac{1}{2}$ and therefore $X_U \equiv 1$. Hence $H = \frac{1}{L} h' \mathbf{1}_{[0,U]}$ is a 1-admissible integrand such that $(H \cdot S)_\infty \equiv X_U \equiv 1$, a contradiction to (NA).

So we will look at the process S under the conditional probability measure $\mathbb{P}_F$.

Our strategy will be to verify that S satisfies the property (NFLVR) with respect to $\mathbb{P}_F$ which will imply the existence of a local martingale measure $\mathbb{Q}$ for S which is equivalent to $\mathbb{P}_F$ and therefore absolutely continuous with respect to P. But there are difficulties:

Under the measure $\mathbb{P}_F$ the Doob-Meyer decomposition will change, there will be more admissible integrands and the verification of the No Free Lunch Property with Vanishing Risk for general admissible integrands (under $\mathbb{P}_F$!!) is by no means trivial.

We are now ready to reformulate the main theorem stated in the introduction in a more precise way and to abord the proof:

4.2 Main Theorem. *If the continuous semimartingale S satisfies the No Arbitrage property with respect to general admissible integrands, then with the notation introduced above, it satisfies the No Free Lunch Property with Vanishing Risk with respect to $\mathbb{P}_F$.*

As a consequence there is an absolutely continuous local martingale measure that is equivalent to $\mathbb{P}_F$, i.e. it is precisely supported by the set F.

The proof of the theorem still needs some auxiliary steps which will be stated below.

We first deal with the problem of the usual hypotheses under the measure $\mathbb{P}_F$. The σ-algebras $\tilde{\mathcal{F}}_t$ of the $\mathbb{P}_F$-augmented filtration are obtained from $\mathcal{F}_t$ by adding all $\mathbb{P}_F$ null sets. It is easily seen that the new filtration is still right continuous and satisfies the usual hypotheses for the new measure $\mathbb{P}_F$. The following technical results are proved in Delbaen and Schachermayer (1994d).

4.3 Proposition. *If $\tilde{\tau}$ is a stopping time with respect to the filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ then there is a stopping time τ with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathbb{P}_F$ a.s. we have $\tilde{\tau} = \tau$. If $\tilde{\tau}$ is finite or bounded then τ may be chosen to be finite or bounded.*

4.4 Proposition. *If $\tilde{H}$ is a predictable process with respect to the filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ then there is a predictable process H with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, such that $\mathbb{P}_F$ a.s. we have $\tilde{H} = H$.*

This settles the problem of the usual hypotheses. Each time we need a $\tilde{\mathcal{F}}$-predictable process, we can without danger, replace it by a predictable process for $\mathcal{F}$. Without further notice we will do this.

The process S is a semimartingale with respect to the system $(\tilde{\mathcal{F}}, \mathbb{P}_F)$. This is well known, see Protter (1990).

Remark also that for $\mathbb{P}_F$ we have that $\int_0^\infty h'_u \, d\langle M, M\rangle_u \, h_u < \infty$ a.s.. We will need this later on.

As a first step we will decompose S into a sum of a $\mathbb{P}_F$-local martingale and a predictable process of finite variation. Because $\mathbb{P}_F$ is only absolutely continuous with respect to $\mathbb{P}$ we need an extension of the Girsanov-Maruyama formula for this case. The generalisation was given by Lenglart (1977). We need the cadlag martingale U defined as

$$U_t = \mathbf{E}\left[\frac{\mathbf{1}_F}{\mathbb{P}[F]} \mid \mathcal{F}_t\right]$$

Note that U is not necessarily continuous, as we only assumed that S is continuous and not that each $\mathcal{F}_t$-martingale is continuous.

Together with the process U we need the stopping time

$$\nu = \inf\{t \mid U_t = 0\} = \inf\{t > 0 \mid U_{t-} = 0\}$$

(see Dellacherie and Meyer (1980) for this equality).

4.5 Lemma. $\nu = T$ *$\mathbb{P}$-almost surely.*

Proof of the lemma. We first show that for an arbitrary stopping time σ we have that $L_\sigma > 0$ on the set $\{U_\sigma > 0\}$. Let A be a set in $\mathcal{F}_\sigma$ such that $\mathbb{P}[A \cap \{U_\sigma > 0\}] > 0$. This already implies that $\mathbb{P}[A \cap F] > 0$. Indeed we have that

$$\mathbf{E}[\mathbf{1}_A \mathbf{1}_F \mid \mathcal{F}_\sigma] = \mathbb{P}[F] \mathbf{1}_A U_\sigma$$

and hence we necessarily have that $\mathbb{P}[A \cap F] > 0$. The following chain of equalities is almost trivial

$$\int_{A\cap\{U_\sigma>0\}} L_\sigma = \int_A L_\sigma \, \mathbf{1}_{\{U_\sigma>0\}} \geq \mathbb{P}[F] \int_A L_\sigma \, U_\sigma = \int_A L_\sigma \mathbf{1}_F = \int_{A\cap F} L_\sigma.$$

The last term is strictly positive since $L_\sigma > 0$ on F. This proves that for each set A such that $\mathbb{P}[A \cap \{U_\sigma > 0\}] > 0$ we must have $\int_{A\cap\{U_\sigma>0\}} L_\sigma > 0$. This implies that $L_\sigma > 0$ on the set $\{U_\sigma > 0\}$, hence $\nu \leq T$.

The converse inequality is less trivial and requires the use of the (NA) property of S. We proceed in the same way. Take $G \in \mathcal{F}_\sigma$ such that $G \subset \{L_\sigma > 0\}$ and $\mathbb{P}[G] > 0$. Suppose that

$U_\sigma = 0$ on G. We will show that this leads to a contradiction. If $U_\sigma = 0$ on G then clearly $G \cap F = \emptyset$. But on F^c we have that L_t tends to zero and hence $\frac{1}{L_t}$ tends to ∞. We know that $\frac{1}{L_t} - 1$ can be obtained as a stochastic integral with respect to S. We take the stopping time $\mu = \infty$ on G^c and equal to $\inf\{t \mid L_t \le \frac{1}{2} L_\sigma\}$ on the set G. The outcome

$$\mathbf{1}_G = \left(\frac{1}{L_\mu} - \frac{1}{L_\sigma}\right) L_\sigma \, \mathbf{1}_G$$

is the result of a 1-admissible strategy and clearly produces arbitrage. We may therefore suppose that $\mathbb{P}[G \cap F] > 0$ and hence we also have $\int_G U_\sigma > 0$. Again this suffices to show that $U_\sigma > 0$ on the set $\{L_\sigma > 0\}$ and again implies that $T \le \nu$. The proof of the lemma is complete now. $\square$

Proof of the Main Theorem. We now calculate the decomposition of the continuous semimartingale S under $\mathbb{P}_F$. If $S = M + A$ is the Doob-Meyer decomposition of S under $\mathbb{P}$ then, under $\mathbb{P}_F$ we write $S = \tilde{M} + \tilde{A}$ where $\tilde{A}_t = A_t + \int \frac{d\langle M, U\rangle_s}{U_s}$, see Lenglart (1977). This integral exists for the measure $\mathbb{P}_F$ since on F the process U is bounded away from 0. A more explicit formula for $\tilde{A}$ can be found if we use the structure of $\langle M, U\rangle$. We thereto use the Kunita-Watanabe decomposition of the L^2 martingale U with respect to the martingale M. This is done in the following way (see Jacod (1979)). The space of all L^2 martingales of the form $\alpha \cdot M$ is a stable space and in fact we have $\|(\alpha \cdot M)_\infty\|_2 = \mathbf{E}[\int \alpha' \, d\langle M, M\rangle \, \alpha]$. The orthogonal projection of U_∞ on this space is given by $(\beta \cdot M)_\infty$ for some predictable process β, where of course

$$\mathbf{E}[\int \beta' \, d\langle M, M\rangle \, \beta] < \infty.$$

In this notation we may write:

$$d\langle M, U\rangle = d\langle M, M\rangle \beta.$$

It follows that also $\int \beta' \, d\langle M, M\rangle \, \beta < \infty$ a.s. for the measure $\mathbb{P}_F$ and the measure $d\tilde{A}$ can be written as

$$d\tilde{A} = d\langle M, M\rangle (h_t + \frac{\beta_t}{U_t}) = d\langle M, M\rangle \, k_t.$$

Here we have put $k = h + \frac{\beta}{U}$ to simplify notation.

To prove the $NFLVR$ property for S under $\mathbb{P}_F$ we use the criterion of theorem 1.3 above.

Step 1: the set of 1-admissible integrands for $\mathbb{P}_F$ is bounded in $L^0(F)$

From the properties of β and h we deduce that for the measure $\mathbb{P}_F$, the integral

$$\int_0^\infty k_t' d\langle M, M\rangle k_t < \infty \quad \mathbb{P}_F \text{ a.s.}$$

The $\mathbb{P}_F$ local martingale $\tilde{L}$ is now defined as

$$\tilde{L}_t = \exp\left(-\int_0^t k_u' \, d\tilde{M}_u - \frac{1}{2}\int_0^t k_u' \, d\langle M, M\rangle_u \, k_u\right)$$

It follows that

$$\tilde{L}_\infty > 0 \ \ \mathbb{P}_F \text{ a.s.}$$

It is chosen in such a way that $\tilde{L}S$ is a $\mathbb{P}_F$ local martingale and therefore the set $\tilde{\mathcal{K}}_1$ constructed with the 1-admissible, with respect to $\mathbb{P}_F$!!!, integrands, is bounded in $L^0(\mathbb{P}_F)$.

In particular this also excludes the possibility of immediate arbitrage for S with respect to $\mathbb{P}_F$.

Step 2: S satisfies (NA) with respect to $\mathbb{P}_F$ (and with respect to general admissible integrands).

Since by step 1 immediate arbitrage is excluded the violation of the (NA) property would, by lemma 3.1, give us a predictable integrand H such that for $\mathbb{P}_F$ the integrand is of finite support, is S-integrable and 1-admissible. When the support of H is contained in $]\!]\sigma_1, \sigma_2]\!]$ it gives an outcome at least ε on the set $\{\sigma_1 < \infty\}$. All this, of course, with respect to $\mathbb{P}_F$.

The rest of the proof is devoted to the transformation of this phenomenon to a situation valid for $\mathbb{P}$.

Without loss of generality we may suppose that for the measure $\mathbb{P}$ we have $\sigma_1 \le \sigma_2 \le T$, we replace e.g. the stopping time σ_2 by $\max(\sigma_1, \sigma_2)$ and then we replace σ_1 and σ_2 by respectively $\min(T, \sigma_1)$ and $\min(T, \sigma_2)$. All these substitutions have no effect when seen under the measure $\mathbb{P}_F$. Since $\mathbb{P}_F[\{\sigma_1 < \sigma_2 < \infty\}] > 0$, we certainly have that $\mathbb{P}[\{\sigma_1 < \sigma_2 < T\}] > 0$.

Roughly speaking we will now use the strategy H to construct arbitrage on the set F and we use the process $\frac{1}{L}$ to construct a sure win on the set F^c, as on the interval $[\![0, T[\![$, the process $\frac{1}{L} - 1$ equals $K \cdot S$ for a well chosen integrand K. When we add the two integrands, H and K, we should obtain an integrand that gives arbitrage on Ω with respect to $\mathbb{P}$ and this will provide the desired contradiction.

Let the sequence of stopping times τ_n be defined as

$$\tau_n = \inf\left\{ t \mid L_t \le \frac{1}{n} \right\}.$$

We have that $\tau_n \uparrow T$ for $\mathbb{P}$ and $\tau_n \uparrow \infty$ for the measure $\mathbb{P}_F$. Since we have that $L_{\tau_n} > 0$ a.s. we also have that $U_{\tau_n} > 0$ a.s.. It follows that on the σ-algebra $\mathcal{F}_{\tau_n}$ the two measures, $\mathbb{P}$ and $\mathbb{P}_F$ are equivalent. We can therefore conclude that for each n the integrand $H\mathbf{1}_{[\![0,\tau_n]\!]}$ as well as the integrand $K\mathbf{1}_{[\![0,\tau_n]\!]}$ is S integrable and 1-admissible for $\mathbb{P}$. The last integrand still has to be renormalised.

In fact on the set F itself, the lower bound -1 for the process $K \cdot S$ is too low since it will be compensated at most by ε. We therefore transform K in such a way that it will stay above $\varepsilon/2$ but will nevertheless give outcomes that are very big on the set F^c. Let us define

$$\begin{aligned}
\tilde{K} &= K\mathbf{1}_{\{\sigma_1 < T\}}\frac{\varepsilon}{2}L_{\sigma_1} \\
\tilde{K}^n &= \tilde{K}\,\mathbf{1}_{[\![0,\tau_n]\!]} \\
\tilde{H} &= H\mathbf{1}_{\{\sigma_1 < T\}} \\
\tilde{H}^n &= \tilde{H}\,\mathbf{1}_{[\![0,\tau_n]\!]}.
\end{aligned}$$

From the preceding considerations it follows that the integrands $\tilde{H}^n$ are all 1-admissible for $\mathbb{P}$ and that the integrands $\tilde{K}^n$ are $\varepsilon/2$-admissible for $\mathbb{P}$. The outcomes $(\tilde{K}^n \cdot S)_{\tau_n}$ tend to ∞ on $F^c \cap \{\sigma_1 < T\}$, and the outcomes $(\tilde{H}^n \cdot S)_{\tau_n}$ become larger than ε on the set $F \cap \{\sigma_1 < T\}$. When we add them we see that on the set $\{\sigma_1 < T\}$ we have

$$\liminf_{n\to\infty}((\tilde{H} + \tilde{K}) \cdot S)_{\tau_n} = \liminf_{n\to\infty}((\tilde{H}^n + \tilde{K}^n) \cdot S)_{\tau_n} \geq \frac{\varepsilon}{2}.$$

Define now the stopping time μ as

$$\mu = \tau_n \text{ if } n \text{ is the first number such that } ((\tilde{H}^n + \tilde{K}^n) \cdot S)_{\tau_n} \geq \frac{\varepsilon}{4}.$$

The stopping time μ is finite on the set $\{\sigma_1 < \tau\}$. The integrand $J = (\tilde{H} + \tilde{K})\mathbf{1}_{]0,\mu]}$ is now S integrable and is certainly $1 + \frac{\varepsilon}{2}$ admissible. By the definition of the stopping time μ we have that $(J \cdot S)_\mu \geq \frac{\varepsilon}{4}\mathbf{1}_{\{\sigma_1 < T\}}$, producing arbitrage. Since the process S satisfied the (NA) property, we arrived at a contradiction.

Step 2 is therefore completed and this ends the proof of the theorem. $\square$

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