THE FUNDAMENTAL THEOREM OF ASSET PRICING FOR UNBOUNDED STOCHASTIC PROCESSES

F. DELBAEN AND W. SCHACHERMAYER

Departement für Mathematik, Eidgenössische Technische Hochschule Zürich,
ETH-Zentrum, CH-8092 Zürich, Schweiz,
Institut für Statistik der Universität Wien,
Brünnerstr. 72, A-1210 Wien, Austria.

ABSTRACT. The *Fundamental Theorem of Asset Pricing* states - roughly speaking - that the absence of arbitrage possibilities for a stochastic process S is equivalent to the existence of an equivalent martingale measure for S. It turns out that it is quite hard to give precise and sharp versions of this theorem in proper generality, if one insists on modifying the concept of "no arbitrage" as little as possible. It was shown in [DS94] that for a *locally bounded* $\mathbb{R}^d$-valued semi-martingale S the condition of *No Free Lunch with Vanishing Risk* is equivalent to the existence of an *equivalent local martingale measure* for the process S. It was asked whether the local boundedness assumption on S may be dropped.

In the present paper we show that if we drop in this theorem the local boundedness assumption on S the theorem remains true if we replace the term *equivalent local martingale measure* by the term *equivalent sigma-martingale measure.* The concept of sigma-martingales was introduced by Chou and Emery — under the name of "semimartingales de la classe (Σ_m)".

We provide an example which shows that for the validity of the theorem in the non locally bounded case it is indeed necessary to pass to the concept of sigma-martingales. On the other hand, we also observe that for the applications in Mathematical Finance the notion of sigma-martingales provides a natural framework when working with non locally bounded processes S.

The duality results which we obtained earlier are also extended to the non locally bounded case. As an application we characterize the hedgeable elements.

1. INTRODUCTION

The topic of the present paper is the statement and proof of the subsequent *Fundamental Theorem of Asset Pricing* in a *general version for not necessarily locally bounded semi-martingales*:

1.1 Main Theorem. *Let $S=(S_t)_{t\in\mathbb{R}_+}$ be an $\mathbb{R}^d$-valued semi-martingale defined on the stochastic base $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}_+},\mathbb{P})$.*

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Then S satisfies the condition of No Free Lunch with Vanishing Risk if and only if there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that S is a sigma-martingale with respect to $\mathbb{Q}$.

This theorem has been proved under the additional assumption that the process S is locally bounded in [DS94]. Under this additional assumption one may replace the term "sigma-martingale" above by the term "local martingale".
We refer to [DS94] for the history of this theorem, which goes back to the seminal work of Harrison, Kreps and Pliska ([HK79], [HP81], [Kr81]) and which is of central importance in the applications of stochastic calculus to Mathematical Finance. We also refer to [DS94] for the definition of the concept of *No Free Lunch with Vanishing Risk* which is a mild strengthening of the concept of *No Arbitrage.*
On the other hand, to the best of our knowledge, the second central concept in the above theorem, the notion of a *sigma-martingale* (see def. 2.1 below) has not been considered previously in the context of Mathematical Finance. In a way, this is surprising, as we shall see in 2.4 below that this concept is very well-suited for the applications in Mathematical Finance, where one is interested not so much in the process S itself but rather in the family $(H \cdot S)$ of *stochastic integrals* on the process S, where H runs through the S-integrable predictable processes satisfying a suitable admissibility condition (see [HP81], [DS94] and section 4 and 5 below).
The concept of sigma-martingales, which relates to martingales similarly as sigma-finite measures relate to finite measures, has been introduced by C.S. Chou and M. Emery ([C77], [E78]) under the name "semi-martingales de la classe (Σ_m)". We shall show in section 2 below (in particular in example 2.3) that this concept is indeed natural and unavoidable in our context if we consider processes S with unbounded jumps.

The paper is organized as follows: In section 2 we recall the definition and basic properties of sigma-martingales. In section 3 we present the idea of the proof of the main theorem by considering the (very) special case of a two-step process $S = (S_0, S_1) = (S_t)_{t=0}^1$. This presentation is mainly for expository reasons in order to present the basic idea without burying it under the technicalities needed for the proof in the general case. But, of course, the consideration of the two-step case only yields the $(n+1)$'th proof of the Dalang-Morton-Willinger theorem [DMW90], i.e., the fundamental theorem of asset pricing in finite discrete time (for alternative proofs see [S92], [KK94], [R94]). We end section 3 by isolating in lemma 3.5 the basic idea of our approach in an abstract setting.
Section 4 is devoted to the proof of the main theorem in full generality. We shall use the notion of the *jump measure* associated to a stochastic process and its *compensator* as presented, e.g., in [JS87].
Section 5 is devoted to a generalization of the duality results obtained in [DS95]. These results are then used to identify the hedgeable elements as maximal elements in the cone of w-admissible outcomes. The concept of w-admissible integrand is a natural generalization to the non locally bounded case of the previously used concept of admissible integrand.
In [Ka97] Y. Kabanov also proves our Main Theorem. The idea of the proof is the same but worked out differently. The rechnique is to change the characteristics of the big jumps. Kabanov also repeats the proof, given in [DS94].
For unexplained notation and for further background on the main theorem we refer to [DS94].

2. Sigma-Martingales

In this section we recall a concept which has been introduced by C.S. Chou [C77] and M. Emery [E78] under the name "semi-martingales de la classe (Σ_m)". This notion will play a central role in the present context. We take the liberty to baptize this notion as "*sigma-martingales*". We choose this name as the relation between martingales and sigma-martingales is somewhat analogous to the relation between finite and sigma-finite measures (compare [E78], prop. 2). Other researchers prefer the name martingale transform.

2.1 Definition. An $\mathbb{R}^d$-valued semimartingale $X = (X_t)_{t\in\mathbb{R}_+}$ is called a *sigma-martingale* if there exists an $\mathbb{R}^d$-valued martingale M and an M-integrable predictable $\mathbb{R}_+$-valued process φ such that $X = \varphi \cdot M$.

We refer to ([E78], prop. 2) for several equivalent reformulations of this definition and we now essentially reproduce the basic example given by M. Emery ([E78], p.152) which highlights the difference between the notion of a martingale (or, more generally, a local martingale) and a sigma-martingale.

2.2 Example. [E78]:*A sigma-martingale which is not a local martingale.*
Let the stochastic base $(\Omega, \mathcal{F}, \mathbb{P})$ be such that there are two independent stopping times T and U defined on it, both having an exponential distribution with parameter 1.
Define M by

$$M_t = \begin{cases} 0 & \text{for } t < T \wedge U \\ 1 & \text{for } t \geq T \wedge U \text{ and } T = T \wedge U \\ -1 & \text{for } t \geq T \wedge U \text{ and } U = T \wedge U \end{cases}$$

It is easy to verify that M is almost surely well-defined and is indeed a martingale with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by M. The deterministic (and therefore predictable) process $\varphi_t = \frac{1}{t}$ is M-integrable (in the sense of Stieltjes) and $X = \varphi \cdot M$ is well-defined:

$$X_t = \begin{cases} 0 & \text{for } t < T \wedge U \\ 1/(T \wedge U) & \text{for } t \geq T \wedge U \text{ and } T = T \wedge U \\ -1/(T \wedge U) & \text{for } t \geq T \wedge U \text{ and } U = T \wedge U \end{cases}$$

But X fails to be a martingale as $\mathbb{E}[|X_t|] = \infty$, for all $t > 0$, and it is not hard to see that X also fails to be a local martingale (see [E78]), as $\mathbb{E}[|X_T|] = \infty$ for each stopping time T that is not identically zero. But, of course, X is a sigma-martingale. □

We shall be interested in the class of semimartingales S which *admit an equivalent measure under which they are a sigma-martingale.* We shall present an example of an $\mathbb{R}^2$-valued process S which admits an equivalent sigma-martingale measure (which in fact is unique) but which does not admit an equivalent local martingale measure. This example will be a slight extension of Emery's example.
The reader should note that in Emery's example 2.2 above one may replace the measure $\mathbb{P}$ by an equivalent measure $\mathbb{Q}$ such that X is a true martingale under $\mathbb{Q}$. For example, choose $\mathbb{Q}$ such that under this new measure T and U are independent and distributed according to a law μ on $\mathbb{R}_+$ such that μ is equivalent to the exponential law (i.e., equivalent to Lebesgue-measure on $\mathbb{R}_+$) and such that $\mathbb{E}_\mu\left[\frac{1}{t}\right] < \infty$.

2.3 Example. *A sigma-martingale S which does not admit an equivalent local martingale measure.*
With the notation of the above example define the $\mathbb{R}^2$-valued process $S = (S^1, S^2)$ by letting $S^1 = X$ and S^2 the compensated jump at time $T \wedge U$ i.e.,

$$S_t^2 = \begin{cases} -2t & \text{for } t < T \wedge U \\ 1 - 2(T \wedge U) & \text{for } t \geq T \wedge U \end{cases}$$

(Observe that $T \wedge U$ is exponentially distributed with parameter 2).
Clearly S^2 is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by S. Denoting by $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$ the filtration generated by S^2, it is a well-known property of the Poisson-process (c.f. [J79], p. 347) that on $\mathcal{G}$ the restriction of $\mathbb{P}$ to $\mathcal{G} = \bigvee_{t\in\mathbb{R}_+} \mathcal{G}_t$ is the unique probability measure equivalent to $\mathbb{P}$ under which S^2 is a martingale. It follows that $\mathbb{P}$ is the only probability measure on $\mathcal{F} = \bigvee_{t\in\mathbb{R}_+} \mathcal{F}_t$ equivalent to $\mathbb{P}$ under which $S = (S^1, S^2)$ is a sigma-martingale.
As S fails to be a local martingale under $\mathbb{P}$ (it's first coordinate fails to beso) we have exhibited a sigma-martingale for which there does not ex ist an equivalent martingale measure. $\square$

2.4 Remark. In the applications to Mathematical Finance and in particular in the context of *pricing and hedging derivative securities by no-arbitrage arguments* the object of central interest is the set of *stochastic integrals* $H \cdot S$ on a given stock price process S, where H runs through the S-integrable predictable processes such that the process $H \cdot S$ satisfies appropriate regularity condition. In the present context this regularity condition is the admissibility condition $H \cdot S \geq -M$ for some $M \in \mathbb{R}_+$ (see [HP81], [DS94] and section 4 below). In different contexts one might impose an $L^p(\mathbb{P})$-boundedness condition on the stochastic integral $H \cdot S$ (see, e.g., [Kr81], [DH86], [St90], [DMSSS96]). In section 5, we shall deal with a different notion of admissibility, which is adjusted to the case of big jumps.
Now make the trivial (but nevertheless crucial) observation: passing from S to $\varphi \cdot S$, where φ is a strictly positive S-integrable predictable process, *does not change the set of stochastic integrals.* Indeed, we may write

$$H \cdot S = (H\varphi^{-1}) \cdot (\varphi \cdot S)$$

where the predictable $\mathbb{R}^d$-valued process H is S-integrable iff $H\varphi^{-1}$ is $\varphi \cdot S$-integrable. The moral of this observation: when we are interested only in the set of *stochastic integrals* $H \cdot S$ the requirement that S is a sigma-martingale is just as good as the requirement that S is a true martingale.

We end this section with two observations which are similar to the results in [E78]. The first one stresses the distinction between the notions of a local martingale and a sigma-martingale.

2.5 Proposition. *For a semi-martingale X the following assertions are equivalent.*

(i) *X is a local martingale.*
(ii) *$X = \varphi \cdot M$ where the M-integrable, predictable $\mathbb{R}_+$-valued process φ is increasing and M is a local martingale.*
(ii') *$X = \varphi \cdot M$ where the M-integrable, predictable $\mathbb{R}_+$-valued process φ is locally bounded and M is a local martingale.*

(iii) $X = \varphi \cdot M$ *where the* M*-integrable, predictable* $\mathbb{R}_+$*-valued process* φ *is increasing and* M *is a martingale.*
(iii') $X = \varphi \cdot M$ *where the* M*-integrable, predictable* $\mathbb{R}_+$*-valued process* φ *is locally bounded and* M *is a martingale.*
(iv) $X = \varphi \cdot M$ *where the* M*-integrable, predictable* $\mathbb{R}_+$*-valued process* φ *is increasing and* M *is a martingale in* $\mathcal{H}^1$*.*
(iv) $X = \varphi \cdot M$ *where the* M*-integrable, predictable* $\mathbb{R}_+$*-valued process* φ *is locally bounded and* M *is a martingale in* $\mathcal{H}^1$*.*

We will not prove this proposition as its proof is similar to the proof of the next proposition.

2.6 Proposition. *For a semi-martingale* X *the following are equivalent*

(i) X *is a sigma-martingale.*
(ii) $X = \varphi \cdot M$ *where the* M*-integrable, predictable* $\mathbb{R}_+$*-valued process* φ *is strictly positive and* M *is a local martingale.*
(iii) $X = \varphi \cdot M$ *where the* M*-integrable, predictable* $\mathbb{R}_+$*-valued process* φ *is strictly positive and* M *is a martingale.*
(iv) $X = \varphi \cdot M$ *where the* M*-integrable, predictable* $\mathbb{R}_+$*-valued process* φ *is strictly positive and* M *is a martingale in* $\mathcal{H}^1$*.*

Proof. Since (iv) implies (iii) implies (ii) and since obviously (ii) implies (i), we only have to prove that (i) implies (iv). For simplicity we assume that $M_0 = 0$, leaving the necessary alterations to the reader. If X is a sigma-martingaletingale, then there is a local martingale M as well as a nonnegative M-integrable predictable process ϕ such that $X = \phi \cdot M$. Let $(T_n)_{n\geq 1}$ be a sequence that localizes M in the sense that T_n is increasing, tends to ∞ and for each n, M^{T_n} is in $\mathcal{H}^1$. Put $T_0 = 0$ and for $n \geq 1$, define N^n as the $\mathcal{H}^1$ martingale $N^n = (\phi \mathbf{1}_{]T_{n-1},T_n]}) \cdot M^{T_n}$. Let now $N = \sum_{n\geq 1} a_n N^n$, where the strictly positive sequence a_n is chosen such that $\sum a_n \|N^n\|_{\mathcal{H}^1} < \infty$. The process N is an $\mathcal{H}^1$ martingale. We now put $\psi = \mathbf{1}_{\{\phi=0\}} + \phi \sum_n a_n^{-1} \mathbf{1}_{]T_{n-1},T_n]}$. It is easy to check that $X = \psi \cdot N$ and that ψ is strictly positive.

Corollary 2.7. *A local sigma-martingale is a sigma-martingale. More precisely, if* X *is a semi-martingale and if* $(T_k)_{k\geq 1}$ *is an increasing sequence of stopping times, tending to* ∞ *such that each stopped process* X^{T_k} *is a sigma-martingale, then* X *itself is a sigma-martingale.*

Proof. For each k take ϕ^k, X^{T_k} integrable such that $\phi^k > 0$ on $[\![0, T_k]\!]$, $\phi^k \cdot X^{T_k}$ is a uniformly integrable martingale and $\|\phi^k \cdot X^{T_k}\|_{\mathcal{H}^1} < 2^{-k}$. Put $T_0 = 0$ and $\phi^0 = \phi^1 \mathbf{1}_{[0]}$. It is now obvious that $\phi = \phi^0 + \sum_{k\geq 1} \phi^k \mathbf{1}_{]T_{k-1},T_k]}$ is strictly positive, is X integrable and is such that $\phi \cdot X$ is an $\mathcal{H}^1$ martingale. □

3. One-period processes

In this section we shall present the basic idea of the proof of the main theorem in the easy context of a process consisting only of one jump. Let $S_0 \equiv 0$ and $S_1 \in L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ be given and consider the stochastic process $S = (S_t)_{t=0}^1$; as filtration we choose $(\mathcal{F}_t)_{t=0}^1$ where $\mathcal{F}_1 = \mathcal{F}$ and $\mathcal{F}_0$ is some sub-σ-algebra of $\mathcal{F}$. At a first stage we shall in addition make the simplifying assumption that $\mathcal{F}_0$ is trivial, i.e., consists only of null-sets and their complements. In this setting the definition

of the *No-Arbitrage* condition (NA) (see [DMW90] or [DS94]) for the process S boils down to the requirement that, for $x \in \mathbb{R}^d$, the condition $(x, S) \geq 0$ a.s. implies that $(x, S) = 0$ a.s., where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^d$.
From the theorem of Dalang-Morton-Willinger [DMW90] we deduce that the No Arbitrage condition (NA) implies the existence of an equivalent martingale measure for S, i.e., a measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}), \mathbb{Q} \sim \mathbb{P}$, such that $\mathbb{E}_{\mathbb{Q}}[S_1] = 0$.
By now there are several alternative proofs of the Dalang-Morton-Willinger theorem known in the literature ([S92], [KK94], [R94]) and we shall present yet another proof of this theorem in the subsequent lines. While some of the known proofs are very elegant (e.g., [R94]) our subsequent proof is rather clumsy and heavy. But it is this method which will be extensible to the general setting of an $\mathbb{R}^d$-valued (not necessarily locally bounded) semi-martingale and will allow us to prove the main theorem in full generality.
Let us fix some notation: by *Adm* we denote the convex cone of *admissible elements* of $\mathbb{R}^d$ which consists of those $x \in \mathbb{R}^d$ such that the random variable (x, S) is (almost surely) uniformly bounded from below.
By K we denote the convex cone in $L^0(\Omega, \mathcal{F}, \mathbb{P})$ formed by the admissible stochastic integrals on the process S, i.e.,

$$K = \{(x, S_1) : x \in Adm\} \tag{3.1}$$

and we denote by C the convex cone in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ formed by the uniformly bounded random variables dominated by some element of K, i.e.,

$$\begin{aligned} C &= (K - L^0_+(\Omega, \mathcal{F}, \mathbb{P})) \cap L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \\ &= \{f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) : \text{ there is } g \in K, f \leq g\}. \end{aligned} \tag{3.2}$$

Under the assumption that S satisfies (NA), i.e. $K \cap L^0_+ = \{0\}$, we want to find an equivalent martingale measure $\mathbb{Q}$ for the process S. The first argument is well-known in the present context (compare [S92] and theorem 4.1 below for a general version of this result; we refer to [S94] for an account on the history of this result, in particular on the work of J.A. Yan [Y80] and D. Kreps [Kr81]):

3.1 Lemma. *If S satisfies (NA) the convex cone C is weak* closed in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, and $C \cap L^\infty_+(\Omega, \mathcal{F}, \mathbb{P}) = \{0\}$. Therefore there is a probability measure $\mathbb{Q}_1$ on $\mathcal{F}, \mathbb{Q}_1 \sim \mathbb{P}$ such that*

$$\mathbb{E}_{\mathbb{Q}_1}[f] \leq 0 \qquad \text{for } f \in C. \quad \square$$

In the case, when S_1 is uniformly bounded, the measure $\mathbb{Q}_1$ is already the desired equivalent martingale measure. Indeed, in this case the cone *Adm* of admissible elements is the entire space $\mathbb{R}^d$ and therefore

$$\mathbb{E}_{\mathbb{Q}_1}[(x, S_1)] \leq 0, \qquad \text{for } x \in \mathbb{R}^d,$$

which implies that

$$\mathbb{E}_{\mathbb{Q}_1}[(x, S_1)] = 0, \qquad \text{for } x \in \mathbb{R}^d,$$

whence $\mathbb{E}_{\mathbb{Q}_1}[S] = 0$.

But if Adm is only a sub-cone of $\mathbb{R}^d$ (possibly reduced to $\{0\}$), we can only say much less: first of all, S_1 need not be $\mathbb{Q}_1$-integrable. But even assuming that $\mathbb{E}_{\mathbb{Q}_1}[S_1]$ exists we cannot assert that this value equals zero; we can only assert that

$$(x, \mathbb{E}_{\mathbb{Q}_1}[S_1]) = \mathbb{E}_{\mathbb{Q}_1}[(x, S_1)] \leq 0, \qquad \text{for } x \in Adm,$$

which means that $\mathbb{E}_{\mathbb{Q}_1}[S_1]$ lies in the cone Adm^0 polar to Adm, i.e.,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_1}[S_1] \in Adm^0 &= \{y \in \mathbb{R}^d : (x,y) \leq 1 \text{ for } x \in Adm\} \\ &= \{y \in \mathbb{R}^d : (x,y) \leq 0 \text{ for } x \in Adm\}.\end{aligned}$$

The next lemma will imply that, by passing from $\mathbb{Q}_1$ to an equivalent probability measure $\mathbb{Q}$ *with distance* $\|\mathbb{Q} - \mathbb{Q}_1\|$ *in total variation norm less than* $\varepsilon > 0$, we may remedy both possible defects of $\mathbb{Q}_1$: under $\mathbb{Q}$ the expectation of S_1 is well-defined and it equals zero.
The idea for the proof of this lemma goes back in the special case $d = 1$ and $Adm = \{0\}$ to the work of D. McBeth [M91].

3.2 Lemma. *Let $\mathbb{Q}_1$ be a probability measure as in lemma 3.1 and $\varepsilon > 0$. Denote by B the set of barycenters*

$$B = \{\mathbb{E}_{\mathbb{Q}}[S_1] : \mathbb{Q} \text{ ¿probability on } \mathcal{F}, \mathbb{Q} \sim \mathbb{P}, \|\mathbb{Q} - \mathbb{Q}_1\| < \varepsilon, \text{ and } S_1 \text{ is } \mathbb{Q}\text{-integrable}\}$$

Then B is a convex subset of $\mathbb{R}^d$ containing 0 in its relative interior. In particular, there is $\mathbb{Q} \sim \mathbb{Q}_1, \|\mathbb{Q} - \mathbb{Q}_1\| < \varepsilon$, such that $\mathbb{E}_{\mathbb{Q}}[S_1] = 0$.

Proof. Clearly B is convex. Let us also remark that it is nonempty. To see this let us take $\delta > 0$ and let us define $\frac{d\mathbb{Q}}{d\mathbb{Q}_1} = \frac{\exp(-\delta\|S_1\|)}{\mathbb{E}_{\mathbb{Q}_1}[\exp(-\delta\|S_1\|)]}$. Clearly S_1 is $\mathbb{Q}$ integrable and from Lebesgue's dominated convergence theorem we deduce that for δ small enough $\|\mathbb{Q} - \mathbb{Q}_1\| < \varepsilon$.
If 0 were not in the relative interior of B we could find by the Minkowski separation theorem, an element $x \in \mathbb{R}^d$, such that B is contained in the halfspace $H_x = \{y \in \mathbb{R}^d : (x,y) \geq 0\}$ and such that $(x,y) > 0$ for some $y \in B$. In order to obtain a contradiction we distinguish two cases:

<u>Case 1:</u> x fails to be admissible, i.e., (x, S_1) fails to be (essentially) uniformly bounded from below.
First find, as above, a probability measure $\mathbb{Q}_2 \sim \mathbb{P}, \|\mathbb{Q}_1 - \mathbb{Q}_2\| < \varepsilon/2$ and such that $\mathbb{E}_{\mathbb{Q}_2}[S_1]$ is well-defined.
By assumption the random variable (x, S_1) is not (essentially) uniformly bounded from below, i.e., for $M \in \mathbb{R}_+$, the set

$$\Omega_M = \{\omega : (x, S_1(\omega)) < -M\}$$

has strictly positive $\mathbb{Q}_2$-measure. For $M \in \mathbb{R}_+$ define the measure $\mathbb{Q}^M$ by

$$\frac{d\mathbb{Q}^M}{d\mathbb{Q}_2} = \begin{cases} 1 - \varepsilon/4 & \text{on } \Omega \setminus \Omega_M \\ 1 - \varepsilon/4 + \frac{\varepsilon}{4\mathbb{Q}_2(\Omega_M)} & \text{on } \Omega_M. \end{cases}$$

It is straightforward to verify that $\mathbb{Q}^M$ is a probability measure, $\mathbb{Q}^M \sim \mathbb{P}$, $\|\mathbb{Q}^M - \mathbb{Q}_2\| < \varepsilon/2$, $d\mathbb{Q}^M/d\mathbb{Q}_2 \in L^\infty$ and such that

$$(x, \mathbb{E}_{\mathbb{Q}^M}[S_1]) = \mathbb{E}_{\mathbb{Q}^M}\left[(x, S^1)\right] \leq (1-\varepsilon/4)\mathbb{E}_{\mathbb{Q}_2}[(x, S_1)] - \frac{\varepsilon M}{4}.$$

For $M > 0$ big enough the right hand side becomes negative which gives the desired contradiction.

Case 2: x is admissible, i.e., (x, S_1) is (essentially) uniformly bounded from below. In this case we know from the Beppo-Levi theorem that the random variable (x, S_1) is $\mathbb{Q}_1$-integrable and that $\mathbb{E}_{\mathbb{Q}_1}[(x, S_1)] \leq 0$; (note that, for each $M \in \mathbb{R}_+$, we have that $(x, S_1) \wedge M$ is in C and therefore $\mathbb{E}_{\mathbb{Q}_1}[(x, S_1) \wedge M] \leq 0$).
Also note that (x, S_1) cannot be equal to 0 a.s., because as we saw above there is a $y \in B$ such that $(x, y) > 0$ and hence (x, S_1) cannot equal zero a.s. either. The No Arbitrage property then tells us that $\mathbb{Q}_1[(x, S) > 0]$ as well as $\mathbb{Q}_1[(x, S) < 0]$ are both strictly positive.
We next observe that for all $\eta > 0$ the variable $\exp(\eta(x, S_1)^-)$ is bounded. The measure $\mathbb{Q}_2$, given by $\frac{d\mathbb{Q}_2}{d\mathbb{Q}_1} = \frac{\exp(\eta(x,S_1)^-)}{\mathbb{E}_{\mathbb{Q}_1}[\exp(\eta(x,S_1)^-)]}$ is therefore well defined. For η small enough we also have that $\|\mathbb{Q}_2 - \mathbb{Q}_1\| < \varepsilon$. But $\mathbb{Q}_2$ also satisfies:

$$\mathbb{E}_{\mathbb{Q}_2}[(x, S_1)] < 0.$$

Indeed:

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_1}\left[\exp\left(\eta(x, S_1)^-\right)(x, S_1)\right] \\
&\quad = -\mathbb{E}_{\mathbb{Q}_1}\left[\exp\left(\eta(x, S_1)^-\right)(x, S_1)^-\right] + \mathbb{E}_{\mathbb{Q}_1}\left[(x, S_1)^+\right] \\
&\quad < -\mathbb{E}_{\mathbb{Q}_1}\left[(x, S_1)^-\right] + \mathbb{E}_{\mathbb{Q}_1}\left[(x, S_1)^+\right] \leq 0.
\end{aligned}$$

The measure $\mathbb{Q}_2$ does not necessarily satisfy the requirement that $\mathbb{E}_{\mathbb{Q}_2}[\|S_1\|] < \infty$. We therefore make a last tansformation and we define $d\mathbb{Q} = \frac{\exp(-\delta\|S_1\|)}{\mathbb{E}_{\mathbb{Q}_2}[\exp(-\delta\|S_1\|)]} d\mathbb{Q}_2$. For $\delta > 0$ tending to zero we obtain that $\|\mathbb{Q} - \mathbb{Q}_2\|$ tends to 0 and $\mathbb{E}_{\mathbb{Q}}[(x, S_1)]$ tends to $\mathbb{E}_{\mathbb{Q}_2}[(x, S_1)]$which is strictly negative. So for δ small enough we find a probability mesaure $\mathbb{Q}$ such that $\mathbb{Q} \sim \mathbb{P}$, $\|\mathbb{Q} - \mathbb{Q}_1\| < \varepsilon$, $\mathbb{E}_{\mathbb{Q}}[\|S_1\|] < \infty$ and $\mathbb{E}_{\mathbb{Q}}[(x, S_1)] < 0$, a contradiction to the choice of x. $\square$

Lemma 3.2 in conjunction with lemma 3.1 implies in particular that, given the stochastic process $S = (S_t)_{t=0}^1$ with $S_0 \equiv 0$ and $\mathcal{F}_0$ trivial, we may find a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that S is a $\mathbb{Q}$-martingale. We obtained the measure $\mathbb{Q}$ in two steps: first (lemma 3.1) we found $\mathbb{Q}_1 \sim \mathbb{P}$ which took care of the *admissible integrands*, which means that

$$\mathbb{E}_{\mathbb{Q}_1}[(x, S_1)] \leq 0, \qquad \text{for } x \in Adm.$$

In a second step (lemma 3.2) we found $\mathbb{Q} \sim \mathbb{P}$ such that $\mathbb{Q}$ took care of *all integrands*, i.e.,

$$(x, \mathbb{E}_{\mathbb{Q}}[S_1]) = \mathbb{E}_{\mathbb{Q}}[(x, S_1)] \leq 0 \qquad \text{for } x \in \mathbb{R}^d$$

and therefore

$$\mathbb{E}_{\mathbb{Q}}[S_1] = 0,$$

which means that S is a $\mathbb{Q}$-martingale.
In addition, we could assert in lemma 3.2 that $\|\mathbb{Q}_1 - \mathbb{Q}\| < \varepsilon$, a property which will be crucial in the sequel.
The strategy for proving the main theorem will be similar to the above approach. Given a general semi-martingale $S = (S_t)_{t \in \mathbb{R}_+}$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ we first replace $\mathbb{P}$ by $\mathbb{Q}_1 \sim \mathbb{P}$ such that $\mathbb{Q}_1$ "takes care of the admissible integrands", i.e.,

$$\mathbb{E}_{\mathbb{Q}_1}\left[(H \cdot S)_\infty\right] \leq 0, \qquad \text{for } H \text{ admissible.}$$

For this first step, the necessary technology has been developed in [DS94] and may be carried over almost verbatim.
The new ingredient developed in the present paper is the second step which takes care of the "big jumps" of S. By repeated application of an argument as in lemma 3.2 above we would like to change $\mathbb{Q}_1$ into a measure $\mathbb{Q}, \mathbb{Q} \sim \mathbb{P}$, such that S becomes a $\mathbb{Q}$-martingale. A glance at example 2.3 above reveals that this hope is, in the general setting, too optimistic and we can only try to turn S into a $\mathbb{Q}$-sigma-martingale. This will indeed be possible, i.e., we shall be able to find $\mathbb{Q}$ and a strictly positive predictable process φ, such that, *for every — not necessarily admissible — predictable $\mathbb{R}^d$-valued process H satisfying$\|H\|_{\mathbb{R}^d} \leq \varphi$*, we have that $H \cdot S$ is a $\mathbb{Q}$-marti ngale. In particular $\varphi \cdot S$ will be a $\mathbb{Q}$-martingale.
In order to complete this program we shall isolate in Lemma 3.5 below, the argument proving lemma 3.2 in the appropriate abstract setting. In particular we show that the construction in the proof of lemma 3.2 may be parameterized to depend in a measurable way on a parameter η varying in a measure space $(E, \mathcal{E}, \pi)$. The proof of this lemma is standard but long. One has to check a lot of measurability properties in order to apply the measurable selection theorem. Since the proofs are not really used in the sequel and are standard, the reader can, at a first reading, look at the definition 3.3, convince herself that the two parametrisations given in lemma 3.4 are equivalent and look at lemma 3.5.

Definition 3.3. *We say that a probability measure μ on $\mathbb{R}^d$ satisfies the NA property if for every $x \in \mathbb{R}^d$ we have $\mu(\{a \mid (x, a) < 0\}) > 0$ as soon as $\mu(\{a \mid (x, a) > 0\}) > 0$.*

We start with some notation that we will keep fixed for the rest of this section. We first assume that $(E, \mathcal{E}, \pi)$ is a probability space that is saturated for the null sets, i.e. if $A \subset B \in \mathcal{E}$ and if $\pi(B) = 0$ then $A \in \mathcal{E}$. The probability π can easily be changed into a σ-finite positive measure, but in order not to overload the statements we skip this straightforward generalisation. We recall that a Polish space X is a topological space that is homeomorphic to a complete separable metrisable space. The Borel sigma algebra of X is denoted by $\mathcal{B}(X)$. We will mainly be working in a space $E \times X$ where X is a Polish space. The canonical projection of $E \times X$ onto E is denoted by pr. If $A \in \mathcal{E} \otimes \mathcal{B}(X)$ then $pr(A) \in \mathcal{E}$, see [Au65] and [D72]. Furthermore there is a countable family $(f_n)_{n \geq 1}$ of measurable functions $f_n \colon pr(A) \to X$ such that

(1) for each $n \geq 1$ the graph of f_n is a selection of A, i.e. $\{(\eta, f_n(\eta)) \mid \eta \in pr(A)\} \subset A$,
(2) for each $\eta \in pr(A)$ the set $\{f_n(\eta) \mid n \geq 1\}$ is dense in $A_\eta = \{x \mid (\eta, x) \in A\}$.

We call such a sequence a countable dense selection of A.

The set $\mathcal{P}(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$ is equipped with the topology of convergence in law, also called weak* convergence. It is well known that $\mathcal{P}(\mathbb{R}^d)$ is Polish. If $F: E \to \mathcal{P}(\mathbb{R}^d)$ is a mapping, then the measurability of F can be reformulated as follows: for each bounded Borel function g, we have that the mapping $\eta \to \int_{\mathbb{R}^d} g(y)\, dF_\eta(y)$ is $\mathcal{E}$ measurable. This is easily seen using monotone class arguments. Using such a given measurable function F as a transition kernel, we can define a probability measure λ_F on $E \times \mathbb{R}^d$ as follows. For an element $D \in \mathcal{E} \otimes \mathcal{B}(\mathbb{R}^d)$ of the form $D = A \times B$, we define $\lambda_F(D) = \int_A F_\eta(B)\, \pi(d\eta)$.
For each $\eta \in E$ we define the set $Supp(F_\eta)$ as the support of the measure F_η, i.e. the smallest closed set of full F_η-measure. The set $\mathcal{S}$ is defined as $\{\eta, x) \mid x \in Supp(F_\eta)\}$. The set is an element of $\mathcal{E} \otimes \mathcal{B}(\mathbb{R}^d)$. Indeed, take a countable base $(U_n)_{n \geq 1}$ of the topology of $\mathbb{R}^d$ and write the complement as:

$$\mathcal{S}^c = \bigcup_{n \geq 1} \left(\{\eta \mid F_\eta(U_n) = 0\} \times U_n \right).$$

If $x: E \to \mathbb{R}^d$ is a measurable function then $\varphi: E \to \mathbb{R}_+ \cup \{+\infty\}$ defined as $\varphi(\eta) = \|(x_\eta, .)^-\|_{L^\infty(F_\eta)}$ is $\mathcal{E}$ measurable. Indeed take a countable dense selection $(f_n)_{n \geq 1}$ of $\mathcal{S}$ and observe that $\varphi(\eta) = \inf\{(x_\eta, f_n(\eta))^- \mid n \geq 1\}$.
For each $\eta \in E$ we denote by $\mathcal{A}dm(\eta)$ the cone in $\mathbb{R}^d$ consisting of elements $x \in \mathbb{R}^d$ so that $(x, .)^- \in L^\infty(F_\eta)$. The set $\mathcal{A}dm$ is then defined as $\{(\eta, x) \mid x \in \mathcal{A}dm(\eta)\}$. This set is certainly in $\mathcal{E} \otimes \mathcal{B}(\mathbb{R}^d)$. Indeed $\mathcal{A}dm = \{(\eta, x) \mid \inf_{n \geq 1}(x, f_n) > -\infty\}$ where the sequence $(f_n)_{n \geq 1}$ is a countable dense selection of $\mathcal{S}$.

3.4 Lemma. *If F is a measurable mapping from $(E, \mathcal{E}, \pi)$ into the probability measures on $\mathbb{R}^d$, then the following are equivalent:*

(1) *For almost every $\eta \in E$, the probability measure F_η satisfies the No Arbitrage property*
(2) *For every measurable selection x_η of $\mathcal{A}dm$, we have $\lambda_F\left[(\eta, a) \mid x_\eta(a) < 0\right] > 0$ as soon as $\lambda_F\left[(\eta, a) \mid x_\eta(a) > 0\right] > 0$.*

Proof. The implication $1 \Rightarrow 2$ is almost obvious since for each $\eta \in E$ we have that $F_\eta\left[\{a \mid (x_\eta, a) < 0\}\right] > 0$ as soon as $F_\eta\left[\{a \mid (x_\eta, a) > 0\}\right] > 0$. Therefore if $\lambda_F\left[(\eta, a) \mid x_\eta(a) > 0\right] > 0$, we have that $\pi(B) > 0$ where B is the set

$$B = \{\eta \in E \mid F_\eta\left[\{a \mid (x_\eta, a) > 0\}\right] > 0\}.$$

For the elements $\eta \in B$ we then also have that $F_\eta\left[\{a \mid (x_\eta, a) < 0\}\right] > 0$ and integration with respect to π then gives the result:

$$\lambda_F\left[(\eta, a) \mid (x_\eta, a) < 0\right] = \int_E \pi(d\eta) F_\eta\left[\{a \mid (x_\eta, a) < 0\}\right] > 0.$$

Let us now prove the reverse implication $2 \Rightarrow 1$.
We consider the set

$$A = \{(\eta, x) \mid F_\eta\left[a \mid (x, a) \geq 0\right] = 1 \text{ and } F_\eta\left[a \mid (x, a) > 0\right] > 0\}.$$

The reader can check that this set is in $\mathcal{E} \otimes \mathcal{B}(\mathbb{R}^d)$ and therefore the set $B = pr(A) \in \mathcal{E}$. Suppose that $\pi(B) > 0$ and take a measurable selection x_η of A. Outside B we define $x_\eta = 0$. Clearly $\lambda_F\left(\{(\eta, a) \mid (x_\eta, a) > 0\}\right) > 0$ and hence we have that $\lambda_F\left(\{(\eta, a) \mid (x_\eta, a) < 0\}\right) > 0$, a contradiction since $(x_\eta, a) \geq 0$, λ_F a.s.. So we see that $B = \emptyset$ a.s. or what is the same for almost every $\eta \in E$ the measure F_η satisfies the No Arbitrage property. $\square$

3.5 The Crucial Lemma. *Let $(E,\mathcal{E},\pi)$ be a probability measure space and let $(F_\eta)_{\eta\in E}$ be a family of probability measures on $\mathbb{R}^d$ such that the map $\eta \to F_\eta$ is $\mathcal{E}$-measurable.*
Let us assume that F satisfies the property that for each measurable map $x: E \to \mathbb{R}^d; \eta \to x_\eta$ with the property that for every $\eta \in E$ we have $(x_\eta, y) \geq -1$, for F_η almost every y, we also have that $\int_{\mathbb{R}^d} F_\eta(dy)(x_\eta, y) \leq 0$.
Let $\varepsilon: E \to \mathbb{R}_+ \setminus \{0\}$ be $\mathcal{E}$ measurable and strictly positive.
Then, we may find an $\mathcal{E}$-measurable map $\eta \to G_\eta$ from E to the probability measures on $\mathcal{B}(\mathbb{R}^d)$ such that, for π-almost every $\eta \in E$,

(i) $F_\eta \sim G_\eta$ *and* $\|F_\eta - G_\eta\| < \varepsilon_\eta$,
(ii) $\mathbb{E}_{G_\eta}[\|y\|] < \infty$ *and* $\mathbb{E}_{G_\eta}[y] = 0$.

Proof. As observed above the set $\mathcal{P}(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$, endowed with the weak* topology is a Polish space. We will show that the set

$$\left\{(\eta,\mu) \mid \int_{\mathbb{R}^d} \|x\|\, d\mu < \infty; \int_{\mathbb{R}^d} x\, d\mu = 0; F_\eta \sim \mu; \|\mu - F_\eta\| < \varepsilon(\eta)\right\}$$

is in $\mathcal{E} \otimes \mathcal{B}(\mathcal{P}(\mathbb{R}^d))$. Since, by lemma 3.4, for almost all η, the measure F_η satisfies the No Arbitrage assumption of Definition 3.3, we obtain that, for almost all η, the vertical section is nonempty. We can therefore find a measurable selection G_η and this will then end the proof.
The proof of the measurability property is easy but requires some arguments.
First we observe that the set $M = \{\mu \mid \int_{\mathbb{R}^d} \|x\|\, d\mu < \infty\}$ is in $\mathcal{B}(\mathcal{P}(\mathbb{R}^d))$. This follows from the fact that $\mu \to \int \|x\|\, d\mu$ is Borel measurable as it is an increasing limit of the weak* continuous functionals $\mu \to \int \min(\|x\|, n)\, d\mu$.
Next we observe that $M \to \mathbb{R}^d; \mu \to \int_{\mathbb{R}^d} x\, d\mu$ is Borel measurable.
The third observation is that $\{(\eta,\mu) \mid \|\mu - F_\eta\| < \varepsilon(\eta)\}$ is in $\mathcal{E} \otimes \mathcal{B}(\mathcal{P}(\mathbb{R}^d))$.
Finally we show that $\{(\eta,\mu) \mid \mu \sim F_\eta\}$ is also in $\mathcal{E} \otimes \mathcal{B}(\mathcal{P}(\mathbb{R}^d))$. This will then end the proof of the measurability property.
We take an increasing sequence of finite sigma-algebras $\mathcal{D}_n$ such that $\mathcal{B}(\mathbb{R}^d)$ is generated by $\cup_n \mathcal{D}_n$. For each n we see that the mapping $(\eta,\mu,x) \to \left.\frac{d\mu}{dF_\eta}\right|_{\mathcal{D}_n}(x) = q_n(\eta,\mu,x)$ is $\mathcal{E} \otimes \mathcal{B}(\mathcal{P}(\mathbb{R}^d)) \otimes \mathcal{B}(\mathbb{R}^d)$ measurable. The mapping

$$q(\eta,\mu,x) = \liminf q_n(\eta,\mu,x)$$

is clearly $\mathcal{E}\otimes\mathcal{B}(\mathcal{P}(\mathbb{R}^d))\otimes\mathcal{B}(\mathbb{R}^d)$ measurable. By the martingale convergence theorem we have that for each μ, the mapping q defines the Radon-Nikodym density of the part of μ that is absolutely continuous with respect to F_η. Now we have that

$$\{(\eta,\mu) \mid \mu \sim F_\eta\} = \left\{(\eta,\mu) \mid \int_{\mathbb{R}^d} q(\eta,\mu,x)\, dF_\eta(x) = 1; \lim_n \int_{\mathbb{R}^d} (nq) \wedge 1\, dF_\eta(x) = 1\right\}$$

and this shows that $\{(\eta,\mu) \mid \mu \sim F_\eta\}$ is in $\mathcal{E} \otimes \mathcal{B}(\mathcal{P}(\mathbb{R}^d))$.

Next suppose that $(E,\mathcal{E},\pi)$ is not necessarily complete. In that case we first complete the space $(E,\mathcal{E},\pi)$ by replacing the sigma-algebra $\mathcal{E}$ by $\tilde{\mathcal{E}}$ generated by $\mathcal{E}$ and all the null sets. We then obtain an $\tilde{\mathcal{E}}$ measurable mapping $\tilde{F}_\eta$ which can easily be replaced by a $\mathcal{E}$ measurable mapping F_η such that π almost surely $F_\eta = \tilde{F}_\eta$. $\square$

Remark. We have not striven for maximal generality in the formulation of lemma 3.5: for example, we could replace the probability measures F_η by finite non-negative measures on $\mathbb{R}^d$. In this case we may obtain the G_η in such a way that the total mass $G_\eta(\mathbb{R}^d)$ equals $F_\eta(\mathbb{R}^d)$, π-almost surely.

To illustrate the meaning of the Crucial Lemma we note a little observation in the spirit of [M91] which shows in particular the limitations of the no-arbitrage-theory when applied e.g. to Gaussian models for the stock returns in finite discrete time.

3.6 Proposition. *Let $(S_t)_{t=0}^T$ be an adapted $\mathbb{R}^d$-valued process based on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ such that for every predictable process $(h_t)_{t=1}^T$ we have that $(h \cdot S)_T = \sum_{t=1}^T h_t \Delta S_t$ is unbounded from above and from below as soon as $(h \cdot S)_T \not\equiv 0$. For example, this assumption is satisfied if the $\mathcal{F}_{t-1}$-conditional distributions of the jumps ΔS_t are normally distributed on $\mathbb{R}^d$.*
Then, for $\varepsilon > 0$, there is a measure $\mathbb{Q} \sim \mathbb{P}, \|\mathbb{Q} - \mathbb{P}\| < \varepsilon$, such that S is a $\mathbb{Q}$-martingale.
As a consequence, the set of equivalent martingale measures is dense with respect to the variation norm in the set of $\mathbb{P}$-absolutely continuous measures.

Proof. Suppose first that $T = 1$. Contrary to the setting of the motivating example at the beginning of this section we do not assume that $\mathcal{F}_0$ is trivial.
Let $(E, \mathcal{E}, \pi)$ be $(\Omega, \mathcal{F}_0, \mathbb{P})$ and denote by $(F_\omega)_{\omega \in \Omega}$ the $\mathcal{F}_0$-conditional distribution of $\Delta S_1 = S_1 - S_0$. The assumption of lemma 3.5 is (trivially) satisfied as by hypothesis the $\mathcal{F}_0$-measurable functions $x(\omega)$ such that $\mathbb{P}$-a.s. we have $(x_\omega, y) \geq -1, F_\omega$-a.s., satisfy $(x_\omega, y) = 0, F_\omega$-a.s., for $\mathbb{P}$-a.e. $\omega \in \Omega$.
Choose $\varepsilon(\omega) \equiv \varepsilon$ and find G_η as in the lemma. To translate the change of the conditional distributions of ΔS_1 into a change of the measure $\mathbb{P}$, find $Y : \Omega \times \mathbb{R}^d \to \mathbb{R}_+$,

$$Y(\omega, x) = \frac{dG_\omega}{dF_\omega}(x) \qquad x \in \mathbb{R}^d, \omega \in \Omega$$

such that, for $\mathbb{P}$-a.e. $\omega \in \Omega, Y(\omega, \cdot)$ is a version of the Radon-Nikodym derivative of G_ω with respect to F_ω, and such that $Y(\cdot, \cdot)$ is $\mathcal{F}_0 \otimes \mathrm{Borel}(\mathbb{R}^d)$-measurable.
Letting

$$\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}(\omega) = Y(\omega, \Delta S_1(\omega))$$

we obtain an $\mathcal{F}_1$-measurable density of a probability measure. Assertion (i) of lemma 3.5 implies that $\hat{\mathbb{Q}} \sim \mathbb{P}$ and $\|\hat{\mathbb{Q}} - \mathbb{P}\| < \varepsilon$. Assertion (ii) implies that

$$E_{\hat{\mathbb{Q}}}\left[\|\Delta S_1\|_{\mathbb{R}^d} | \mathcal{F}_0\right] < \infty \qquad \text{a.s.}$$

and

$$\mathbb{E}_{\hat{\mathbb{Q}}}\left[\Delta S_1 | \mathcal{F}_0\right] = 0. \qquad \text{a.s.}$$

We are not quite finished yet as this only shows that $(S_t)_{t=0}^1$ is a $\hat{\mathbb{Q}}$-sigma-martingale but not necessarily a $\hat{\mathbb{Q}}$-martingale as it may happen that $\mathbb{E}[\|\Delta S_1\|_{\mathbb{R}^d}] = \infty$. But it is easy to overcome this difficulty: find a strictly positive $\mathcal{F}_0$-measurable function $w(\omega)$, normalised so that $\mathbb{E}_{\hat{\mathbb{Q}}}[w] = 1$ and such that $\mathbb{E}_{\hat{\mathbb{Q}}}[w(\omega)\mathbb{E}[\|\Delta S_1\|_{\mathbb{R}^d} | \mathcal{F}_0]] < \infty$. We can construct w is such a way that the probability measure $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}(\omega)}{d\hat{\mathbb{Q}}(\omega)} = w(\omega),$$

still satisfies $||\mathbb{Q} - \mathbb{P}|| < \varepsilon$. Then

$$\mathbb{E}_{\mathbb{Q}}\left[||\Delta S_1||_{\mathbb{R}^d}\right] < \infty$$

and

$$\mathbb{E}_{\mathbb{Q}}\left[\Delta S_1 | \mathcal{F}_0\right] = 0, \qquad \text{a.s.,}$$

i.e., S is a $\mathbb{Q}$-martingale.

To extend the above argument from $T = 1$ to arbitrary $T \in \mathbb{N}$ we need yet another small refinement: an inspection of the proof of lemma 3.5 above reveals that in addition to assertions (i) and (ii) of lemma 3.5, and given $M > 1$, we may choose G_η such that

$$\text{(iii)} \qquad \left\| \frac{dG_\eta}{dF_\eta} \right\|_{L^\infty(\mathbb{R}^d, F_\eta)} \leq M, \qquad \pi\text{- a.s.}$$

We have not mentioned this additional assertion in order not to overload lemma 3.5 and as we shall only need (iii) in the present proof.
Using (iii), with $M = 2$ say, and, choosing w above also uniformly bounded by 2, the argument in the first part of the proof yields a probability $\mathbb{Q} \sim \mathbb{P}, ||\mathbb{Q} - \mathbb{P}|| < \varepsilon$, such that $||\frac{d\mathbb{Q}}{d\mathbb{P}}||_{L^\infty(\mathbb{P})} \leq 4$.
Now let $T \in \mathbb{N}$ and $(S_t)_{t=0}^T$, based on $(\Omega, (\mathcal{F}_t)_{t=0}^T, \mathcal{F}, \mathbb{P})$, be given. By backward induction on $t = T, \ldots, 1$ apply the first part of the proof to find $\mathcal{F}_t$-measurable densities Z_t such that, defining the probability measure $\mathbb{Q}^{(t)}$ by

$$\frac{d\mathbb{Q}^{(t)}}{d\mathbb{P}} = Z_t,$$

we have that the two-step process $(S_u \prod_{v=t+1}^T Z_v)_{u=t-1}^t$ is a $\mathbb{Q}^{(t)}$-martingale with respect to the filtration $(\mathcal{F}_u)_{u=t-1}^t$, $\mathbb{Q}^{(t)} \sim \mathbb{P}$, $||\mathbb{Q}^{(t)} - \mathbb{P}||_1 < \varepsilon 4^{-T} T^{-1}$, and such that $||Z_t||_{L^\infty(\mathbb{P})} \leq 4$.
Defining

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \prod_{t=1}^T Z_t,$$

we obtain a probability measure $\mathbb{Q}, \mathbb{Q} \sim \mathbb{P}$ such that $(S_t)_{t=0}^T$ is a martingale under $\mathbb{Q}$. Indeed,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\Delta S_t | \mathcal{F}_{t-1}\right] &= \mathbb{E}_{\mathbb{P}}\left[\Delta S_t \prod_{u=1}^T Z_u | \mathcal{F}_{t-1}\right] \\
&= \left(\prod_{u=1}^{t-1} Z_u\right) \mathbb{E}_{\mathbb{P}}\left[Z_t \Delta S_t \prod_{u=t+1}^T Z_u | \mathcal{F}_{t-1}\right] \\
&= \left(\prod_{u=1}^{t-1} Z_u\right) \mathbb{E}_{\mathbb{Q}^{(t)}}\left[\Delta S_t \prod_{u=t+1}^T Z_u | \mathcal{F}_{t-1}\right] = 0
\end{aligned}$$

and

$$\mathbb{E}_{\mathbb{Q}}\left[||\Delta S_t||_{\mathbb{R}^d}\right] \leq ||\prod_{u=1}^{t-1} Z_u||_\infty \cdot \mathbb{E}_{\mathbb{Q}^{(t)}}\left[||\Delta S_t \prod_{u=t+1}^T Z_u||_{\mathbb{R}^d}\right] < \infty.$$

Finally we may estimate $\|\mathbb{Q}-\mathbb{P}\|_1$ by

$$\begin{aligned}
\|\mathbb{Q}-\mathbb{P}\|_1 &= \mathbb{E}_{\mathbb{P}}\left[|\prod_{t=1}^{T} Z_t - 1|\right] \\
&\leq \mathbb{E}_{\mathbb{P}}\left[\sum_{t=1}^{T} |\prod_{u=1}^{t} Z_u - \prod_{u=1}^{t-1} Z_u|\right] \\
&\leq \sum_{t=1}^{T} \|\prod_{u=1}^{t-1} Z_u\|_{L^\infty(\mathbb{P})} \mathbb{E}_{\mathbb{P}}\left[|Z_t - 1|\right] \\
&\leq T \cdot 4^T \varepsilon 4^{-T} T^{-1} = \varepsilon.
\end{aligned}$$

The proof of the first part of proposition 3.6 is thus finished and we have shown in the course of the proof that we may find $\mathbb{Q}$ such that, in addition to the assertions of the proposition, $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is uniformly bounded.
As regards the final assertion, let $\mathbb{P}'$ be any $\mathbb{P}$-absolutely continuous measure. For given $\varepsilon > 0$, first take $\mathbb{P}'' \sim \mathbb{P}$ such that $\|\mathbb{P}''-\mathbb{P}'\| < \varepsilon$. Now apply the first assertion with $\mathbb{P}''$ replacing $\mathbb{P}$. As a result we get an equivalent martingale measure $\mathbb{Q}$ such that $\|\mathbb{Q}-\mathbb{P}''\| < \varepsilon$, hence also $\|\mathbb{Q}-\mathbb{P}'\| < 2\varepsilon$.
This finishes the proof of proposition 3.6 $\square$

4. The General $\mathbb{R}^d$-valued Case

In this section $S = (S_t)_{t\in\mathbb{R}_+}$ denotes a general $\mathbb{R}^d$-valued càdlàg semi-martingale based on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$ where we assume that the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ satisfies the usual conditions of completeness and right continuity.
Similarly as in [DS94] we define an S-integrable $\mathbb{R}^d$-valued predictable process $H = (H_t)_{t\in\mathbb{R}_+}$ to be an *admissible integrand* if the stochastic process

$$(H \cdot S)_t = \int_0^t (H_u, dS_u) \qquad\qquad t \in \mathbb{R}_+$$

is (almost surely) uniformly bounded from below.
It is important to note that, similarly as in proposition 3.6 above, it may happen that the cone of admissible integrands is rather small and possibly even reduced to zero: consider, for example, the case when S is a compound Poisson process with (two-sided) unbounded jumps, i.e., $S_t = \sum_{i=1}^{N_t} X_i$, where $(N_t)_{t\in\mathbb{R}_+}$ is a Poisson process and $(X_i)_{i=1}^{\infty}$ an i.i.d. sequence of real random variables such that $\|X_i^+\|_\infty = \|X_i^-\|_\infty = \infty$. Clearly, a predictable process H, such that $H \cdot S$ remains uniformly bounded from below, must vanish almost surely.

Continuing with the general setup we denote by K the convex cone in $L^0(\Omega, \mathcal{F}, \mathbb{P})$ given by

$$K = \{f = (H \cdot S)_\infty : H \text{ admissible}\}$$

where this definition requires in particular that the random variable $(H \cdot S)_\infty := \lim_{t\to\infty}(H \cdot S)_t$ is (almost surely) well-defined (compare [DS94], definition 2.7).
Again we denote by C the convex cone in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ formed by the uniformly bounded random variables dominated by some element of K, i.e.,

$$
\begin{aligned}
C &= (K - L^0_+(\Omega, \mathcal{F}, \mathbb{P})) \cap L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \\
&= \{f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) : \text{ there is } g \in K, f \leq g\}. \qquad (4.1)
\end{aligned}
$$

We say [DS94], p.467, that the semi-martingale S satisfies the condition of *No Free Lunch with Vanishing Risk* (NFLVR) if the closure $\overline{C}$ of C, taken with respect to the norm-topology $\|\cdot\|_\infty$ of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ intersects $L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$ only in 0, i.e.,

$$S \text{ satisfies (NFLVR)} \Longleftrightarrow \overline{C} \cap L^\infty_+ = \{0\}.$$

For the economic interpretation of this concept, which is a very mild strengthening of the "no arbitrage" concept, we refer to [DS94].
The subsequent crucial theorem 4.1 was proved in ([DS94], th.4.2) under the additional assumption that S is bounded. An inspection of the proof given in [DS94] reveals that — for the validity of the subsequent theorem 4.1 — the boundedness assumption on S may be dropped.

4.1 Theorem. *Under the assumption (NFLVR) the cone C is weak* closed in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Hence there is a probability measure $\mathbb{Q}_1 \sim \mathbb{P}$ such that*

$$\mathbb{E}_{\mathbb{Q}_1}[f] \leq 0, \qquad \text{for } f \in C. \quad \square$$

4.2 Remark. In the case, when S is bounded, $\mathbb{Q}_1$ is already a martingale measure for S, and when S is locally bounded, $\mathbb{Q}_1$ is a local martingale measure for S (compare [DS94], theorem 1.1 and corollary 1.2).
To take care of the non locally bounded case we have to take care of the "big jumps" of S. We shall distinguish between the jumps of S occurring at accessible stopping times and those occurring at totally inaccessible stopping times.
We start with an easy lemma which will allow us to change the measure $\mathbb{Q}_1$ countably many times without loosing the equivalence to $\mathbb{P}$.

4.3 Lemma. *Let $(\mathbb{Q}_n)_{n=1}^\infty$ be a sequence of probability measures on $(\Omega, \mathcal{F}, \mathbb{P})$ such that each $\mathbb{Q}_n$ is equivalent to $\mathbb{P}$. Suppose further that the sequence of strictly positive numbers $(\varepsilon_n)_{n\geq 2}$ is such that*

(1) $\|\mathbb{Q}_n - \mathbb{Q}_{n+1}\| < \varepsilon_{n+1}$,
(2) *if* $\mathbb{Q}_n[A] \leq \varepsilon_{n+1} 2^n$ *then* $\mathbb{P}[A] \leq 2^{-n}$.

Then the sequence $(\mathbb{Q}_n)_{n\geq 1}$ converges with respect to the total variation norm to a probability measure $\mathbb{Q}$, which is equivalent to $\mathbb{P}$.

Proof of lemma 4.3. Clearly the second assumption implies that $\varepsilon_{n+1} \leq 2^{-n}$ and hence the sequence $(\mathbb{Q}_n)_{n\geq 1}$ converges in variation norm to a probability measure $\mathbb{Q}$. We have to show that $\mathbb{Q} \sim \mathbb{P}$. For each n we let q_{n+1} be defined as the Radon-Nikodym derivative of $\mathbb{Q}_{n+1}$ with respect to $\mathbb{Q}_n$. Clearly for each $n \geq 2$ we then have $\int |1 - q_{n+1}|\, d\mathbb{Q}_n \leq \varepsilon_{n+1}$ and hence the Markov inequality implies that $\mathbb{Q}_n[|1 - q_{n+1}| \geq 2^{-n}] \leq 2^n \varepsilon_{n+1}$. The hypothesis on the sequence $(\varepsilon_n)_{n\geq 2}$ then implies that $\mathbb{P}[|1 - q_{n+1}| \geq 2^{-n}] \leq 2^{-n}$. From the Borel Cantelli lemma it also follows that a.s. the series $\sum_{n\geq 2} |1 - q_n|$ converges and hence the product $\prod_{n\geq 2} q_n$ converges to a function q a.s. different from 0. Clearly $q = \frac{d\mathbb{Q}}{d\mathbb{Q}_1}$ which shows that $\mathbb{Q} \sim \mathbb{Q}_1 \sim \mathbb{P}$. $\square$

We are now ready to take the crucial step in the proof of the main theorem. To make life easier we still make the simplifying assumption that S does not jump at predictable times. In 4.6 below we finally shall also deal with the case of the predictable jumps.

4.5 Proposition. *Let $S = (S_t)_{t\in\mathbb{R}_+}$ be an $\mathbb{R}^d$-valued semi-martingale which is quasi-left-continuous, i.e., such that, for every predictable stopping time T we have $S_T = S_{T-}$ almost surely.*
Suppose, as in 4.1 above, that $\mathbb{Q}_1 \sim \mathbb{P}$ is a probability measure verifying

$$\mathbb{E}_{\mathbb{Q}_1}[f] \leq 0 \qquad \text{for } f \in C.$$

Then there is, for $\varepsilon > 0$, a probability measure $\mathbb{Q} \sim \mathbb{P}$, $\|\mathbb{Q} - \mathbb{Q}_1\| < \varepsilon$, such that S is a sigma-martingale with respect to $\mathbb{Q}$.
In addition, for every predictable stopping time T, the probabilities $\mathbb{Q}$ and $\mathbb{Q}_1$ on $\mathcal{F}_T$, coincide , conditionally on $\mathcal{F}_{T-}$, i.e.,

$$\frac{d\mathbb{Q}_{|\mathcal{F}_T}}{d\mathbb{Q}_{1|\mathcal{F}_T}} = \frac{d\mathbb{Q}_{|\mathcal{F}_{T-}}}{d\mathbb{Q}_{1|\mathcal{F}_{T-}}} \qquad a.s.$$

Proof. Step 1: Define the stopping time T by

$$T = \inf\{t : \|\Delta S_t\|_{\mathbb{R}^d} \geq 1\}$$

and first suppose that S remains constant after time T, hence S has at most one jump bigger than 1.
Similarly as in [JS87], II.2.4 we decompose S into

$$S = X + \check{X}$$

where X equals "S stopped at time $T-$", i.e.,

$$X_t = \begin{cases} S_t & \text{for } t < T \\ S_{T-} & \text{for } t \geq T \end{cases}$$

and $\check{X}$ the jump of S at time T, i.e.,

$$\check{X}_t = \Delta S_T \cdot \mathbf{1}_{[T,\infty[}.$$

As X is bounded, it is a special semi-martingale, and we can find its Doob-Meyer decomposition with respect to $\mathbb{Q}_1$

$$X = M + B$$

where M is a local $\mathbb{Q}_1$-martingale and B a predictable process of locally finite variation.
We shall now find a probability measure $\mathbb{Q}_2$ on $\mathcal{F}, \mathbb{Q}_2 \sim \mathbb{P}$, s.t.

(i) $\|\mathbb{Q}_2 - \mathbb{Q}_1\| < \varepsilon/2$,
(ii) $\mathbb{Q}_2|_{\mathcal{F}_{T-}} = \mathbb{Q}_1|_{\mathcal{F}_{T-}}$ and $\frac{d\mathbb{Q}_2}{d\mathbb{Q}_1}$ is $\mathcal{F}_T$-measurable,
(iii) S is a sigma-martingale under $\mathbb{Q}_2$.

We introduce the jump measure μ associated to $\check{X}$,

$$\mu(\omega, dt, dx) = \delta_{(T(\omega), \Delta S_T(\omega))},$$

where $\delta_{t,x}$ denotes Dirac-measure at $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and we denote by ν the $\mathbb{Q}_1$-compensator of μ (see [JS87], prop. II.1.6). Similarly as in ([JS87], prop. II.2.9) we may find a locally $\mathbb{Q}_1$-integrable, predictable and increasing process A such that

$$B = b \cdot A$$
$$\nu(\omega, dt, dx) = F_{\omega,t}(dx) dA_t(\omega)$$

where $b = (b^i)_{i=1}^d$ is a predictable process and $F_{\omega,t}(dx)$ a transition kernel from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}^d,$ Borel $(\mathbb{R}^d))$, i.e., a $\mathcal{P}$-measurable map $(\omega, t) \to F_{\omega,t}(dx)$ from $\Omega \times \mathbb{R}_+$ into the nonnegative Borel measures on $\mathbb{R}^d$. Since the processes X and $\check{X}$ arequasi left continuous, the processes A and B can be chosen to be continuous, but this is not really needed.
The process ν and μ are such that for each nonnegative $\mathcal{P} \otimes \text{Borel}(\mathbb{R}^d)$ measurable function g we have that

$$\int_{\Omega \times \mathbb{R}_+ \times \mathbb{R}^d} g(\omega, t, y) \mu(\omega, dt, dy) \, \mathbb{P}(d\omega) = \int_{\Omega \times \mathbb{R}_+ \times \mathbb{R}^d} g(\omega, t, y) \nu(\omega, dt, dy) \, \mathbb{P}(d\omega)$$

To stay in line with the notation used in [JS87], $H_{\omega,t} * F_{\omega,t}$, where H is a predictable $\mathbb{R}^d$ valued process and F is the kernel described above, denotes the predictable $\mathbb{R}$ valued process $E_{F_{\omega,t}}[(H_{\omega,t}\ , \cdot)] = \int_{\mathbb{R}^d} (H_{\omega,t}, y)\, F_{\omega,t}[dy]$.
We may assume that A is constant after T, $\mathbb{Q}_1$-integrable and its integral is bounded by one, i.e.,

$$\mathbb{E}_{\mathbb{Q}_1}[A_\infty] = dA(\Omega \times \mathbb{R}_+) \leq 1,$$

where dA denotes the measure on $\mathcal{P}$ defined by $dA(]\!]T_1, T_2]\!]) = \mathbb{E}_{\mathbb{Q}_1}[A_{T_2} - A_{T_1}]$, for stopping times $T_1 \leq T_2$.
We now shall find a $\mathcal{P}$-measurable map $(\omega, t) \to G_{\omega,t}$ such that for dA-almost each (ω, t),

(a) $F_{\omega,t}(dx) \sim G_{\omega,t}(dx), F_{\omega,t}(\mathbb{R}^d) = G_{\omega,t}(\mathbb{R}^d)$ and $\|F_{\omega,t} - G_{\omega,t}\| < \frac{\varepsilon}{2}$,
(b) $\mathbb{E}_{G_{\omega,t}}[\|y\|_{R^d}] < \infty$ and $\mathbb{E}_{G_{\omega,t}}[y] = -b(\omega, t)$.

This is a task of the type of "martingale problem" or rather "semi-martingale problem" as dealt with, e.g., in [JS87], def. III.2.4.
We apply lemma 3.5 and the remark following it: as measure space $(E, \mathcal{E}, \omega)$ we take $(\Omega \times \mathbb{R}_+, \mathcal{P}, dA)$ and we shall consider the map

$$\eta = (\omega, t) \to \tilde{F}_{\omega,t} := F_{\omega,t} \bigstar \delta_{b(\omega,t)}$$

where $\delta_{b(\omega,t)}$ denotes the Dirac measure at $b(\omega, t) \in \mathbb{R}^d$, $\bigstar$ denotes convolution and therefore $\tilde{F}_{\omega,t}$ is the measure $F_{\omega,t}$ on $\mathbb{R}^d$ translated by the vector $b(\omega, t)$.
We claim that the family $(\tilde{F}_{\omega,t})_{(\omega,t) \in \Omega \times \mathbb{R}_+}$ satisfies the assumptions of lemma 3.5 above. Indeed, let $H_{\omega,t}$ be any $\mathcal{P}$-measurable function such that dA-almost surely $H_{\omega,t} \in \mathcal{A}dm(\tilde{F}_{\omega,t}) = \mathcal{A}dm(F_{\omega,t})$.
By multiplying H with a predictable strictly positive process v, we may eventually assume that $\|H_{\omega,t}\|_{\mathbb{R}^d} \leq 1$ and that, at least dA a.e., also the predictable process

$\|\langle H_{\omega,t}, .\rangle^-\|_{L^\infty(F_{\omega,t}}$ is bounded by 1. That the latter process is predictable follows from the discussion preceding the Crucial Lemma and essentially follows from the measurable selection theorem.
The boundedness property translates to the fact that $H = (H_t(\omega))_{t\in\mathbb{R}_+}$ is an *admissible integrand* for the process $\check{X}$. This follows from the definition of the compensator ν in the following way. For each natural number n we have, according to the definition of the compensator that

$$\begin{aligned}
&\mathbb{E}\left[\left(\langle H_{\omega,t}, \Delta S_T(\omega)\rangle^-\right)^n \mathbf{1}_{T<\infty}\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}_+\times\mathbb{R}^d} \left(\langle H_{\omega,t}, y\rangle^-\right)^n \mu(\omega, dt, dy)\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}_+\times\mathbb{R}^d} \left(\langle H_{\omega,t}, y\rangle^-\right)^n \nu(\omega, dt, dy)\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}_+}\int_{\mathbb{R}^d} \left(\langle H_{\omega,t}, y\rangle^-\right)^n F_{\omega,t}(dy)\, dA_t(\omega)\right] \\
&\leq \mathbb{E}\left[\int_{\mathbb{R}_+}\int_{\mathbb{R}^d} 1\, F_{\omega,t}(dy)\, dA_t(\omega)\right] \\
&\leq \mathbb{E}\left[\int_{\mathbb{R}_+\times\mathbb{R}^d} 1\, \nu(\omega, dt, dy)\right] \\
&\leq \mathbb{E}\left[\int_{\mathbb{R}_+\times\mathbb{R}^d} 1\, \mu(\omega, dt, dy)\right] \leq \mathbb{Q}_1[T<\infty].
\end{aligned}$$

Since the inequality holds for each n we necessarily have that

$$(H\cdot\check{X})_t \geq -1 \qquad \text{a.s., for all } t\in\mathbb{R}_+.$$

Noting that M is a (locally bounded) local martingale and B is of locally bounded variation, we may find a sequence of stopping times $(U_j)_{j=1}^\infty$ increasing to infinity, such that, for each $j\in\mathbb{N}$,

(1) M^{U_j} is a martingale, bounded in the Hardy space $\mathcal{H}^1(\mathbb{Q}_1)$, and
(2) B^{U_j} is of bounded variation.

Hence, for each predictable set P contained in $[\![0, U_j]\!]$, for some $j\in\mathbb{N}$, we have that $H\mathbf{1}_P$ is an admissible integrand for S and $H\mathbf{1}_P\cdot M$ is a martingale bounded in $\mathcal{H}^1(\mathbb{Q}_1)$ and therefore

$$\mathbb{E}_{\mathbb{Q}_1}\left[(H\mathbf{1}_P\cdot M)_\infty\right] = 0.$$

As by hypothesis

$$\mathbb{E}_{\mathbb{Q}_1}\left[(H\mathbf{1}_P\cdot S)_\infty\right] \leq 0$$

we obtain

$$\mathbb{E}_{\mathbb{Q}_1}\left[(H\mathbf{1}_P\cdot(\check{X}+B))_\infty\right] \leq 0.$$

Using the identities

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_1}\left[(H\mathbf{1}_P\cdot(\check{X}+B))_\infty\right] &= \int_{\Omega\times\mathbb{R}_+} (H_{\omega,t} * F_{\omega,t} + (H_{\omega,t}, b_{\omega,t}))\mathbf{1}_P dA(\omega,t) \\
&= \int_{\Omega\times\mathbb{R}_+} (H_{\omega,t} * \tilde{F}_{\omega,t})\mathbf{1}_P dA(\omega,t) \leq 0
\end{aligned}$$

which hold true for each $P \in \mathcal{P}$ contained in $[\![0, U_j]\!]$, for some $j \in \mathbb{N}$, we conclude that for dA-almost each (ω, t) we have

$$H_{\omega,t} * \tilde{F}_{\omega,t} = \mathbb{E}_{\tilde{F}_{\omega,t}}\left[(H_{\omega,t}\ ,\cdot)\right] \leq 0.$$

This inequality implies that assumption 2 in Lemma 3.4 is satisfied and hence $F_{\omega,t}$ satisfies the no arbitrage property, i.e. the hypothesis of lemma 3.5 is satisfied.
Hence we may find a transition kernel $\tilde{G}_{\omega,t}$ as described by lemma 3.5 — with ε replaced by $\varepsilon/2$ — and letting $G_{\omega,t} = \tilde{G}_{\omega,t} \bigstar \delta_{-b(\omega,t)}$ we obtain a transition kernel satisfying (a) and (b) above.
We now have to translate the change of transition kernels from $F_{\omega,t}$ to $G_{\omega,t}$ into a change of measures from $\mathbb{Q}_1$ to $\mathbb{Q}_2$ on the sigma-algebra $\mathcal{F}_T$ which will be done by defining the Radon-Nikodym derivative $\frac{d\mathbb{Q}_2}{d\mathbb{Q}_1}$. We refer to [JS87], III.3 for a treatment of the relevant version of Girsanov's theorem for random measures.
For (ω, t) fixed denote by $Y(\omega, t\ , \cdot)$ the Radon-Nikodym derivative of $G_{\omega,t}$ with respect to $F_{\omega,t}$, i.e.

$$Y(\omega, t, x) = \frac{dG_{\omega,t}}{dF_{\omega,t}}(x), \qquad\qquad x \in \mathbb{R}^d,$$

which is $F_{\omega,t}$-almost surely well-defined and strictly positive. We may and do choose for dA-almost each (ω, t), a version $Y(\omega, t, x)$ such that $Y(\cdot, \cdot, \cdot)$ is $\mathcal{P} \otimes$ Borel$(\mathbb{R}^d)$-measurable.
We now define

$$\frac{d\mathbb{Q}_2}{d\mathbb{Q}_1}(\omega) = Z_\infty(\omega) = Y(\omega, T(\omega), \Delta S_{T(\omega)}(\omega))\mathbf{1}_{\{T<\infty\}} + \mathbf{1}_{\{T=\infty\}}$$
$$\text{and} \qquad Z_t(\omega) = Y(\omega, T(\omega), \Delta S_{T(\omega)}(\omega))\mathbf{1}_{\{T\leq t\}} + \mathbf{1}_{\{T>t\}}.$$

The intuitive interpretation of these formulas goes as follows: for fixed $\omega \in \Omega$ we look at time $T(\omega)$ which is the unique "big" jump of $(S_t(\omega))_{t\in\mathbb{R}_+}$. The density $Y(\omega, T(\omega), x)$ gives the density of the distribution of the compensated jump measure $G_{\omega,t}$ with respect to $F_{\omega,t}$, if the jump equals x and therefore we evaluate $Y(\omega, T(\omega), x)$ at the point $x = \Delta S_{T(\omega)}(\omega)$ to determine the density of $\mathbb{Q}_2$ with respect to $\mathbb{Q}_1$. If $T(\omega) = \infty$ the density $\frac{d\mathbb{Q}_2}{d\mathbb{Q}_1}(\omega)$ is simply equal to 1. The variable $Y(\omega, T(\omega), \Delta S_T(\omega))$ is certainly integrable. Indeed

$$\begin{aligned}
\mathbb{E}[Y(\omega, T(\omega), \Delta S_T(\omega))\mathbf{1}_{T<\infty}] \quad &= \mathbb{E}\left[\int_{\mathbb{R}_+\times\mathbb{R}^d} Y(\omega, t, y)\mu(\omega, dt, dy)\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}_+\times\mathbb{R}^d} Y(\omega, t, y)\nu(\omega, dt, dy)\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}_+}\int_{\mathbb{R}^d} Y(\omega, t, y)F_{\omega,t}(dy)dA_t(\omega)\mathbb{Q}_1(d\omega)\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}_+}\int_{\mathbb{R}^d} F_{\omega,t}(\mathbb{R}^d)dA_t(\omega)\mathbb{Q}_1(d\omega)\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}_+\times\mathbb{R}^d} \nu(\omega, dt, dy)\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}_+\times\mathbb{R}^d} \mu(\omega, dt, dy)\right] = \mathbb{Q}_1[T < \infty].
\end{aligned}$$

The process Z can also be written as

$$Z = Y(\omega, T, \Delta S_T)\mathbf{1}_{[T,\infty[} + \mathbf{1}_{[0,T[},$$

from which it follows that Z is a process of integrable variation. The maximal function Z^* of Z is therefore integrable.
In order to show that $\mathbb{Q}_2$ is indeed a probability measure and that $Z_t = \frac{d\mathbb{Q}_2|_{\mathcal{F}_t}}{d\mathbb{Q}_1|_{\mathcal{F}_t}}$ we shall show that $(Z_t)_{t\in\mathbb{R}_+}$ is a uniformly integrable martingale closed by Z_∞.
We may write $Z = (Z_t)_{t\in\mathbb{R}_+}$ as

$$Z = 1 + (Y(\omega, t, x) - 1) * \mu.$$

From the definition of the compensator ν ([JS87], II.1.8) we deduce thatwe may write the compensator Z^p of Z

$$\begin{aligned} Z^p &= 1 + (Y(\omega, t, x) - 1) * \nu \\ &= 1 + ((Y(\omega, t, x) - 1) * F_{\omega,t}) \cdot A \end{aligned}$$

Noting that, for dA-almost each (ω, t) we have that $(Y(\omega, t, x) - 1) * F_{\omega,t} = \mathbb{E}_{F_{\omega,t}}[Y(\omega, t, x) - 1] = 0$ we deduce that the compensator Z^p is constant. Since $Z - Z^p$ is a martingale, by defintion of the compensator of processes of integrable variation, it follows that Z is a martingale as well.
To estimate the distance $||\mathbb{Q}_2 - \mathbb{Q}_1||$, note that

$$\begin{aligned} ||\mathbb{Q}_2 - \mathbb{Q}_1|| &= \mathbb{E}_{\mathbb{Q}_1}\left[\left|1 - \frac{d\mathbb{Q}_2}{d\mathbb{Q}_1}\right|\right] \\ &\leq \mathbb{E}_{\mathbb{Q}_1}\left[(|Y(\omega, t, x) - 1| * \nu)_\infty\right] \\ &\leq \mathbb{E}_{\mathbb{Q}_1}(||F_{\omega,t} - G_{\omega,t}|| \cdot A)_\infty \leq (\varepsilon/2)\,\mathbb{E}_{\mathbb{Q}_1}[A_\infty] \leq \varepsilon/2. \end{aligned}$$

Next observe that $\mathbb{Q}_2|_{\mathcal{F}_{T-}} = \mathbb{Q}_1|_{\mathcal{F}_{T-}}$: indeed, we have to show that $\mathbb{Q}_1$ and $\mathbb{Q}_2$ coincide on the sets of the form $A \cap \{T > t\}$, where $A \in \mathcal{F}_t$, as these sets generate $\mathcal{F}_{T-}$. Noting that Z_t is equal to 1 on $\{T > t\}$, this becomes obvious.
Finally we show that S is a sigma-martingale under $\mathbb{Q}_2$. First note that M remains a local martingale under $\mathbb{Q}_2$ as M is continuous at time T, i.e., $M_{T-} = M_T$, and $\mathbb{Q}_1$ and $\mathbb{Q}_2$ coincide on $\mathcal{F}_{T-}$.
As regards the remaining part $\check{X} + B$ of the semi-martingale S we have by (b) above that, for dA-almost each (ω, t), $\mathbb{E}_{G_{\omega,t}}[||y||_{\mathbb{R}^d}] < \infty$ and $\mathbb{E}_{G_{\omega,t}}[y] = -b(\omega, t)$. This *does not necessarily* imply that $\check{X} + B$ is already a martingale (or a local martingale) under $\mathbb{Q}_2$ as a glance at example 2.2 reveals. We may only conclude that $\check{X} + B$ is a $\mathbb{Q}_2$-sigma-martingale, as we presently shall see.
Define

$$\varphi_t(\omega) = (\mathbb{E}_{G_{\omega,t}}[||y||_{\mathbb{R}^d}])^{-1} \wedge 1,$$

which is a predictable dA-almost surely strictly positive process. The process $\varphi \cdot (\check{X} + B)$ is a process of $\mathbb{Q}_2$-integrable variation as

$$\begin{gathered} \mathbb{E}_{\mathbb{Q}_2}\left[\, \mathrm{var}_{||\cdot||_{\mathbb{R}^d}}(\varphi \cdot (\check{X} + B))\right] = \\ \mathbb{E}_{\mathbb{Q}_2}\left[\varphi \cdot (||y||_{\mathbb{R}^d} * G_{\omega,t} \cdot A + ||b_{\omega,t}||_{\mathbb{R}^d} \cdot A)\right] \leq 2\mathbb{E}_{\mathbb{Q}_2}[A_\infty] = 2\mathbb{E}_{\mathbb{Q}_1}[A_\infty] \leq 2, \end{gathered}$$

where the last equality follows from the fact that $\mathbb{Q}_1$ and $\mathbb{Q}_2$ coincide on $\mathcal{F}_{T-}$ and that, A being predictable, A_∞ is $\mathcal{F}_{T-}$ measurable.
Hence $\varphi\cdot(\check{X}+B)$ is a process of integrable variation whose compensator is constant and therefore $\varphi\cdot(\check{X}+B)$ is a $\mathbb{Q}_2$-martingale of integrable variation, whence in particular a $\mathbb{Q}_2$-martingale. Therefore $\check{X}+B$ as well as S are $\mathbb{Q}_2$-sigma-martingales.

Summing up: We have proved proposition 4.5 under the additional hypothesis that S remains constant after the first time T when S jumps by at least 1 with respect to $\|\cdot\|_{\mathbb{R}^d}$.
Step 2: Now we drop this assumption and assume w.l.g. that $S_0=0$. Let $T_0=0, T_1=T$ and define inductively the stopping times

$$T_k=\inf\{t>T_{k-1}:\|\Delta S_t\|_{\mathbb{R}^d}\geq 1\},\qquad k=2,3,\ldots$$

so that $(T_k)_{k=1}^\infty$ increases to infinity. Let

$$S^{(k)}=\mathbf{1}_{]T_{k-1},T_k]}\cdot S\qquad k=1,2,\ldots.$$

Note that $S^{(1)}$ satisfies the assumptions of the first part of the proof, where we have shown that there is a measure $\mathbb{Q}_2\sim\mathbb{P}$, satisfying (i), (ii), (iii) above for $T=T_1$. Now repeat the above argument to choose inductively, for $k=2,3,\ldots$, measures $\mathbb{Q}_{k+1}\sim\mathbb{P}$ such that

(i) $\|\mathbb{Q}_{k+1}-\mathbb{Q}_k\|<\varepsilon/2^{k-1}\wedge\inf\{\frac{2^{-k}\mathbb{Q}_k[A]}{\mathbb{P}[A]}:A\in\mathcal{F},\mathbb{P}[A]\geq 2^{-k}\}$.
(ii) $\mathbb{Q}_{k+1}|_{\mathcal{F}_{T_k-}}=\mathbb{Q}_k|_{\mathcal{F}_{T_k-}}$ and $\frac{d\mathbb{Q}_{k+1}}{d\mathbb{Q}_k}$ is $\mathcal{F}_{T_k}$-measurable.
(iii) $S^{(k)}$ is a sigma-martingale under $\mathbb{Q}_{k+1}$.

The condition in (i) above is chosen such that we may apply lemma 4.3 to conclude that

$$\mathbb{Q}=\lim_{k\to\infty}\mathbb{Q}_k$$

exists and is equivalent to $\mathbb{P}$. From (ii) and (iii) it follows that each $S^{(k)}$ is a sigma-martingale under $\mathbb{Q}^l$, for each $l\geq k$. It follows that each $S^{(k)}$ is a $\mathbb{Q}$ sigma-martingale and hence S, being a local sigma-martingale is then a sigma-martingale. This proves the first part of proposition 4.5.

As regards the final assertion of proposition 4.5 note that, for any predictable stopping time U, the random times

$$U_k=\begin{cases}U & \text{if } T_{k-1}<U\leq T_k\\ \infty & \text{otherwise}\end{cases}$$

are predictable stopping times, for $k=1,2,\ldots$. Indeed, as easily seen, the set $\{T_k<U\leq T_{k+1}\}$ is in $\mathcal{F}_{U-}$, showing that U_k is predictable.
By our construction and property (ii) above we infer that, for $k=1,2,\ldots$,

$$\frac{d\mathbb{Q}_{|\mathcal{F}_{U_k}}}{d\mathbb{Q}_{1|\mathcal{F}_{U_k}}}=\frac{d\mathbb{Q}_{|\mathcal{F}_{(U_k)-}}}{d\mathbb{Q}_{1|\mathcal{F}_{(U_k)-}}}\qquad\text{a.s.}$$

which implies that

$$\frac{d\mathbb{Q}_{|\mathcal{F}_U}}{d\mathbb{Q}_{1|\mathcal{F}_U}}=\frac{d\mathbb{Q}_{|\mathcal{F}_{U-}}}{d\mathbb{Q}_{1|\mathcal{F}_{U-}}}\qquad\text{a.s.}$$

The proof of proposition 4.5 is complete now. $\square$

Proposition 4.5 contains the major part of the proof of the main theorem. The missing ingredient is still the argument for the predictable jumps of S. The argument for the predictable jumps given below will be similar to (but technically slightly easier than) the proof of proposition 4.5.

4.6 Proof of the Main Theorem. Let S be an $\mathbb{R}^d$-valued semi-martingale satisfying the assumption (NFLVR). By theorem 4.1 we may find a probability measure $\mathbb{Q}_1 \sim \mathbb{P}$ such that,

$$\mathbb{E}_{\mathbb{Q}_1}[f] \leq 0 \qquad \text{for } f \in C.$$

We also may find a sequence $(T_k)_{k=1}^{\infty}$ of predictable stopping times exhausting the accessible jumps of S, i.e., such that for each predictable stopping time T with $\mathbb{P}[T = T_k < \infty] = 0$, for each $k \in \mathbb{N}$, we have that $S_{T-} = S_T$ almost surely. We may and do assume that the stopping times $(T_k)_{k=1}^{\infty}$ are disjoint, i.e., that $\mathbb{P}[T_k = T_j < \infty] = 0$ for $k \neq j$.
Denote by D the predictable set

$$D = \bigcup_{k \geq 1} [\![T_k]\!] \subseteq \Omega \times \mathbb{R}_+$$

and split S into $S = S^a + S^i$, where

$$S^a = \mathbf{1}_D \cdot S \qquad S^i = \mathbf{1}_{(\Omega \times \mathbb{R}_+) \setminus D} \cdot S$$

where the letters "a" and "i" refer to "accessible" and "inaccessible". S^a and S^i are well-defined semimartingales and in view of the above construction S^i is quasi-left-continuous.
Denote by C^a and C^i the cones in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ associated by (4.1) to S^a and S^i, and observe that C^a and C^i are subsets of C (obtained by considering only integrands supported by D or $((\Omega \times \mathbb{R}_+) \setminus D$ respectively) hence

$$\mathbb{E}_{\mathbb{Q}_1}[f] \leq 0, \qquad \text{for } f \in C^a \text{ and for } f \in C^i.$$

Hence S^i satisfies the assumptions of proposition 4.5 with respect to the probability measure $\mathbb{Q}_1$ and we therefore may find a probability measure, now denoted by $\hat{\mathbb{Q}}, \hat{\mathbb{Q}} \sim \mathbb{P}$, which turns S^i into a sigma-martingale and such that, for each predictable stopping time T, we have

$$\frac{d\hat{\mathbb{Q}}_{|\mathcal{F}_T}}{d\mathbb{Q}_{1|\mathcal{F}_T}} = \frac{d\hat{\mathbb{Q}}_{|\mathcal{F}_{T-}}}{d\mathbb{Q}_{1|\mathcal{F}_{T-}}} \tag{4.3}$$

By assumption we have, for each $k = 1, 2, \ldots$, and for each admissible integrand H supported by $[\![T_k]\!]$, that

$$\mathbb{E}_{\mathbb{Q}_1}[(H \cdot S)_\infty] = \mathbb{E}_{\mathbb{Q}_1}\left[H_{T_k}(S_{T_k} - S_{(T_k)_-})\right] \leq 0.$$

Noting that the inequality remains true if we replace H by $H\mathbf{1}_A$, for any $\mathcal{F}_{(T_k)_-}$-measurable set A, and using (4.3) we obtain

$$\mathbb{E}_{\hat{\mathbb{Q}}}\left[(H\cdot S)_\infty\right] = \mathbb{E}_{\hat{\mathbb{Q}}}\left[H_{T_k}(S_{T_k} - S_{T_k-})\right] \le 0 \tag{4.4}$$

for each admissible integrand supported by $[\![T_k]\!]$.
We now shall proceed inductively on k: suppose we have chosen, for $k \ge 0$, probability measures $\tilde{\mathbb{Q}}_0 = \hat{\mathbb{Q}}, \tilde{\mathbb{Q}}_1, \dots, \tilde{\mathbb{Q}}_k$ such that

$$\mathbb{E}_{\tilde{\mathbb{Q}}_k}\left[S_{T_j}|\mathcal{F}_{T_j-}\right] = S_{T_j-} \qquad j = 1, \dots, k$$

and such that, for

$$\varepsilon_j = \varepsilon/2^{j+1} \wedge \inf\left\{\frac{2^{-j}\tilde{\mathbb{Q}}_j[A]}{\mathbb{P}[A]} : A \in \mathcal{F}, \mathbb{P}[A] \ge 2^{-j}\right\},$$

we have

$$\|\tilde{\mathbb{Q}}_j - \tilde{\mathbb{Q}}_{j+1}\| < \varepsilon_j \qquad j = 0, \dots, k-1.$$

In addition we assume that $\tilde{\mathbb{Q}}_j$ and $\tilde{\mathbb{Q}}_{j-1}$ agree "before T_j- and after T_j"; this means that $\tilde{\mathbb{Q}}_j$ and $\tilde{\mathbb{Q}}_{j-1}$ coincide on the σ-algebra $\mathcal{F}_{T_j-}$ and that the Radon-Nikodym derivative $d\tilde{\mathbb{Q}}_j/d\tilde{\mathbb{Q}}_{j-1}$ is $\mathcal{F}_{T_j}$-measurable.
Now consider the stopping time T_{k+1}: denote on the set $\{T_{k+1} < \infty\}$ by F_ω the jump measure of the jump $S^a_{T_{k+1}} - S^a_{T_{k+1}-}$ conditional on $\mathcal{F}_{T_{k+1}-}$. By (4.4) this $(\Omega, \mathcal{F}_{T_{k+1}-}, \mathbb{P})$-measurable family of probability measures on $\mathbb{R}^d$ satisfies the assumptions of lemma 3.5 and we therefore may find an $\mathcal{F}_{T_{k+1}-}$-measurable family of probability measures G_ω, a.s. defined on $\{T_{k+1} < \infty\}$, such that

(i) $F_\omega \sim G_\omega$ and $\|F_\omega - G_\omega\| < \varepsilon_k$
(ii) $\mathbb{E}_{G_\omega}\left[\|y\|_{\mathbb{R}^d}\right] < \infty$ and $\mathrm{bary}(G_\omega) = \mathbb{E}_{G_\omega}[y] = 0$.

Letting, similarly as in the proof of proposition 4.5 above,

$$Y(\omega, x) = \frac{dF_\omega}{dG_\omega}(x)$$

be a $\mathcal{F}_{T_{k+1}-} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable version of the Radon-Nikodym derivatives $\frac{dF_\omega}{dG_\omega}$ and defining

$$\frac{d\tilde{\mathbb{Q}}_{k+1}}{d\tilde{\mathbb{Q}}_k}(\omega) = \mathbf{1}_{\{T_{k+1}<\infty\}} Y(\omega, \Delta S_{T_{k+1}}(\omega)) + \mathbf{1}_{\{T_{k+1}=\infty\}}$$

we obtain a measure $\tilde{\mathbb{Q}}_{k+1} \sim \mathbb{P}, \|\tilde{\mathbb{Q}}_{k+1} - \tilde{\mathbb{Q}}_k\| < \varepsilon_k$, $\tilde{\mathbb{Q}}_{k+1}|_{\mathcal{F}_{(T_{k+1})-}} = \tilde{\mathbb{Q}}_k|_{\mathcal{F}_{(T_{k+1})-}}$ and $\frac{d\tilde{\mathbb{Q}}_{k+1}}{d\tilde{\mathbb{Q}}_k}$ being $\mathcal{F}_{T_{k+1}}$-measurable. For each $M \in \mathbb{R}_+$

$$\mathbf{1}_{[\![T_{k+1}]\!]\cap\{T_{k+1}<\infty \text{ and } \mathbb{E}_{G_\omega}[\|y\|]\le M\}} \cdot S = \mathbf{1}_{[\![T_{k+1}]\!]\cap\{T_{k+1}<\infty \text{ and } \mathbb{E}_{G_\omega}[\|y\|]\le M\}} \cdot S^a$$

is a martingale under $\tilde{\mathbb{Q}}_{k+1}$ and therefore

$$S^{(k+1)} := \mathbf{1}_{[\![T_{k+1}]\!]\cap\{T_{k+1}<\infty\}} \cdot S$$

is a sigma-martingale under $\tilde{\mathbb{Q}}_{k+1}$.

Letting $\mathbb{Q} = \lim_{k\to\infty} \tilde{\mathbb{Q}}_k$, each of the semi-martingales $S^{(k)} = \mathbf{1}_{[\![T_k]\!]} \cdot S$ is a $\mathbb{Q}$-sigma-martingale. It follows that

$$S^a = \sum_{k=1}^{\infty} S^{(k)}$$

is a $\mathbb{Q}$-sigma-martingale and therefore

$$S = S^a + S^i$$

is a $\mathbb{Q}$-sigma-martingale too.
The proof of the main theorem is complete now. $\square$

For later use, let us resume in the subsequent proposition what we have shown in the above proof.

4.7 Proposition. *Denote by $\mathbf{M}^e_s$ the set of probability measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that for admissible integrands, the process $H \cdot S$ becomes a supermartingale. More precisely*

$$\mathbf{M}^e_s = \{\mathbb{Q} \mid \mathbb{Q} \sim \mathbb{P} \text{ and for each } f \in C : \mathbb{E}_{\mathbb{Q}}[f] \leq 0\}.$$

If S satisfies $NFLVR$, then

$$\mathbf{M}^e_\sigma = \{\mathbb{Q} \mid S \text{ is a } Q \text{ sigma-martingale}\},$$

is dense in $\mathbf{M}^e_s$.

Theorem 4.8. *The set $\mathbf{M}^e_\sigma$ is a convex set.*

Proof. Let $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathbf{M}^e_\sigma$ and let ϕ_1, ϕ_2 be strictly positive real valued S integrable predictable processes, such that for $i = 1, 2$, $\phi_i \cdot S$ is an $\mathcal{H}^1(\mathbb{Q}_i)$-martingale. Take now $\phi = \min(\phi_1, \phi_2)$. Since $0 < \phi \leq \phi_1$, $\phi \cdot S$ is still an $\mathcal{H}^1(\mathbb{Q}_1)$ martingale. Similarly $\phi \cdot S$ is still an $\mathcal{H}^1(\mathbb{Q}_2)$ martingale. From this it follows that $\phi \cdot S$ is an $\mathcal{H}^1((\mathbb{Q}_1 + \mathbb{Q}_2)/2)$ martingale. $\square$

5. Duality Results and Maximal Elements

In this section we suppose without further notice that S is an $\mathbb{R}^d$-valued semi-martingale that satisfies the $NFLVR$ property, so that the set

$$\mathbf{M}^e_\sigma = \{\mathbb{Q} \mid \mathbb{Q} \sim \mathbb{P} \text{ and } S \text{ is a } \mathbb{Q} \text{ sigma- martingale}\}$$

is nonempty. We remark that when the price process S is locally bounded then the set $\mathbf{M}^e_\sigma$ coincides with the set as introduced in [DS94], i.e. the set of all equivalent local martingale measures for the process S.
In the case of locally bounded processes we showed the following duality equality, (see [De92], theorem 6.1 for the case of continuous processes, [DS94], theorem 5.7 for the case of bounded functions and [DS95], theorem 9 for the case of positive functions). The duality argument was used by El Karoui-Quenez [EKQ91]. For a nonnegative random variable g we have:

$$\sup_{\mathbb{Q}\in\mathbf{M}^e_\sigma} \mathbb{E}_{\mathbb{Q}}[g] = \inf\{\alpha \mid \text{ there is } H \text{ admissible and } g \leq \alpha + (H\cdot S)_\infty\}.$$

Using this equality we were able to derive a characterisation of maximal elements, see [DS95] corollary 14.
In the general case, i.e. when the process S is not necessarily locally bounded, the set of admissible integrands might be restricted to the zero integrand, compare proposition 3.6 above. Below we will show that also in this case the above equality remains valid, at least for positive random variables g. This result does not immediately follow from the results in [DS95].
Another approach to the problem is to enlarge the concept of admissible integrand in a similar way as was done in [S94] and [DS96]. Here the idea is to allow for integrands H that are such that the process $H \cdot S$ is controlled from below by an appropriate function w, the so-called w-admissible integrands. We will generalise the above duality equality to the setting of such integrands and we will see that even in the locally bounded case this generalisation yields some new results.
If we want to control a process $H \cdot S$ from below by a function w then, of course, the problem is that w cannot be too big, as this would allow doubling strategies and therefore arbitrage. Also w cannot be too small because this could imply that the only such integrand H is the zero integrand. This idea is made precise in the following definitions of w-admissible integrands and of feasible weight functions.

5.1 Definition. *If $w \geq 1$ is a random variable, if there is $\mathbb{Q}_0 \in \mathbf{M}^e_\sigma$ such that $\mathbb{E}_{\mathbb{Q}_0}[w] < \infty$, if a is a nonegative number, then we say that the integrand H is (a,w)-admissible if for each element $\mathbb{Q} \in \mathbf{M}^e_\sigma$ and each $t \geq 0$, we have $(H \cdot S)_t \geq -a\mathbb{E}_{\mathbb{Q}}[w \mid \mathcal{F}_t]$. We simply say that H is w-admissible if H is (a,w)-admissible for some nonegative a.*

Remark. If we put $w = 1$ we again find the usual concept of 1-admissible integrands. We required that $\operatorname{essinf} w > 0$ so that the admissible integrands become automatically w-admissible. The idea in fact is to allow unbounded functions w and therefore there seems to be no gain to introduce functions w that are too small. Requiring that $w \geq 1$ is by no means a restriction compared to the condition $\operatorname{essinf} w > 0$.

Remark. The present notion of admissible integrand is more suitable for our purposes than the one introcuded in [DS96].
The next lemma, based on a well known stability property of the set $\mathbf{M}^e_\sigma$, shows that in the inequality $(H \cdot S)_t \geq -\mathbb{E}_{\mathbb{Q}}[w \mid \mathcal{F}_t]$, it does not harm to restrict to elements $\mathbb{Q} \in \mathbf{M}^e_\sigma$ such that $\mathbb{E}_{\mathbb{Q}}[w] < \infty$.

5.2 Lemma. *Let $w \geq 0$ be such that $\mathbb{E}_{\mathbb{Q}_0}[w] < \infty$ for some $\mathbb{Q}_0 \in \mathbf{M}^e_\sigma$. Suppose that for some $\mathbb{Q} \in \mathbf{M}^e_\sigma$ and some real constant k the set $A = \{\mathbb{E}_{\mathbb{Q}}[w \mid \mathcal{F}_t] \leq k\}$ has positive probability, then there is $\mathbb{Q}_1 \in \mathbf{M}^e_\sigma$ such that on A we have $\mathbb{E}_{\mathbb{Q}_1}[w \mid \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[w \mid \mathcal{F}_t]$ and $\mathbb{E}_{\mathbb{Q}_1}[w] < \infty$.*

Proof. Let Z_s be a càdlàg version of the density process $Z_s = \mathbb{E}_{\mathbb{Q}_0}\left[\frac{d\mathbb{Q}}{d\mathbb{Q}_0} \mid \mathcal{F}_s\right]$. Now we put

$$
\begin{aligned}
Z^1_s &= 1 \quad \text{for } s < t \\
Z^1_s &= 1 \quad \text{for } s \geq t \text{ and } \omega \notin A \\
Z^1_s &= \frac{Z_s}{Z_t} \quad \text{for } s \geq t \text{ and } \omega \in A.
\end{aligned}
$$

Clearly the probability measure $\mathbb{Q}_1$ defined by $d\mathbb{Q}_1 = Z^1_\infty \, d\mathbb{Q}_0$ is in the set $\mathbf{M}^e_\sigma$ and satisfies the required properties. Indeed on the set A we have $\mathbb{E}_{\mathbb{Q}_1}[w \mid \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[w \mid \mathcal{F}_t]$ and $\mathbb{E}_{\mathbb{Q}_1}[w] \le \mathbb{E}_{\mathbb{Q}_0}[w] + k < \infty$. $\square$

In section 2, we recalled Émery's example showing that a stochastic integral with respect to a martingale is not always a local martingale. In [AS94] Ansel and Stricker gave necessary and sufficient conditions under which a stochastic integral with respect to a local martingale remains a local martingale. We rephrase part of their result in our context of sigma-martingales.

Theorem 5.3. *Let H be S integrable and w-admissible, then $H \cdot S$ is a local martingale (and hence also a super-martingale) for each $\mathbb{Q} \in \mathbf{M}^e_\sigma$ such that $\mathbb{E}_{\mathbb{Q}}[w] < \infty$.*

Proof. Simply write $H \cdot S$ as $(H\varphi^{-1}) \cdot (\varphi \cdot S)$, where the strictly positive predictable real valued process φ is such that $\varphi \cdot S$ is a $\mathcal{H}^1(\mathbb{Q})$ martingale. Then apply the Ansel-Stricker result. $\square$

Remark. The statement of the preceding theorem becomes false if we replace the condition $\mathbb{Q} \in \mathbf{M}^e_\sigma$ by $\mathbb{Q} \in \mathbf{M}^e_s$. To see this, take the process S defined as $S_t = 0$ for $t \le 1$ and $S_t = S_1$, a one dimensional normal variable, for $t \ge 1$. The filtration is simply the filtration generated by S. As there are no admissible integrands, every equivalent probability measure is in $\mathbb{Q} \in \mathbf{M}^e_s$. But it is clearly false that S becomes a $\mathbb{Q}$-supermartingale (i.e. $\mathbb{E}_{\mathbb{Q}}[S_1] \le 0$) as soon as $\mathbb{E}_{\mathbb{Q}}[|S_1|] < \infty$.

5.4 Definition. *A random variable $w\colon \Omega \to \mathbb{R}_+$ such that $w \ge 1$ is called a feasible weight function for the process S, if*

(1) *there is a strictly positve bounded predictable process φ such that the maximal function of the $\mathbb{R}^d$-valued stochastic integral $\varphi \cdot S$ satisfies $(\varphi \cdot S)^* \le w$.*
(2) *there is an element $\mathbb{Q} \in \mathbf{M}^e_\sigma$ such that $\mathbb{E}_{\mathbb{Q}}[w] < \infty$.*

Remark. For feasible weight functions w, it might happen that for some elements $\mathbb{Q} \in \mathbf{M}^e_\sigma$ we have that $\mathbb{E}_{\mathbb{Q}}[w] = \infty$, see the example 5.14 below.
If no confusion can arise to which process the feasibility condition refers, then we will simply say that the weight function is feasible. The first item in the definition requires that w is big enough in order to allow non-trivial integrands H such that both H and $-H$ are w-admissible. The second item requires w to be not too big and as we will see this will avoid arbitrage opportunities. It follows from proposition 2.6, that because $\mathbf{M}^e_\sigma$ is nonempty, the existence of feasible weight functions is guaranteed. For locally bounded processes S, a function $w \ge 1$ is feasible as soon as there is $\mathbb{Q} \in \mathbf{M}^e_\sigma$ with $\mathbb{E}_{\mathbb{Q}}[w] < \infty$.
We can now state the generalisations of the duality theorem mentioned above.

5.5 Theorem. *If w is a feasible weight function and g is a random variable such that $g \ge -w$ then:*

$$\sup_{\mathbb{Q} \in \mathbf{M}^e_\sigma; \mathbb{E}_{\mathbb{Q}}[w] < \infty} \mathbb{E}_{\mathbb{Q}}[g] = \inf \{\alpha \mid \textit{ there is } H \;\; w-\textit{admissible and } g \le \alpha + (H \cdot S)_\infty\}.$$

If the quantities are finite then the infimum is a minimum.

Remark. The reader can see that even in the case of locally bounded processes S the result yields more precise information. Indeed we restrict the supremum to those measures $\mathbb{Q} \in \mathbf{M}^e_\sigma$ such that $\mathbb{E}_{\mathbb{Q}}[w] < \infty$.

For a feasible weight function w, we denote by $\mathcal{K}_w$ the set

$$\mathcal{K}_w = \{(H \cdot S)_\infty \mid H \text{ is } w\text{-admissible}\}.$$

5.6 Definition. *An element $g \in \mathcal{K}_w$ is called maximal if $h \in \mathcal{K}_w$ and $h \geq g$ imply that $h = g$.*

The maximal elements in this set are then characterised as follows:

5.7 Theorem. *If $w \geq 1$ is a feasible weight function, if H is w-admissible and if $h = (H \cdot S)_\infty$, then the following are equivalent:*

(1) *h is maximal*
(2) *there is $\mathbb{Q} \in \mathbf{M}_\sigma^e$ such that $\mathbb{E}_\mathbb{Q}[w] < \infty$ and $\mathbb{E}_\mathbb{Q}[h] = 0$*
(3) *there is $\mathbb{Q} \in \mathbf{M}_\sigma^e$ such that $\mathbb{E}_\mathbb{Q}[w] < \infty$ and $H \cdot S$ is $\mathbb{Q}$-uniformly integrable martingale.*

In the proof of these results we will make frequent use of Theorem D and corollary 4.12 of [DS96]. These two results were proved for the slightly more restrictive notion of admissibility, but the reader can go through the proofs and check that the results remain valid for the present notion of w-admissible integrands. Indeed the lower bound $H \cdot S \geq -w$ is only used to control the negative parts of the possible jumps in the stochastic integral. This can also be achieved by the inequality $H \cdot S \geq -\mathbb{E}_\mathbb{Q}[w \mid F_t]$ where $\mathbb{E}_\mathbb{Q}[w] < \infty$. Compare the formulation of Theorems B and C in [DS96]. For the convenience of the reader let us rephrase the results of [DS96] in the present setting.

Theorem D. *Let $\mathbb{Q}$ be a probability measure, equivalent to $\mathbb{P}$. Let M be an $\mathbb{R}^d$-valued $\mathbb{Q}$-local martingale and $w \geq 1$ a $\mathbb{Q}$-integrable function.*
Given a sequence $(H^n)_{n\geq 1}$ of M-integrable $\mathbb{R}^d$-valued predictable processes such that

$$(H^n \cdot M)_t \geq -\mathbb{E}_\mathbb{Q}[w \mid \mathcal{F}_t], \qquad \text{for all } n, t,$$

then there are convex combinations

$$K^n \in conv\{H^n, H^{n+1}, \dots\},$$

and there is a super-martingale $(V_t)_{t\in\mathbb{R}_+}, V_0 \leq 0$, such that

$$\lim_{\substack{s \to t, s\in\mathbb{Q}_+ \\ >}} \lim_{n\to\infty} (K^n \cdot M)_s = V_t \qquad \text{for } t \in \mathbb{R}_+, \text{ a.s.}, \tag{i}$$

and an M-integrable predictable process H^0 such that

$$((H^0 \cdot M)_t - V_t)_{t\in\mathbb{R}_+} \quad \text{is increasing.} \tag{ii}$$

In addition, $H^0 \cdot M$ is a local martingale and a super-martingale.

Corollary 4.12. *Let S be an $\mathbb{R}^d$- valued semi-martingale such that $\mathbf{M}_\sigma^e(S) \neq \emptyset$ and $w \geq 1$ a weight function such that there is some $\mathbb{Q} \in \mathbf{M}_\sigma^e(S)$ with $\mathbb{E}_\mathbb{Q}[w] < \infty$. Then the convex cone*

$$\{g \mid \text{ there is a } (1, w) \text{ admissible integrand } H \text{ such that } g \leq (H \cdot S)_\infty\}$$

is closed in $L^0(\Omega)$ with respect to the topology of convergence in measure.

Let $w \geq 1$ be such that there is $\mathbb{Q} \in \mathbf{M}^e_\sigma$, with $\mathbb{E}_\mathbb{Q}[w] < \infty$. The set

$$\mathcal{K}_w = \{(H \cdot S)_\infty \mid H \text{ is } w\text{-admissible}\}$$

is a cone in the space of measurable functions L^0. As in [DS94] we need the cone of all elements that are dominated by outcomes of w-admissible integrands:

$$\mathcal{C}^0_w = \{g \mid g \leq (H \cdot S)_\infty \text{ where, } H \text{ is } w\text{-admissible}\}.$$

If H is w-admissible and $\mathbb{E}_\mathbb{Q}[w] < \infty$ for some $\mathbb{Q} \in \mathbf{M}^e_\sigma$, then it follows from the results in [AS92] that the process $H \cdot S$ is a $\mathbb{Q}$-supermartingale. Therefore the limit $(H \cdot S)_\infty$ exists and $\mathbb{E}_\mathbb{Q}[(H \cdot S)_\infty] \leq 0$. It also follows that for elements $g \in \mathcal{C}^0_w$, we have that $-\infty \leq \mathbb{E}_\mathbb{Q}[g] \leq 0$. We will use this result frequently. We also remark that if H is w-admissible and if $(H \cdot S)_\infty \geq -w$ then H is already $(1, w)$ admissible. Indeed because of the supermartingale property of $H \cdot S$ we have that (at least for those $\mathbb{Q} \in \mathbf{M}^e_\sigma$ such that $\mathbb{E}_\mathbb{Q}[w] < \infty$):

$$(H \cdot S)_t \geq \mathbb{E}_\mathbb{Q}[(H \cdot S)_\infty \mid \mathcal{F}_t] \geq \mathbb{E}_\mathbb{Q}[-w \mid \mathcal{F}_t].$$

By lemma 5.2 this means that H is $(1, w)$-admissible.

5.8 Theorem. *If $w \geq 1$ and if there is some $\mathbb{Q} \in \mathbf{M}^e_\sigma$ is such that $\mathbb{E}_\mathbb{Q}[w] < \infty$, then*

$$\mathcal{C}^\infty_w = \left\{h \mid h \in L^\infty \text{ and } hw \in \mathcal{C}^0_w\right\}$$

is weak closed in $L^\infty(\mathbb{Q})$.*

Proof. This is just a reformulation of corollary 4.12 cited above. □

We now prove the duality result stated in theorem 5.5. The proof is broken up into several lemmata. As we will work with functions $w \geq 1$ that are not necessarily feasible weight functions we will make use of a larger class of equivalent measures namely:

$$\mathbf{M}^e_{s,w} = \{\mathbb{Q} \sim \mathbb{P} \mid \mathbb{E}_\mathbb{Q}[w] < \infty \text{ and for each } h \in \mathcal{C}_w : \mathbb{E}_\mathbb{Q}[h] \leq 0\}.$$

The reader can check that $\mathbf{M}^e_{s,w}$ is the set of equivalent probability measures so that w is integrable and with the property that for a w-admissible integrand H, the process $H \cdot S$ is a supermartingale. When we work with admissible integrands, i.e. with w identically equal to 1, then we simply drop, as in proposition 4.7, the subscript w.

5.9 Lemma. *If $w \geq 1$ has a finite expectation for at least one element $\mathbb{Q} \in \mathbf{M}^e_\sigma$, if g is a random variable such that $g \geq -w$ then:*

$$\sup_{\mathbb{Q} \in \mathbf{M}^e_{s,w}\,;\,\mathbb{E}_\mathbb{Q}[w] < \infty} \mathbb{E}_\mathbb{Q}[g] \leq \inf\{\alpha \mid \text{ there is } H \;\; w-\text{admissible and } g \leq \alpha + (H \cdot S)_\infty\}.$$

Proof. The proof follows the same lines as the proof of theorem 9 in [DS95]. If $w \geq 1$ and $\mathbb{Q} \in \mathbf{M}^e_{s,w}$ then as observed above, the process $H \cdot S$ is a $\mathbb{Q}$-super martingale for each that H is w-admissible. Therefore the inequality $g \leq \alpha + (H \cdot S)_\infty$ implies that $\mathbb{E}_\mathbb{Q}[g] \leq \alpha$. □

Remark. If, under the hypothesis of the theorem, $\sup_{\mathbb{Q} \in \mathbf{M}^e_\sigma; \mathbb{E}_\mathbb{Q}[w] < \infty} \mathbb{E}_\mathbb{Q}[g] = \infty$, then also $\inf\{\alpha \mid \text{ there is } H \;\; w-\text{admissible and } g \leq \alpha + (H \cdot S)_\infty\} = \infty$. This simply means that no matter how big the constant a is taken, there is no w-admissible integrand H such that $g \leq a + (H \cdot S)_\infty$.

5.10 Lemma. *If $w \geq 1$, if for some $\mathbb{Q} \in \mathbf{M}^e_\sigma$ we have $\mathbb{E}_{\mathbb{Q}_0}[w] < \infty$, if $\frac{g}{w}$ is bounded, if*

$$\beta < \inf \{\alpha \mid \textit{ there is } H \;\; w - \textit{admissible and } g \leq \alpha + (H \cdot S)_\infty\},$$

then there is a probability measure $\mathbb{Q} \in \mathbf{M}^e_{s,w}$ such that $\mathbb{E}_{\mathbb{Q}}[g] > \beta$.

Proof. The hypothesis on β means that:

$$\left(\frac{g-\beta}{w} + L^\infty_+\right) \cap \mathcal{C}^\infty_w = \{0\}.$$

Because the set $\mathcal{C}^\infty_w$ is weak* closed, we can apply Yan's separation theorem [Y80] and we obtain a strictly positive measure μ, equivalent to $\mathbb{P}$ such that

(1) $\mathbb{E}_\mu\left[\frac{g-\beta}{w}\right] > 0$
(2) for all $h \in \mathcal{C}^\infty_w$ we have $\mathbb{E}_\mu[h] \leq 0$.

If we normalize μ so that the measure $\mathbb{Q}$ defined as $d\mathbb{Q} = \frac{1}{w} d\mu$ becomes a probability measure, then we find that

(1) $\mathbb{Q} \sim \mathbb{P}$ and $\mathbb{E}_{\mathbb{Q}}[w] < \infty$,
(2) $\mathbb{E}_{\mathbb{Q}}[g] > \beta$,
(3) for all $h \in \mathcal{C}^\infty_w$ we have that $\mathbb{E}_{\mathbb{Q}}[hw] \leq 0$.

The latter inequality together with the Beppo-Levi theorem then implies that for each w-admissible integrand H we have that $\mathbb{E}_{\mathbb{Q}}[(H \cdot S)_\infty] \leq 0$. $\square$

5.11 Lemma. *If $w \geq 1$, if some $\mathbb{Q}_0 \in \mathbf{M}^e_\sigma$ we have $\mathbb{E}_{\mathbb{Q}_0}[w] < \infty$, if $g \geq -w$ then*

$$\sup_{\mathbb{Q} \in \mathbf{M}^e_{s,w}} \mathbb{E}_{\mathbb{Q}}[g] \geq \inf \{\alpha \mid \textit{ there is } H \;\; w - \textit{admissible and } g \leq \alpha + (H \cdot S)_\infty\}.$$

Moreover if the quantity on the right hand side is finite then the infimum is a minimum.

Proof. For each $n \geq 1$, we have that $\frac{g \wedge n}{w}$ is bounded and hence we can apply the previous lemma. This tells us that, for each $n \in \mathbb{N}$,

$$\begin{aligned}\alpha_n &= \sup\{\mathbb{E}_{\mathbb{Q}}[g \wedge n] \mid \mathbb{Q} \in \mathbf{M}^e_{s,w}\} \\ &\geq \inf\{\alpha \mid \text{ there is } H \;\; w - \text{admissible and } g \wedge n \leq \alpha + (H \cdot S)_\infty\}.\end{aligned}$$

Because there is nothing to prove when $\lim_n \alpha_n = \infty$ we may suppose that $\sup_n \alpha_n = \lim_n \alpha_n = \alpha < \infty$. So, for each n, we take a w-admissible integrand H^n such that $g \wedge n \leq \alpha_n + \frac{1}{n} + (H^n \cdot S)_\infty$. Let us now fix $\mathbb{Q}_0 \in \mathbf{M}^e_\sigma$ such that $\mathbb{E}_{\mathbb{Q}_0}[w] < \infty$. From Theorem D in [DS96], cited above, we deduce the existence of $K^n \in conv\{H^n, H^{n+1}, \dots\}$ as well as H^0, such that

(1) $V_t = \lim_{s \underset{>}{\to} t, s \in \mathbb{Q}_+} \lim_{n \to \infty} (K^n \cdot S)_s$ exists a.s., for all $t \geq 0$,
(2) $(H^0 \cdot S)_t - V_t$ is increasing,
(3) $V_0 \leq 0$.

From this it follows that $(H^0 \cdot S)_t \geq -V_0 + V_t \geq V_t$. Since H^n is w-admissible (and hence $(1, w)$-admissible) we have that K^n is $(1, w)$-admissible and hence we find that $V_t \geq -\mathbb{E}_{\mathbb{Q}}[w \mid \mathcal{F}_t]$ for all $\mathbb{Q} \in \mathbf{M}^e_\sigma$ such that $\mathbb{E}_{\mathbb{Q}}[w] < \infty$. It is now clear that

H^0 is w-admissible. Since the sequence α_n is increasing we also obtain that for all t and all $\mathbb{Q} \in \mathbf{M}^e$ with $\mathbb{E}_{\mathbb{Q}}[w] < \infty$:

$$(K^n \cdot S)_t + \alpha_n + \frac{1}{n} \geq \mathbb{E}_{\mathbb{Q}}\left[(K^n \cdot S)_\infty \mid \mathcal{F}_t\right] + \alpha_n + \frac{1}{n} \geq \mathbb{E}_{\mathbb{Q}}\left[g \wedge n \mid \mathcal{F}_t\right].$$

This yields that for all t and all n

$$(H^0 \cdot S)_t + \alpha_n + \frac{1}{n} \geq V_t + \alpha_n + \frac{1}{n} \geq \mathbb{E}_{\mathbb{Q}}\left[g \wedge n \mid \mathcal{F}_t\right].$$

If t tends to infinity this gives $(H^0 \cdot S)_\infty + \alpha_n + \frac{1}{n} \geq g \wedge n$ for all n. By taking the limit over n we finally find that

$$(H^0 \cdot S)_\infty + \alpha \geq g.$$

This shows the desired inequality and at the same time also shows that the infimum is a minimum. □

We are now ready to prove the duality results. We start with the case of admissible integrands thus extending theorem 9 of [DS95] to the case of non locally bounded processes S. Recall that we assume throughout this section that S is an $\mathbb{R}^d$ valued semi-martingale satisfying $(NFLVR)$.

5.12 Theorem. *For a nonnegative random variable g we have:*

$$\sup_{\mathbb{Q} \in \mathbf{M}^e_\sigma} \mathbb{E}_{\mathbb{Q}}[g] = \inf\left\{\alpha \mid \text{ there is } H \text{ admissible and } g \leq \alpha + (H \cdot S)_\infty\right\}.$$

Proof. From the previous lemmata it follows that we only have to show that

$$\sup_{\mathbb{Q} \in \mathbf{M}^e_\sigma} \mathbb{E}_{\mathbb{Q}}[g] = \sup_{\mathbb{Q} \in \mathbf{M}^e_s} \mathbb{E}_{\mathbb{Q}}[g].$$

This follows from proposition 4.7 and the fact that g is bounded from below. □

We now complete the proof for the case of feasible weight functions w and w-admissible integrands:

Proof of Theorem 5.5. In this case we show that $\mathbf{M}^e_\sigma = \mathbf{M}^e_{s,w}$. We already observed that $\mathbf{M}^e_\sigma \subset \mathbf{M}^e_{s,w}$. Take now $\mathbb{Q} \in \mathbf{M}^e_{s,w}$.
Since w is now supposed to be a feasible weight function, we have the existence of a strictly positive predictable function φ such that $(\varphi \cdot S)^* \leq w$. It follows that outcomes of the form $\mathbf{1}_A\left(\varphi \cdot S^i_t - \varphi \cdot S^i_s\right)$ or $-\mathbf{1}_A\left(\varphi \cdot S^i_t - \varphi \cdot S^i_s\right)$, where $s < t$, $A \in \mathcal{F}_s$ and $(S^i)_{i=1\ldots d}$ are the coordinates of S, are outcomes of w-admissible integrands. Therefore $\varphi \cdot S$ is a $\mathbb{Q}$ martingale and $\mathbb{Q} \in \mathbf{M}^e_\sigma$. □

5.13 Corollary. *If $w \geq 1$ is a feasible weight function then the set*

$$\{\mathbb{Q} \mid \mathbb{Q} \in \mathbf{M}^e_\sigma,\ \mathbb{E}_{\mathbb{Q}}[w] < \infty\}$$

is dense in $\mathbf{M}^e_\sigma$ for the variation norm.

Proof. If the set would not be dense then by the Hahn Banach theorem, there exists $\mathbb{Q}_0 \in \mathbf{M}^e_\sigma$ and a bounded function g such that

$$\mathbb{E}_{Q_0}[g] > \sup\{\mathbb{E}_{\mathbb{Q}}[g] \mid \mathbb{Q} \in \mathbf{M}^e_\sigma,\ \mathbb{E}_{\mathbb{Q}}[w] < \infty\} = \alpha.$$

This, together with Theorem 5.5, would then imply

$$\begin{aligned}
\alpha_0 &= \inf\{\alpha \mid \text{ there is } H \text{ admissible and } g \leq \alpha + (H \cdot S)_\infty\} \\
&= \sup_{\mathbb{Q}\in\mathbf{M}^e_\sigma} \mathbb{E}_\mathbb{Q}[g] \\
&> \sup_{\mathbb{Q}\in\mathbf{M}^e_\sigma;\mathbb{E}_\mathbb{Q}[w]<\infty} \mathbb{E}_\mathbb{Q}[g] \\
&= \inf\{\alpha \mid \text{ there is } H \text{ } w\text{-admissible and } g \leq \alpha + (H \cdot S)_\infty\}.
\end{aligned}$$

But a w-admissible integrand H such that $(H \cdot S)_\infty \alpha \geq g$ is already admissible, proving that the strict inequality cannoy hold. Indeed the process $H \cdot S$ is a $\mathbb{Q}$ supermartingale for each element $\mathbb{Q} \in \mathbf{M}^e_\sigma$ such that $\mathbb{E}_\mathbb{Q}[w] < \infty$. Therefore the process $H \cdot S$ is bounded below by $-\alpha - \|g\|_\infty$ and this means that H is admissible.

Remark. An interesting question is whether by taking the supremum in theorem 5.5, we have, for general unvbounded functions g, to restrict to those elements $\mathbb{Q} \in \mathbf{M}^e_\sigma$ such that for the feasible weight function w we have $\mathbb{E}_\mathbb{Q}[w] < \infty$. More precisely is there a contingent claim $g \geq -w$ such that

$$\sup_{\mathbb{Q}\in\mathbf{M}^e_\sigma} \mathbb{E}_\mathbb{Q}[g] > \sup_{\mathbb{Q}\in\mathbf{M}^e_\sigma;\mathbb{E}_\mathbb{Q}[w]<\infty} \mathbb{E}_\mathbb{Q}[g].$$

An inspection of the proof of the above theorem shows that we used the $\mathbb{Q}$-intgerability of the feasible weight function w in order to conclude that the w-admissible integrand H defined a $\mathbb{Q}$-supermartingale $H \cdot S$.
The next example, however, shows that it might happen that, for some sigma-martingale measure, $H \cdot S$ is a supermartingale, while for other sigma-martingale measures, it fails to be so.

5.14 Example. *There is a continuous process S, $S_0 = 0$, satisfying $NFLVR$ and such that*

1. $\mathbb{P} \in \mathbf{M}^e_\sigma$,
2. *S is a $\mathbb{P}$ uniformly integrable martingale*
3. *S is bounded above and hence for all $\mathbb{Q} \in \mathbf{M}^e_\sigma$ the process S is a submartingale*
4. *for some $\mathbb{Q} \in \mathbf{M}^e_\sigma$, the process S is not a $\mathbb{Q}$ supermartingale*

This example will then, as we will see, also solve negatively the question whether the two suprema are the same.
The example is based on [DS97]. There we gave an example of a one dimensional, continuous, strictly positive price process X, $X_0 = 1$, such that the set $\mathbf{M}^e_\sigma$ is nonempty, $\mathbb{P} \in \mathbf{M}^e_\sigma$ and X is a $\mathbb{P}$-uniformly integrable martingale, whereas for some other element $\mathbb{Q} \in \mathbf{M}^e_\sigma$ we have that $\mathbb{E}_\mathbb{Q}[X_\infty] < 1$. We now take the following elements $w = |X_\infty| + 1$ and $S = 1 - X$. Clearly $\mathbb{E}_\mathbb{P}[w] < \infty$ and since X is continuous, we can find a predictable, strictly positive process φ such that $\varphi \cdot S$ remains bounded by 1. It follows that w is feasible. Clearly the process S is then the gains process of the w-admissible integrand $H = 1$. If we put $g = S_\infty$ we trivially have that $\mathbb{E}_\mathbb{Q}[g] > 0$. This shows the following two assertions:

1. the process S is a not a $\mathbb{Q}$ supermartingale, however, it is a $\mathbb{P}$ uniformly integrable martingale and a $\mathbb{Q}$ local martingale.
2. $\sup_{\mathbb{Q}\in\mathbf{M}^e_\sigma} \mathbb{E}_\mathbb{Q}[g] > \sup_{\mathbb{Q}\in\mathbf{M}^e;\mathbb{E}_\mathbb{Q}[w]<\infty} \mathbb{E}_\mathbb{Q}[g] = 0.$ □

We now turn to the characterisation of maximal and of attainable elements. The approach is different from the one used in [DS95], which was based on a change of numéraire technique. In order not to overload the statements we henceforth suppose that w is a feasible weight function.

5.15 Lemma. *If $g \in \mathcal{K}_w$, then there is a maximal element $h \in \mathcal{K}_w$ such that $h \geq g$.*

Proof. It is sufficient to show that every increasing sequence in $\mathcal{K}_w$ has an upper bound in $\mathcal{K}_w$. So let h_n, $h_1 = g$, be an increasing sequence in $\mathcal{K}_w$. For each n take H^n, w-admissible so that $h_n = (H^n \cdot S)_\infty$. As in the previous proof we then find, as an application of Theorem D in [DS96], that there is H^0, w-admissible such that $(H^0 \cdot S)_\infty \geq \lim_n h_n$. This concludes the proof of the lemma. □

Proof of Theorem 5.8. If $\mathbb{E}_\mathbb{Q}[w] < \infty$ then $H \cdot S$ is a $\mathbb{Q}$ supermartingale and hence (2) and (3) are equivalent. Also it is clear that (2) implies (1). Indeed if g is the result of a w-admisible integrand then $\mathbb{E}_\mathbb{Q}[g] \leq 0$ for each $\mathbb{Q} \in \mathbf{M}^e_\sigma$ such that also $\mathbb{E}_\mathbb{Q}[w] < \infty$. It follows that h is necessarily maximal.
The only remaining part is that (1) implies (2). Since always $\mathbb{E}_\mathbb{Q}[h] \leq 0$ for $\mathbb{Q} \in \mathbf{M}^e_\sigma$ such that also $\mathbb{E}_\mathbb{Q}[w] < \infty$, we obtain already that for measures $\mathbb{Q}$ satisfying these assumptions, h^+ is $\mathbb{Q}$-integrable. So fix such a measure $\mathbb{Q}$. Now let $w_1 = h^+ + w$. Clearly w_1 is a feasible weight function. We will work with the set $\mathcal{K}_{w_1}$. The problem is, however, that we do not (yet) know that h is still maximal in the bigger cone $\mathcal{K}_{w_1}$. From the construction of w_1 it follows that for elements $\mathbb{Q} \in \mathbf{M}^e_\sigma$, $\mathbb{E}_\mathbb{Q}[w_1] < \infty$ if and only if $\mathbb{E}_\mathbb{Q}[w] < \infty$. Now let $g \geq h$ be the result of a w_1-admissible integrand. Hence $g = (K \cdot S)_\infty$ where K is w_1-admissible. Since $(K \cdot S)_\infty \geq g \geq h \geq -w$ and since K is w_1-admissible we have that K is already w-admissible. (Remember that $\mathbb{E}_\mathbb{Q}[w_1] < \infty$ if and only if $\mathbb{E}_\mathbb{Q}[w] < \infty$) From the maximality of h in $\mathcal{K}_w$ it then follows that $g = h$, i.e. h is maximal in $\mathcal{K}_{w_1}$. This can then be translated into

$$\left(\frac{h}{w_1} + L^\infty_+\right) \cap \mathcal{C}^\infty_{w_1} = \{0\}.$$

Using Yan's separation theorem in the same way as in the proof of theorem 5.5 above, we find a measure $\mathbb{Q}_1$ such that $\mathbb{E}_{\mathbb{Q}_1}[w_1] < \infty$, $\mathbb{Q}_1 \in \mathbf{M}^e_\sigma$ and $\mathbb{E}_{\mathbb{Q}_1}[h] \geq 0$. □

The following theorem generalises a result due to Ansel-Stricker and Jacka, [Jk92] and [AS94].

5.16 Theorem. *Let w be a feasible weight function and let $f \geq -w$. Then are equivalent*

(1) *there is a measure $\mathbb{Q} \in \mathbf{M}^e_\sigma$ such that $\mathbb{E}_\mathbb{Q}[w] < \infty$ and such that*

$$\mathbb{E}_\mathbb{Q}[f] = \sup \mathbb{E}_{R \in \mathbf{M}^e_\sigma; \mathbb{E}_R[w] < \infty}[f] < \infty$$

(2) *f can be hedged, i.e. there is $\alpha \in \mathbb{R}$, $\mathbb{Q} \in \mathbf{M}^e_\sigma$ such that $\mathbb{E}_\mathbb{Q}[w] < \infty$, a w-admissible integrand H such that $H \cdot S$ is a $\mathbb{Q}$ uniformly integrable martingale, and such that $f = \alpha + (H \cdot S)_\infty$.*

Proof. Clearly (2) implies (1) by the previous theorem.
For the reverse implication take now $\mathbb{Q}$ as in (1), then the duality result gives a $\alpha \in \mathbb{R}$ as well as a w-admissible integrand H such that $f \leq \alpha + (H \cdot S)_\infty$, where

$\alpha = \sup_{R \in \mathbf{M}^e ; \mathbb{E}_R[w] < \infty} [f]$. Here we use explicitly that the infimum in the duality theorem is a minimum. But then it follows from $\mathbb{E}_{\mathbb{Q}}[w] < \infty$ and from the equality $\mathbb{E}_{\mathbb{Q}}[f] = \alpha$ that $f = \alpha + (H \cdot S)_\infty$ and that $H \cdot S$ is a $\mathbb{Q}$ uniformly integrable martingale. □

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DEPARTEMENT FÜR MATHEMATIK, ETH ZÜRICH, CH-8092 ZÜRICH, SWITZERLAND
E-mail address: `delbaen@math.ethz.ch`

INSTITUT FÜR STATISTIK, UNIVERSITÄT WIEN, BRÜNNERSTR. 72, A-1210 WIEN, AUSTRIA.
E-mail address: `wschach@stat1.bwl.univie.ac.at`