A characterisation of the closure of H^∞ in BMO

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Abstract. We show that a continuous martingale $M \in BMO$ has a $\|\cdot\|_{BMO_2}$-distance to H^∞ less than $\varepsilon > 0$ iff M may be written as a finite sum $M = \sum_{n=0}^{N} {}^{T_n}M^{T_{n+1}}$ such that, for each $0 \le n \le N$, we have $\| {}^{T_n}M^{T_{n+1}} \|_{BMO_2} < \varepsilon$. In particular, we obtain a characterisation of the BMO-closure of H^∞.

This result was motivated by some problems posed in the survey of N. Kazamaki [K 94]. We also give answers to some other questions, pertaining to BMO-martingales, which have been raised by N. Kazamaki [K 94].

1. Introduction

The celebrated Garnett-Jones theorem — in its martingale version due to N. Varopoulos and M. Emery ([K 94], th. 2.8) — characterizes the BMO-distance of a continuous martingale M from L^∞ in terms of (the inverse of) the critical exponent $a(M)$, defined by

$$a(M) = \sup\{a \in \mathbb{R}_+ | \sup_T \|\mathbb{E}[\exp(a|M_\infty - M_T|)|\mathcal{F}_T]\|_\infty < \infty\},$$

where T runs through all stopping times.

In [K 94] N. Kazamaki proposed the critical exponent

$$b(M) = \sup\{b \in \mathbb{R}_+ | \sup_T \|\mathbb{E}[\exp(b(\langle M\rangle_\infty - \langle M\rangle_T))|\mathcal{F}_T]\|_\infty < \infty\},$$

and raised the question whether (the inverse of) $b(M)$ characterizes the BMO-distance of M to H^∞.

We shall give in section 3 an example of a continuous martingale M in BMO such that $b(M) = \infty$ while M is not in the BMO-closure of H^∞. In the present context H^∞ denotes the space of continuous martingales M on a given stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$ such that $\|M\|_{H^\infty} = \text{ess sup}\langle M\rangle_\infty^{\frac{1}{2}} < \infty$.

This example, which also answers negatively another question of [K 94], provides strong evidence that there is little hope to find a characterization of the BMO-closure of H^∞ analogous to the Garnett-Jones theorem in terms of some critical exponent.

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But we can give a different kind of characterization of $\overline{H^\infty}^{\|\cdot\|_{BMO}}$ and, more precisely, of the BMO_2-distance of a continuous martingale to H^∞. Recall ([K 94], p.25) that, for a continuous local martingale M, the BMO_2-norm is defined as

$$\|M\|_{BMO_2} = \sup_T \{\|\mathbb{E}[\langle M\rangle_\infty - \langle M\rangle_T|\mathcal{F}_T]^{\frac{1}{2}}\|_\infty\},$$

where T runs through all stopping times. (For unexplained notation we refer to the end of the introduction and to [K 94]).

1.1 Theorem. *Let M be a continuous, real-valued martingale in $BMO, M_0 = 0$. For $\varepsilon > 0$, we can find a finite increasing sequence*

$$0 = T_0 \leq T_1 \leq \cdots \leq T_N \leq T_{N+1} = \infty$$

of stopping times such that

$$\|{}^{T_n}M^{T_{n+1}}\|_{BMO_2} < \varepsilon \qquad n = 0, \cdots, N$$

if and only if

$$d_{BMO_2}(M, H^\infty) < \varepsilon.$$

1.2 Corollary. *Under the assumptions of theorem 1.1 we have that $M \in \overline{H^\infty}^{\|\cdot\|_{BMO}}$ iff, for each $\varepsilon > 0$, there are stopping times $0 = T_0^\varepsilon \leq T_1^\varepsilon \leq \cdots T_{N(\varepsilon)}^\varepsilon \leq T_{N(\varepsilon)+1}^\varepsilon = \infty$ such that*

$$\|{}^{T_n^\varepsilon}M^{T_{n+1}^\varepsilon}\|_{BMO} < \varepsilon \qquad n = 0, \cdots, N(\varepsilon).$$

The corollary might be compared to the (trivial) statement, that M is in BMO iff for each $\varepsilon > 0$ we may decompose M into $M = \sum_{n=0}^{N(\varepsilon)} M_n$, such that each M_n satisfies $\|M_n\|_{BMO} < \varepsilon$. The flavor of the situation described by corollary 1.2 is that we require that the decomposition of M should be obtained from a partition of $\Omega \times \mathbb{R}_+$ into finitely many stochastic intervals.

We prove theorem 1.1 in section 2 below and in section 3 we construct the counter-example mentioned above. In fact, this example contains much of the motivation and intuition for theorem 1.1.

Let us also mention that the construction of this example is similar in spirit to example 3.12 in [DMSSS 95].

In section 4 we deal with two other problems on BMO-martingales raised in [K 94] and which are not related to H^∞. We show that, given a continuous martingale M and $1 < p < \infty$, then $p < a(M)$ implies that $\mathcal{E}(M)$ satisfies the reverse Hölder condition R_p (prop. 4.1). This answers positively the question raised in ([K 94], p.68)[1]. We also give a positive answer to the question raised in ([K 94], p. 63): if $\hat{\mathbb{P}}$ is a measure equivalent to $\mathbb{P}$ with continuous density process $\mathbb{E}[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}|\mathcal{F}_t]$, then the Girsanov-transformation induces an isomorphism between $BMO(\mathbb{P})$ and $BMO(\hat{\mathbb{P}})$ if and only if the density process is the exponential of a BMO-martingale (prop. 4.3).

[1] See, however, the note added at the end of this paper and the subsequent paper by P. Grandits.

This note is based on and motivated by the highly informative recent survey of N. Kazamaki [K 94], to which we refer for unexplained notation.

We also use the following standard notation. If M is a martingale and T a stopping time we denote by M^T the martingale "stopped at time T", i.e.,

$$M_t^T = M_{t\wedge T}$$

and by ${}^TM$ the martingale "started at time T", i.e.,

$${}^TM = M - M^T.$$

Throughout this note we shall assume that $M = (M_t)_{t\in\mathbb{R}_+}$ is a continuous real-valued martingale, $M_0 = 0$, based on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$ satisfying the "usual conditions" of completeness and right continuity. We do not, however, assume any kind of left-continuity of the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

2. The BMO-distance from H^∞

This section is devoted to the proof of theorem 1.1. We denote, for $1 \le p < \infty$, by $d_p(M, H^\infty)$ the distance of the continuous BMO-martingale M to H^∞ with respect to the norm of $BMO_p(\mathbb{P})$. We start with the easy implication.

2.1 Proof of necessity in theorem 1.1. Let us assume $d_2(M, H^\infty) < \varepsilon$, so that $M = Y + Z$, where Y, Z are continuous martingales, $Y_0 = Z_0 = 0$, $Y \in H^\infty$ and $\|Z\|_{BMO_2} < \varepsilon$.

For $0 < \eta < \varepsilon - \|Z\|_{BMO_2}$ define the stopping times T_n inductively by $T_0 = 0$ and

$$T_n = \inf\{t > T_{n-1} |\ \langle Y\rangle_t - \langle Y\rangle_{T_{n-1}} \ge \eta^2\}.$$

It is obvious that after at most $\eta^{-2}\|Y\|_{H^\infty}^2$ steps we arrive at $T_n \equiv \infty$ and that $\|{}^{T_n}M^{T_{n+1}}\|_{BMO_2} \le \|{}^{T_n}Y^{T_{n+1}}\|_{BMO_2} + \|{}^{T_n}Z^{T_{n+1}}\|_{BMO_2} < \varepsilon$. $\square$

For the reverse implication we need a preparatory result.

2.2 Lemma. *Let T be a stopping time and X a continuous martingale in BMO which vanishes before T, i.e., $X = {}^TX$.*

For $C \in \mathbb{R}_+$ let

$$T_C = \inf\{t | \langle X\rangle_t \ge C\},$$

and define $Y = X^{T_C}$, $Z = {}^{T_C}X$ so that $X = Y + Z$. Then $Y \in H^\infty$ and Z satisfies

$$\mathbb{E}[\langle Z\rangle_\infty | \mathcal{F}_T] \le \frac{\|X\|_{BMO_2}^4}{C} \qquad \text{(a.s.)}$$

Proof. We only have to show the final inequality. Note that

$$\mathbb{E}[\langle Z\rangle_\infty | \mathcal{F}_{T_C}] \le \mathbb{E}[\langle X\rangle_\infty - \langle X\rangle_{T_C} | \mathcal{F}_{T_C}] \le \|X\|_{BMO_2}^2 \qquad \text{(a.s.)}$$

and that

$$\{\mathbb{E}[\langle Z\rangle_\infty|\mathcal{F}_{T_C}] > 0\} \subseteq \{T_C < \infty\}$$
$$\subseteq \{\langle X\rangle_{T_C} = C\}.$$

Whence

$$\mathbb{E}[\langle Z\rangle_\infty|\mathcal{F}_{T_C}] \le \|X\|^2_{BMO_2} \cdot \frac{\langle X\rangle_{T_C}}{C}$$
$$\le \frac{\|X\|^2_{BMO_2}}{C} \cdot \mathbb{E}[\langle X\rangle_\infty|\mathcal{F}_{T_C}]. \qquad \text{(a.s.)}$$

Taking conditional expectations and using $\mathbb{E}[\langle X\rangle_\infty|\mathcal{F}_T] \le \|X\|^2_{BMO_2}$ we get

$$\mathbb{E}[\langle Z\rangle_\infty|\mathcal{F}_T] \le \frac{\|X\|^4_{BMO_2}}{C}. \quad \square \qquad \text{(a.s.)}$$

2.3 PROOF OF SUFFICIENCY IN THEOREM 1.3. Suppose that there is a finite increasing sequence $0 = T_0 \le T_1 \le \cdots \le T_N \le T_{N+1} = \infty$ of stopping times such that

$$\|{}^{T_n}M^{T_{n+1}}\|_{BMO_2} < \varepsilon \qquad n = 0, \cdots, N.$$

We apply lemma 2.2 to each ${}^{T_n}M^{T_{n+1}}$, with $C > N\varepsilon^4/\eta$ where $\eta = \varepsilon^2 - \max(\|{}^{T_n}M^{T_{n+1}}\|^2_{BMO_2})$, to find a decomposition

$${}^{T_n}M^{T_{n+1}} = Y_n + Z_n$$

with $Y_n \in H^\infty$ and

$$\mathbb{E}[\langle Z_n\rangle_\infty|\mathcal{F}_{T_n}] < \frac{\|{}^{T_n}M^{T_{n+1}}\|^4_{BMO_2}}{C} < \eta/N. \tag{1}$$

Note that $Y_n = {}^{T_n}Y^{T_{n+1}}$ and $Z_n = {}^{T_n}Z^{T_{n+1}}$. Letting

$$Y = \sum_{n=0}^{N} Y_n \quad \text{and} \quad Z = \sum_{n=0}^{N} Z_n$$

we clearly have that $Y \in H^\infty$. The crucial point is to show that

$$\|Z\|_{BMO_2} < \varepsilon, \tag{2}$$

which will finish the proof. To show (2) it suffices to show

$$\mathbb{E}[\langle Z\rangle_\infty - \langle Z\rangle_U|\mathcal{F}_U] < \varepsilon^2 \tag{3}$$

for each stopping time U such that there is some $0 \le n \le N$ for which we have

$$T_n \le U \le T_{n+1}.$$

We then may estimate

$$\mathbb{E}[\langle Z\rangle_\infty - \langle Z\rangle_U|\mathcal{F}_U] \le \mathbb{E}[\langle Z\rangle_{T_{n+1}} - \langle Z\rangle_U|\mathcal{F}_U] + \sum_{j=n+1}^{N} \mathbb{E}[\langle Z\rangle_{T_{j+1}} - \langle Z\rangle_{T_j}|\mathcal{F}_U]$$

From (1) we infer that, for $j \ge n+1$,

$$\mathbb{E}[\langle Z\rangle_{T_{j+1}} - \langle Z\rangle_{T_j}|\mathcal{F}_U] = \mathbb{E}[\mathbb{E}[\langle Z\rangle_{T_{j+1}} - \langle Z\rangle_{T_j}|\mathcal{F}_{T_j}]|\mathcal{F}_U] \le \eta/N,$$

which gives

$$\mathbb{E}[\langle Z\rangle_\infty - \langle Z\rangle_U|\mathcal{F}_U] \le \|{}^{T_n}M^{T_{n+1}}\|^2_{BMO_2} + N(\eta/N) < \varepsilon^2,$$

showing (3) and finishing the proof. $\square$

We end this section by indicating how to obtain a sequence $(T_n)_{n=0}^{N+1}$ of stopping times, satisfying the requirements of theorem 1.1, by backward induction.

2.4 Lemma. *For a martingale $M \in BMO$ and $\varepsilon > 0$ denote by $\mathcal{T}$ the family of all stopping times T such that*

$$\|{}^{T}M\|_{BMO_2} \leq \varepsilon$$

Then there exists a minimal element $\hat{T} \in \mathcal{T}$, in the sense that, for each $T \in \mathcal{T}$, we have $T \geq \hat{T}$ almost surely.

PROOF. First observe that $T_1, T_2 \in \mathcal{T}$ implies that $T_1 \wedge T_2 \in \mathcal{T}$. Indeed $\|{}^{T_1 \wedge T_2}M \mathbb{1}_{\{T_1 \leq T_2\}}\|_{BMO_2} = \|{}^{T_1}M \mathbb{1}_{\{T_1 \leq T_2\}}\|_{BMO_2} \leq \varepsilon$ and similarly $\|{}^{T_1 \wedge T_2}M \mathbb{1}_{\{T_2 \leq T_1\}}\|_{BMO_2} \leq \varepsilon$. It follows from the definition of the norm $\|\cdot\|_{BMO_2}$ that this implies that $\|{}^{T_1 \wedge T_2}M\|_{BMO_2} \leq \varepsilon$.

Now it is a standard exhaustion argument to show that there is a decreasing sequence $(T_j)_{j=1}^{\infty}$ in $\mathcal{T}$ such that, for every $C \in \mathbb{R}_+$ and $T \in \mathcal{T}$,

$$\lim_{j \to \infty} (\mathbb{P} \otimes \lambda)]\!]T \wedge C, C]\!] \backslash]\!]T_j \wedge C, C]\!] = 0$$

where λ denotes Lebesgue-measure on $\mathbb{R}_+$.

As we assumed that the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is right continuous we get that

$$\hat{T} = \inf_j T_j$$

is a stopping time and obviously this is the desired minimal element. $\square$

Lemma 2.4 may be used to determine whether $d_2(M, H^\infty) \leq \varepsilon_0$, for $\varepsilon_0 \geq 0$ and a given continuous martingale $M \in BMO$. For $\varepsilon > \varepsilon_0$ define ${}_1T$ as in lemma 2.4 to be the smallest stopping time such that $\|{}^{{}_1T}M\|_{BMO_2} \leq \varepsilon$. Then apply lemma 2.4 to $M^{{}_1T}$ to find a smallest stopping time ${}_2T$ such that $\|{}^{{}_2T}M^{{}_1T}\|_{BMO_2} \leq \varepsilon$. Continuing in an obvious way the process either arrives after finitely many steps at the stopping time zero or we have ${}_nT \not\equiv 0$, for each $n \in \mathbb{N}$. Obviously the first alternative holds true for each $\varepsilon > \varepsilon_0$, iff $d_2(M, H^\infty) \leq \varepsilon_0$.

Finally we remark that we have proved and stated the "isometric" theorem 1.1 in terms of the norm of BMO_2. If we define the norm $\|\cdot\|_{BMO_p}$, for $1 \leq p < \infty$, as the smallest constant C for which

$$\sup_T \mathbb{E}[(\langle M \rangle_\infty - \langle M \rangle_T)^{\frac{p}{2}} | \mathcal{F}_T]^{\frac{1}{p}} \leq C,$$

then an inspection of the above proofs shows that theorem 1.1 also holds true with BMO_2 replaced by BMO_p.

3. A martingale which is not in the BMO-closure of H^∞

In this section we give an example which will provide some motivation and intuition for theorem 1.1 above and also answer negatively two questions of N. Kazamaki ([K 94], p.48 and 70).

As in ([K 94], p.70) we define for an L^2-bounded martingale M the martingale

$$q(M) = \mathbb{E}[\langle M \rangle_\infty | \mathcal{F}_t] - \mathbb{E}[\langle M \rangle_\infty | \mathcal{F}_0]. \tag{1}$$

N. Kazamaki has asked, whether $q(M) \in \overline{L^\infty}^{\|\cdot\|_{BMO}}$ implies that $M \in \overline{H^\infty}^{\|\cdot\|_{BMO}}$; similarly, he raised the question whether $b(M) = \infty$ implies that $M \in \overline{H^\infty}^{\|\cdot\|_{BMO}}$. Both conjectures turn out to be wrong.

3.1 Example. *There is a continuous real-valued martingale M defined on the natural base $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$ of a standard Brownian motion W with the following properties.*

(i) *M is not in $\overline{H^\infty}^{\|\cdot\|_{BMO}}$.*

(ii) *$a(M) = b(M) = a(q(M)) = b(q(M)) = \infty$. Whence M as well as $q(M)$ are in $\overline{L^\infty}^{\|\cdot\|_{BMO}}$ and $\mathcal{E}(M)$ as well as $\mathcal{E}(-M)$ satisfy (A_p), for each $1 < p < \infty$ ([K 94], th. 2.8 and 3.11).*

PROOF. We shall first define a martingale N and then find a martingale M such that we obtain $N = q(M)$ via (1) above.

Fix a sequence $(\alpha_n)_{n=0}^\infty$ tending sufficiently fast to zero. For example, we may choose $\alpha_n = 2^{-2^n}$.

Let W be a standard Brownian motion and define the martingale differences of N on the odd intervals $([2n, 2n+1])_{n=0}^\infty$ by

$$N_t - N_{2n} = (W_t - W_{2n})^{T_n} \qquad t \in [2n, 2n+1]$$

where the stopping time T_n is defined by

$$T_n = \inf\{t \geq 2n \,|\, W_t - W_{2n} = 1 \text{ or } -\alpha_n\}.$$

We finish the definition of N by letting $N_0 = 0$ and letting N be constant on the even intervals $([2n+1, 2n+2])_{n=0}^\infty$.

We now are ready to define the martingale M by specifying its martingale differences on the even intervals: for $n \geq 0$ let

$$M_t - M_{2n+1} = (N_{2n+1} - N_{2n} + \alpha_n)^{\frac{1}{2}}(W_t - W_{2n+1}) \qquad t \in [2n+1, 2n+2].$$

Defining again $M_0 = 0$ and letting M be constant on the odd intervals $([2n, 2n+1])_{n=0}^\infty$ we complete the definition of M.

First note that $q(M) = N$. Indeed,

$$\langle M\rangle_\infty = \sum_{n=0}^\infty (N_{2n+1} - N_{2n} + \alpha_n),$$

so that

$$\begin{aligned} q(M)_t &= \mathbb{E}[\langle M\rangle_\infty | \mathcal{F}_t] - \mathbb{E}[\langle M\rangle_\infty | \mathcal{F}_0] \\ &= \sum_{n=0}^\infty (N_{t\wedge(2n+1)} - N_{t\wedge 2n} + \alpha_n) - \sum_{n=0}^\infty \alpha_n \\ &= N_t. \end{aligned}$$

Let us now show that (i) and (ii) are satisfied.

(ii) We shall show that $b(M) = \infty$. Note that, for $b > 0$,

$$\mathbb{E}[\exp(b\langle M\rangle_\infty)] = \mathbb{E}[\exp(b\sum_{n=0}^{\infty}(N_{2n+1} - N_{2n} + \alpha_n))]$$
$$= \prod_{n=0}^{\infty} \mathbb{E}[\exp(b(N_{2n+1} - N_{2n} + \alpha_n))].$$

If $(\alpha_n)_{n=1}^{\infty}$ tends sufficiently fast to zero, e.g. $\alpha_n = 2^{-2^n}$, we obtain that this infinite product is finite for every $b \in \mathbb{R}_+$.

Similarly, for $k \geq 0$ and $t \in [2k, 2k+2]$ we get

$$\begin{aligned}\mathbb{E}[\exp(b(\langle M\rangle_\infty &- \langle M\rangle_t))]|\mathcal{F}_t] \\ &= \mathbb{E}[\exp(b(\langle M\rangle_\infty - \langle M\rangle_{2k+2}))] \cdot \mathbb{E}[\exp(b(\langle M\rangle_{2k+2} - \langle M\rangle_t))]|\mathcal{F}_t] \\ &\leq \prod_{n=k+1}^{\infty} \mathbb{E}[\exp(b(N_{2n+1} - N_{2n} + \alpha_n))] \cdot \exp(b),\end{aligned}$$

which clearly is uniformly bounded in t, for each $b > 0$. This readily shows that $b(M) = \infty$. (Note that it makes no difference whether we consider conditional expectations with respect to $\mathcal{F}_t$, for each $t \in \mathbb{R}_+$, or with respect to $\mathcal{F}_T$, for every stopping time T, in the above estimate).

The verifications of $a(M) = a(q(M)) = b(q(M)) = \infty$ are similar and left to the reader.

(i) We shall show that $d_2(M, H^\infty) \geq 1$. Assuming the contrary we could find, by theorem 1.1, a finite sequence $(T_n)_{n=0}^{N+1}, 0 = T_0 \leq T_1 \leq \cdots \leq T_{N+1} = \infty$ such that

$$\|{}^{T_n}M^{T_{n+1}}\|_{BMO_2} < 1 \qquad n = 0, \cdots, N.$$

We shall verify inductively that

$$(2) \qquad \mathbb{P}[T_n \leq 2n] > 0 \qquad n = 0, \cdots, N+1$$

which will give the desired contradiction.

The assertion is true for $n = 0$; let us assume it holds true for n and let

$$A_n = \{T_n \leq 2n\},$$

which is an element of $\mathcal{F}_{2n}$. The set

$$B_n = A_n \cap \{N_{2n+1} - N_{2n} = 1\}$$

is in $\mathcal{F}_{2n+1}$ and still has strictly positive measure.

Suppose now that $T_{n+1} \geq 2n+2$ a.s., so that

$${}^{T_n \vee (2n+1)}M^{T_{n+1} \wedge (2n+2)} \mathbb{1}_{B_n} = {}^{2n+1}M^{2n+2} \mathbb{1}_{B_n}$$

would be a martingale of BMO_2-norm less than 1. But this is absurd as

$$\|{}^{2n+1}M^{2n+2}\mathbb{1}_{B_n}\|_{BMO_2}^2 \geq \mathbb{P}[B_n]^{-1} \cdot \mathbb{E}[(\langle M\rangle_{2n+2} - \langle M\rangle_{2n+1})\mathbb{1}_{B_n}] = 1,$$

a contradiction showing (2) and thus finishing the proof. $\square$

3.2 REMARK.

(1) Let us note that in the above example we even have that

$$\sup_T \mathbb{E}[\exp((b(\langle M\rangle_\infty - \langle M\rangle_T)^{\frac{p}{2}})|\mathcal{F}_T] < \infty,$$

for each $b > 0$ and $0 < p < \infty$. This indicates that there seems to be little hope to find a characterisation of $\overline{H^\infty}^{\|\cdot\|_{BMO}}$ similar to the Garnett-Jones theorem.

(2) It turns out, that $N = q(M)$ too is not in $\overline{H^\infty}^{\|\cdot\|_{BMO}}$. The proof is similar to the above proof that $M \notin \overline{H^\infty}^{\|\cdot\|_{BMO}}$.

One also can show that $q(q(M))$ satisfies $a(q(q(M))) = b(q(q(M))) = \infty$ and more generally, denoting by $N^{(k)}$ the k-th iteration $q(q(\cdots q(M)\cdots))$ then we have that $N^{(k)} \notin \overline{H^\infty}^{\|\cdot\|_{BMO}}$ while $a(N^{(k)}) = b(N^{(k)}) = \infty$.

Having made this observation it also becomes clear that we could have constructed the example of a martingale N as above without introducing M and without splitting $\mathbb{R}_+$ into odd and even intervals. But, for expository reasons, we preferred to present the example in terms of the "announcing" martingale N and the "running" martingale M.

4. Solution of two other questions of Kazamaki

4.1 Proposition. *Let M be a continuous real-valued martingale in BMO and define, as above,*

$$a(M) = \sup\{a \in \mathbb{R}_+ | \sup_T \|\mathbb{E}[\exp(a|M_\infty - M_T|)|\mathcal{F}_T]\|_\infty < \infty\}.$$

Then, for $1 < p < a(M)$, the exponential $\mathcal{E}(M)$ satisfies the reverse Hölder condition $(R_p(\mathbb{P}))$.

PROOF. Let $1 < a < a(M)$ and set $a = p$. We have to show that

$$\sup_T \|\mathbb{E}[\mathcal{E}({}^T M)^p_\infty|\mathcal{F}_T]\|_\infty < \infty$$

where T runs through all stopping times. For T fixed, we get

$$\begin{aligned}
\mathbb{E}[\mathcal{E}({}^T M)^p_\infty|\mathcal{F}_T] &= \mathbb{E}[(\exp(M_\infty - M_T - \frac{1}{2}(\langle M\rangle_\infty - \langle M\rangle_T)))^p|\mathcal{F}_T] \\
&= \mathbb{E}[(\exp(M_\infty - M_T - \frac{1}{2}(\langle M\rangle_\infty - \langle M\rangle_T)))^p \cdot \mathbb{1}_{\{M_\infty \geq M_T\}}|\mathcal{F}_T] \\
&+\mathbb{E}[(\exp(M_\infty - M_T - \frac{1}{2}(\langle M\rangle_\infty - \langle M\rangle_T)))^p \cdot \mathbb{1}_{\{M_\infty < M_T\}}|\mathcal{F}_T] \\
&\leq \mathbb{E}[\exp(p|M_\infty - M_T|)|\mathcal{F}_T] + 1.
\end{aligned}$$

By assumption the last expression is uniformly bounded which readily proves that $\mathcal{E}(M)$ satisfies $R_p(\mathbb{P})$. $\square$

4.2 Remark. *(1) The proposition answers the question raised in ([K 94], p.68), (see, however, the note added at the end of this paper and the subsequent paper by P. Grandits): By the Garnett-Jones theorem in its martingale version (N. Varopoulos and M. Emery, [K 94], th. 2.8) we get, for a continuous real-valued martingale $M \in BMO$, the implication $p < (4d_1(M, L^\infty))^{-1} \Rightarrow \mathcal{E}(M)$ satisfies $R_p(\mathbb{P})$.*

(2) There is no reverse to the proposition, i.e., a control on $R_p(\mathbb{P})$ for $\mathcal{E}(M)$ does not imply a control on $a(M)$.

To see this, simply remark that $M \in BMO$ implies that $R_p(\mathbb{P})$ holds true for some $p > 1$ while $a(M)$ may become arbitrarily close to zero.

We now turn to the conjecture raised in ([K 94], p.63) which will turn out to hold true.

Let M be a real-valued continuous local martingale, such that $\mathcal{E}(M)$ is uniformly integrable, and denote by $\hat{\mathbb{P}}$ the probability measure with density $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}(M)_\infty$. To each continuous real-valued local $\mathbb{P}$-martingale X we associate the local $\hat{\mathbb{P}}$-martingale $\hat{X} = -X + \langle X, M\rangle$ and we denote this map by $\phi : \mathcal{L}(\mathbb{P}) \to \mathcal{L}(\hat{\mathbb{P}})$ (see [K 94], p.62).

4.3 Proposition. *If $M \notin BMO(\mathbb{P})$, then ϕ does not map $BMO(\mathbb{P})$ into $BMO(\hat{\mathbb{P}})$.*

PROOF. We shall use the norm $\|\cdot\|_{BMO_2}$ in the subsequent calculations. Fix a standard Brownian motion $W = (W_t)_{t\in\mathbb{R}_+}$.

Step 1: In order to make the idea of the proof transparent we first assume that M simply equals $W^{T_N} = (W_t^{T_N})_{t\in\mathbb{R}_+}$ where T_N denotes the stopping time

$$T_N = \inf\{t : \mathcal{E}(W)_t = 2^N\},$$

where $N \in \mathbb{N}$ will be specified below.

For $n = 0, \cdots, N$ denote

$$T_n = \inf\{t : \mathcal{E}(W)_t = 2^n\},$$

and note that $\mathbb{P}[T_n < \infty] = 2^{-n}$.

The measure $\hat{\mathbb{P}}$ then is given by $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}(M)_\infty = 2^N \cdot \mathbb{I}_{\{T_N<\infty\}}$.

Define the $\mathbb{P}$-martingale $X^{(N)}$ by

$$X^{(N)} = \sum_{n=1}^{N-1} X^n, \qquad \text{where } X^n = {}^{T_{n-1}}W^{(T_{n-1}+1)\wedge T_n}.$$

Note that the $BMO_2(\mathbb{P})$-norm of $X^{(N)}$ is bounded by $\sqrt{2}$, for all $N \in \mathbb{N}$. Indeed, let us first calculate the $L^2(\mathbb{P})$ norm of $X^{(N)}$:

$$\begin{aligned}
\|X^{(N)}\|^2_{L^2(\mathbb{P})} &= \sum_{n=1}^{N-1} \mathbb{E}\left[(T_{n-1}+1)\wedge T_n - T_{n-1}\right] \\
&\leq \sum_{n=1}^{N-1} \mathbb{E}\left[(T_{n-1}+1) - T_{n-1}\right] \\
&= \sum_{n=1}^{N-1} 2^{-n+1} \leq 2.
\end{aligned}$$

In a completely analogous way we obtain, for every $0 \leq j \leq N-1$

$$\mathbb{E}\left[\langle X^{(N)}\rangle_\infty - \langle X^{(N)}\rangle_{T_j} | \mathcal{F}_{T_j}\right] \leq 2 \qquad \text{(a.s.,)}$$

and a moment's reflexion reveals that the above estimate also holds true if we replace T_j above by an arbitrary stopping time T, whence

$$\|X^{(N)}\|^2_{BMO_2(\mathbb{P})} \leq 2.$$

On the other hand, the $BMO(\hat{\mathbb{P}})$-norm of $\hat{X}^{(N)}$ tends to infinity as $N \to \infty$. Indeed

$$\begin{aligned}\mathbb{E}_{\hat{\mathbb{P}}}[\langle \hat{X}^{(N)}\rangle_\infty] &= \mathbb{E}_{\hat{\mathbb{P}}}[\sum_{n=1}^{N-1} \langle X^n\rangle_{T_n} - \langle X^n\rangle_{T_{n-1}}] \\ &= \sum_{n=1}^{N-1} \mathbb{E}_{\hat{\mathbb{P}}}[\langle X^n\rangle_{T_n} - \langle X^n\rangle_{T_{n-1}}] \\ &= (N-1)\mathbb{E}_{\hat{\mathbb{P}}}[\langle X^1\rangle_{T_1} - \langle X^1\rangle_{T_0}],\end{aligned}$$

the last equality being a consequence of the homogeneity of the definition of $X^{(N)}$. Indeed, under $\hat{\mathbb{P}}$, the distribution of the random variables $\langle X^n\rangle_{T_n} - \langle X^n\rangle_{T_{n-1}} = (T_{n-1}+1) \wedge T_n - T_{n-1}$ is identical for $n = 1, \ldots, N-1$.

Summing up what we have shown in step 1: If $M = W^{T_N}$ then there are martingales $X^{(N)}$ as above such that the ratio $\|X^{(N)}\|_{BMO_2(\hat{\mathbb{P}})} / \|X^{(N)}\|_{BMO_2(\mathbb{P})}$ tends to infinity as $N \to \infty$.

Step 2: Now suppose that $M = W^T$, i.e., Brownian motion stopped at some stopping time T. Recall that we also assume that $\mathcal{E}(M)$ is uniformly integrable. We shall show that, for every constant $C > 0$, there is $K > 0$, $\varepsilon > 0$ and $N \in \mathbb{N}$ such that the inequality

$$\mathbb{P}[T \geq K] \geq 1 - \varepsilon$$

implies that there is a martingale X of the form

$$X = \sum_{n=1}^{N-1} {}^{T_{n-1}\wedge T}W^{(T_{n-1}+1)\wedge T_n \wedge T}$$

such that

$$\|X\|^2_{BMO_2(\hat{\mathbb{P}})} / \|X\|^2_{BMO_2(\mathbb{P})} \geq C.$$

Indeed, define $(T_n)_{n=0}^N$ as in step 1, where we choose, with the notation of step 1, N sufficiently big so that

$$\mathbb{E}_{\mathbb{P}}[\langle X^{(N)}\rangle_\infty \cdot \mathcal{E}(W)_{T_N}] / \|X^{(N)}\|^2_{BMO_2(\mathbb{P})} \geq 2Cc^2,$$

where $c > 0$ is the bound on the $BMO(\mathbb{P})$-norm of $X^{(N)}$ obtained in step 1.

Then the martingale X defined above equals just $X^{(N)}$ stopped at time T. Clearly

$$\|X\|^2_{BMO_2(\mathbb{P})} \leq \|X^{(N)}\|^2_{BMO_2(\mathbb{P})},$$

the latter being bounded by the uniform constant c^2.

On the other hand,

$$\begin{aligned}\|X\|^2_{BMO_2(\hat{\mathbb{P}})} &\geq \mathbb{E}_{\hat{\mathbb{P}}}[\langle X\rangle_\infty] \\ &= \mathbb{E}_{\mathbb{P}}[\langle X\rangle_\infty \mathcal{E}(M)_T] \\ &= \mathbb{E}_{\mathbb{P}}[\langle X\rangle_{T\wedge T_N} \mathcal{E}(M)_{T\wedge T_N}].\end{aligned}$$

If $K \to \infty$ and $\varepsilon \to 0$, then this expression converges to $\mathbb{E}_{\mathbb{P}}[\langle X^{(N)}\rangle_{T_N} \mathcal{E}(W)_{T_N}]$ hence, for $K > 0$ sufficiently big and $\varepsilon > 0$ sufficiently small, we obtain

$$\|X\|^2_{BMO_2(\hat{\mathbb{P}})} / \|X\|^2_{BMO_2(\mathbb{P})} \geq C.$$

Step 3: We now pass to the general case. Let M be a continuous real-valued local martingale such that $\mathcal{E}(M)$ is uniformly integrable and such that $M \notin BMO(\mathbb{P})$. We shall show that, for every $C > 0$, there is a martingale X in $BMO(\mathbb{P})$, which is a stochastic integral on M, i.e., $X = H \cdot M$, where the predictable integrand H assumes only the values 0 and 1, such that

$$\|X\|^2_{BMO_2(\hat{\mathbb{P}})} / \|X\|^2_{BMO_2(\mathbb{P})} \geq C.$$

This will readily imply the assertion of the proposition (by the closed graph theorem).

Let $K = K(C) > 0$ and $\varepsilon = \varepsilon(C) > 0$ be the constants given by step 2. As $M \notin BMO(\mathbb{P})$ we may find a stopping time $U, \mathbb{P}[U < \infty] > 0$ such that

$$\mathbb{P}[\langle M\rangle_\infty - \langle M\rangle_U \geq K | \mathcal{F}_U] \geq 1 - \varepsilon \qquad \text{a.s. on } \{U < \infty\}.$$

(see, e.g., [RY]).

Now we are exactly in the situation of step 2: Define the stopping times $(T_n)_{n=0}^N$, where the number $N = N(C)$ is given by step 2, by $T_0 = U$ and

$$T_n = \inf\{t | \mathcal{E}({}^U M)_t \geq 2^n\},$$

where ${}^U M = M - M^U$ is the martingale M starting at U. Define the stopping times $(S_n)_{n=0}^{N-1}$ by

$$S_n = \inf\{t \geq T_n | \langle M\rangle_t - \langle M\rangle_{T_n} \geq 1\} \wedge T_{n+1}$$

and the martingale X by

$$X = \sum_{n=0}^{N-1} \mathbb{1}_{]\!]T_n, S_n]\!]} \cdot M.$$

The arguments of step 2 imply that

$$\mathbb{E}_{\hat{\mathbb{P}}}[\langle X\rangle_\infty - \langle X\rangle_U | \mathcal{F}_U] / \|X\|^2_{BMO_2(\mathbb{P})} \geq C \quad \text{a.s. on } \{U < \infty\}$$

and, in particular

$$\|X\|^2_{BMO_2(\hat{\mathbb{P}})} / \|X\|^2_{BMO_2(\mathbb{P})} \geq C \quad \square$$

4.4 Remark. *The condition that $\mathcal{E}(M)$ is uniformly integrable can be omitted if we are careful to give $\|\cdot\|_{BMO_2(\hat{\mathbb{P}})}$ a proper meaning. If we only assume that M is a real-valued continuous local martingale let $(T_n)_{n=1}^{\infty}$ be an increasing sequence of stopping times tending to infinity which localizes the local martingale $\mathcal{E}(M)$. Denote by $\hat{\mathbb{P}}_n$ the probability measure on $\mathcal{F}_{T_n}$ with density $\frac{d\hat{\mathbb{P}}_n}{d\mathbb{P}_n} = \mathcal{E}(M)_{T_n}$ and define, for a local martingale X, the sequence $c_n = \|X^{T_n}\|_{BMO_2(\hat{\mathbb{P}}_n)}$. Then $(c_n)_{n=1}^{\infty}$ is an increasing sequence in $[0,\infty]$ and if we replace $\|X\|^2_{BMO_2(\hat{\mathbb{P}})}$ by $\lim_{n\to\infty} c_n$ then the assertion of the proposition remains valid.*

NOTE ADDED IN PROOF. After this paper has been finished and accepted for publication I received some comments from N. Kazamaki and M. Kikuchi. They pointed out that there was a mis-understanding with respect to the question raised in ([K], p.68): we have shown in proposition 4.1 and remark 4.2 above that, letting

$$\hat{\Phi}(p) = (4p)^{-1},$$

we have

$$d_1(M, L^\infty) < \hat{\Phi}(p) \Rightarrow \mathcal{E}(M) \text{ satisfies } R_p(\mathbb{P}).$$

However, the proper understanding of the problem posed in ([K], p.68) pertains to the question, whether there exists a function $\Phi : (1,\infty) \to (0,\infty)$ satisfying $\lim_{p\to 1} \Phi(p) = \infty$ such that the above implication holds true with $\hat{\Phi}$ replaced by Φ. The result given in proposition 4.1 above therefore is not satisfactory, as $\hat{\Phi}$ has its singularity at $p = 0$ instead of $p = 1$.

The (properly understood) question of N. Kazamaki ([K], p.68) ultimately was solved negatively by P. Grandits and his counterexample is presented in the subsequent paper : On a conjecture of Kazamaki.

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