IS THERE A PREDICTABLE CRITERION FOR MUTUAL SINGULARITY OF TWO PROBABILITY MEASURES ON A FILTERED SPACE?

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ABSTRACT. The theme of providing predictable criteria for absolute continuity and for mutual singularity of two density processes on a filtered probability space is extensively studied, e.g., in the monograph by J. Jacod and A. N. Shiryaev [JS]. While the issue of absolute continuity is settled there in full generality, for the issue of mutual singularity one technical difficulty remained open ([JS], p210): "We do not know whether it is possible to derive a *predictable* criterion (necessary and sufficient condition) for $P'_T \perp P_T, \ldots$". It turns out that to this question raised in [JS] which we also chose as the title of this note, there are two answers: on the negative side we give an easy example, showing that in general the answer is no, even when we use a rather wide interpretation of the concept of "predictable criterion". The difficulty comes from the fact that the density process of a probability measure P with respect to another measure P' may suddenly jump to zero.

On the positive side we can characterize the set, where P' becomes singular with respect to P — provided this does not happen in a sudden but rather in a continuous way — as the set where the Hellinger process diverges, which certainly is a "predictable criterion". This theorem extends results in the book of J. Jacod and A. N. Shiryaev [JS].

1. INTRODUCTION

We adopt the notation of [JS], which means that we are given two fixed probability measures P, P' on a filtered space $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathcal{F})$ with right continuous filtration and $\mathcal{F} = \bigvee_t \mathcal{F}_t$, $Q = \frac{P+P'}{2}$, z and z' denote the density processes of P and P', relative to Q. We define the process $(Y_t)_{t\geq 0}$ by $Y_t = \sqrt{z_t z'_t}$ and let $h = h(\frac{1}{2})$ denote the Hellinger process of order $\frac{1}{2}$, i. e., the predictable increasing process h such that $h_0 = 0$ and

$$M = Y + Y_- \bullet h \tag{1}$$

is a Q-martingale. For a stopping time T we denote by P_T and P'_T the restrictions of P and P' to $\mathcal{F}_T$. We shall investigate the following question:

1.1 Problem. *Under which conditions can we assert that $P'_T \ll P_T$ or $P'_T \perp P_T$? More generally, can we find a Hahn-decomposition of Ω into two sets, such that P'_T is absolutely continuous (resp. singular) with respect to P_T on these sets?*

The answer to this question should be in terms of a "predictable criterion". By this concept we mean that the answer should be in terms of the values of a predictable process, such as the Hellinger process h, evaluated at time T.

Before proceeding to answering this question we pause for some remarks: In order to avoid irrelevant complications at $t = 0$ we suppose that $P_0 \sim P'_0$. We define the stopping time S as the first moment when either z or z' vanishes,

$$S = \inf\{t : z_t = 0 \text{ or } z'_t = 0\}. \tag{2}$$

Noting that (1) determines the Hellinger process h only up to time S, we define h to be constant after S, i. e., we consider the "Hellinger process in the strict sense" in the terminology of [JS]. We also introduce the Hellinger process $h(0)_t$ of order 0 as the compensator of the process $\mathbb{1}_{[\![S,\infty[\![}$.

Now we review the known results: there is a very satisfactory answer to our problem as regards the question of absolute continuity:

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1.2 Theorem. ([JS], Thm IV 2.6) *In the above setting we have, for every stopping time T,*

$$P'_T \ll P_T \Leftrightarrow P'(G_T) = 1,$$

where $G_T = \{h_T < \infty\} \cap \{h(0)_T = 0\}$.

Note that the set G_T is defined in terms of the values of the two predictable processes h and $h(0)$ at time T. To compare this result with the situation of mutual singularity below we make the following (trivial) reformulation: denote by H the predictable, increasing, $[0, \infty]$-valued process $H_t = h_t + \infty \cdot h(0)_t$ (where $\infty \cdot 0 = 0$). Then

$$P'_T \ll P_T \Leftrightarrow P'(H_T < \infty) = 1,$$

For the question of mutual singularity (or, more generally, for the question of the Hahn-decomposition) the situation is more subtle, the difficulty arising from the fact that z_t may jump to zero as we shall presently see. Of course, we can always get rid of this difficulty by shifting it into the assumptions of a theorem: we thus obtain the subsequent result which directly follows from ([JS], lemma IV 2.12 a), b) and d)):

1.3 Theorem. *In the above setting suppose in addition that*

$$P'_T \left[S \leq T \text{ and } z_{S-} > z_S = 0 \right] = 0.$$

Then the restriction of P'_T to $\{h_T < \infty\}$ is absolutely continuous with respect to P_T while the restriction of P'_T to $\{h_T = \infty\}$ is singular with respect to P_T.

In particular

$$P'_T \perp P_T \Leftrightarrow P'(h_T = \infty) = 1.$$

Our main task in this note is to analyze what we still can say when we don't use the simplifying assumption that z_t is not allowed to jump to zero; unfortunately there is no hope for a complete analogue to theorem 1.2, as the following elementary example, which will be constructed in section 3 below, shows.

1.4 Example. There is a filtered space $(\Omega, (\mathcal{F}_t)_{t=0}^2, \mathcal{F})$ equipped with two probability measures P and P' with the following property:

There is no $[0, \infty]$-valued predictable process H such that, for every stopping time T,

$$P'_T \perp P_T \Leftrightarrow P'(H_T = \infty) = 1.$$

In fact Ω may be chosen to consist only of 4 elements.

Despite this discouraging example we can formulate an interesting positive result, where we shift the problem, that z_t may jump to zero from the assumption (as in theorem 1.3 above) into the conclusion of the theorem:

1.5 Theorem. *Under the assumption that $P_0 \sim P'_0$ we have, for every stopping time T,*

$$\{S \leq T, z_{S-} = 0\} = \{h_T = \infty\} \quad P'\text{-a.s.} \tag{3}$$

The set on the left hand side may be interpreted as the set where P'_T is singular with respect to P_T, but in such a way that, as $t \nearrow S \leq T$, this singularity was not obtained by a "sudden jump", but rather in a continuous way. The assertion of the theorem is that — even in the presence of jumps of z_t to zero — it is precisely this set which is characterized by the divergence of the Hellinger process.

Remark. We tried to formulate Thm 1.5 in a manner that suits best for comparison with the results in [JS], but there are other ways to state it. Actually, the assertion of Thm 1.5 is equivalent to the assertion

$$\{z_{S-} = 0\} = \{h_S = \infty\} \quad P'\text{-a.s.} \tag{3'}$$

To verify (3) $\Rightarrow$ (3′), we take $T := S$, and (3′) $\Rightarrow$ (3) follows from

$$\{S \le T, z_{S-} = 0\} = \{S \le T, h_S = \infty\} = \{S \le T, h_T = \infty\} = \{h_T = \infty\} \quad P'-\text{a.s.}$$

where we used the constancy of h from time S on and the fact that $h_T < \infty$ for $T < S$, $P'-$a.s., which follows e.g. from ([JS], Thm IV 1.18).

Theorem 1.5 and example 1.4 answer the question raised in [JS] right after corollary IV 2.8 and also sharpen the assertions of ([JS], lemma IV 2.12). Our proof is quite different from the methodology used in [JS]: it uses a close monitoring of those paths of z_t, for which $z_{S-} = 0$, and an extension of the Borel-Cantelli lemma due to P. Lévy. Although the proof is elementary it is somewhat labourious and technical.

The paper is organized as follows: In section 2 we give the proof of theorem 1.5 and in section 3 we construct example 1.4.

2. Proof of theorem 1.5

The inclusion $\{0 < S \le T, z_{S-} = 0\} \supseteq \{h_T = \infty\}$, P'-a.s. is proved in ([JS], lemma IV 2.12a). The reverse inclusion can be deduced from ([JS], lemma IV 2.12b+d) in the case, where z isn't allowed to jump to 0 up to time T, i. e., under the assumption of theorem 1.3

$$P'_T\left[0 < S \le T, z_{S-} > z_S = 0\right] = 0. \tag{4}$$

It remains to prove

$$\{0 < S \le T, z_{S-} = 0\} \subseteq \{h_T = \infty\} \quad P'-\text{a.s.} \tag{5}$$

without assuming (4). As our proof will rely heavily on the fact that the local martingale M given by (1) is a Q-martingale, we are aiming at Q-almost sure results. So the first thing to do is replace (5) by

$$\{0 < S \le T, z_{S-} = 0 \text{ or } z'_{S-} = 0\} \subseteq \{h_T = \infty\} \quad Q-\text{a.s.}, \tag{5′}$$

which is indeed equivalent to (5). We need the following lemma:

Lemma 2.1. *Let $(A_t)_{t\in[0,\infty[}$ be an adapted increasing process on a filtered space $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{P})$, $(W_n)_{n\in\mathbb{N}}$ an increasing sequence of stopping times, such that for all $n \in \mathbb{N}$*

$$\mathbb{P}\left[W_{n-1} < \infty, \mathbb{P}\left[A_{W_n} - A_{W_{n-1}} \ge \alpha | \mathcal{F}_{W_{n-1}}\right] < \alpha\right] \le 2^{-n}, \quad \text{for some } \alpha > 0. \tag{6}$$

Then

$$\bigcap_{n=1}^{\infty} \{W_n < \infty\} \subseteq \{A_\infty = \infty\} \quad \mathbb{P}-\text{a.s.} \tag{7}$$

Proof. Let $B_n = \{W_n < \infty\}$, $B = \bigcap_{n=1}^\infty \{W_n < \infty\}$, $E_n = \{A_{W_n} - A_{W_{n-1}} \ge \alpha\}$ and $\xi_n = \mathbb{P}\left[E_n | \mathcal{F}_{W_{n-1}}\right]$. If $\mathbb{P}[B] = 0$, (7) is trivially satisfied, so we assume $\mathbb{P}[B] = a > 0$. Since $B_n \searrow B$, we have

$$\mathbb{P}[\xi_n < \alpha | B] \le \frac{\mathbb{P}[\xi_n < \alpha, B_n]}{\mathbb{P}[B]} \le a^{-1} 2^{-n}.$$

The Borel-Cantelli lemma yields $\mathbb{P}[\xi_n \ge \alpha \text{ i.o.} | B] = 1$, therefore

$$B \subseteq \left\{ \sum_{k=1}^{n} \xi_k \to \infty \right\} \quad \mathbb{P}-\text{a.s.}$$

By Levy's extension of the Borel-Cantelli lemmas (cf. [S], p518 or [W], Thm 12.15)

$$\left\{\sum_{k=1}^{\infty} \xi_k = \infty\right\} = \left\{\sum_{k=1}^{\infty} \mathbb{I}_{E_k} = \infty\right\} \quad \mathbb{P}\text{−a.s.},$$

and the observation $A_{W_n} \geq \alpha \sum_{k=1}^{n} \mathbb{I}_{E_k}$ completes the proof. □

Remark. Given $\alpha > 0$ and a nonnegative random variable X, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we point out the following simple implications

$$\mathbb{E}[X \wedge 2] \geq 3\alpha \;\Rightarrow\; \mathbb{P}[X \wedge 2 \geq \alpha] \geq \alpha \;\Rightarrow\; \mathbb{E}[X \wedge 2] \geq \alpha^2,$$

which justify replacing hypothesis (6) by the equivalent hypothesis

$$(6') \qquad \mathbb{P}\left[W_{n-1} < \infty, \mathbb{E}\left[(A_{W_n} - A_{W_{n-1}}) \wedge 2 | \mathcal{F}_{W_{n-1}}\right] < \alpha'\right] \leq 2^{-n}, \quad \text{for some } \alpha' > 0.$$

We now proceed to the proof of theorem 1.5. First we define an increasing sequence of stopping times $(U_n)_{n\in\mathbb{N}}$ by

$$U_0 = 0, \quad U_n = \inf\left\{t : U_{n-1} \leq t \leq T, z_t \neq 0, z'_t \neq 0, \left\{\tfrac{z_t}{z_{U_{n-1}}}, \tfrac{z'_t}{z'_{U_{n-1}}}\right\} \not\subset \left]\tfrac{1}{2}, 2\right[\right\}, \text{ for } n \geq 1$$

where $\inf\emptyset := \infty$. This definition ensures, that

$$(8) \qquad 0 \leq Y_{\tau-} \leq 2Y_{U_{n-1}} \quad Q\text{-a.s. for stopping times } \tau \in]\!]U_{n-1}, U_n]\!]$$

and that on the set $\{U_n < \infty\}$ we have $\frac{z_{U_n}}{z_{U_{n-1}}} \leq \frac{1}{2}$ or $\frac{z_{U_n}}{z_{U_{n-1}}} \geq 2$ or $\frac{2-z_{U_n}}{2-z_{U_{n-1}}} \leq \frac{1}{2}$ or $\frac{2-z_{U_n}}{2-z_{U_{n-1}}} \geq 2$. Therefore

$$z_{U_n} \in C(z_{U_{n-1}}) \quad Q\text{-a.s. on the set } \{U_n < \infty\},$$

where

$$(9) \qquad C(x) := \left]0, \tfrac{x}{2} \vee 2x - 2\right] \cup \left[2x \wedge 1 + \tfrac{x}{2}, 2\right[.$$

If $Q[\bigcap_{n=1}^{\infty}\{U_n < \infty\}] > 0$, then for any $\mu_0 > 0$

$$(10) \qquad Q[z_{U_n} \in]0, \mu_0[\cup]2 - \mu_0, 2[\,|\, U_n < \infty] \to 1, \quad \text{as } n \to \infty,$$

since on the set $\bigcap_{n=1}^{\infty}\{U_n < \infty\}$ the sequence $(z_{U_n})_{n\in\mathbb{N}}$ converges Q-a.s. to a random variable, which takes values in the two element set $\{0, 2\}$ (other values are not possible by the definition of $(U_n)_{n\in\mathbb{N}}$), and since $Q[U_n < \infty] \to Q[\bigcap_{n=1}^{\infty}\{U_n < \infty\}]$ as $n \to \infty$. In particular, we have

$$\left\{0 < S \leq T, z_{S-} = 0 \text{ or } z'_{S-} = 0\right\} = \bigcap_{n=1}^{\infty}\{U_n < \infty\} \quad Q\text{-a.s.},$$

so $(5')$ and thus theorem 1.5 will follow from lemma 2.1, if we can establish the truth of the subsequent lemma 2.2. □

Lemma 2.2. *There exists* $\gamma > 0$ *such that*

$$Q\left[U_{n-1} < \infty, \mathbb{E}\left[(h_{U_n} - h_{U_{n-1}}) \wedge 2 | \mathcal{F}_{U_{n-1}}\right] < \gamma^2\right] \to 0$$

as $n \to \infty$.

Proof. Suppose to the contrary that, for any $0 < \gamma < 1$, there is a constant $\beta = \beta(\gamma) > 0$ and for infinitely many n (depending on γ) an $\mathcal{F}_{U_{n-1}}$-measurable set $E_{n-1} = E_{n-1}(\gamma) \subset \{U_{n-1} < \infty\}$, such that $Q[E_{n-1}] \geq \beta$ and on E_{n-1} we have

$$(11) \qquad Q\left[h_{U_n} - h_{U_{n-1}} \geq \gamma | \mathcal{F}_{U_{n-1}}\right] < \gamma.$$

For given small $\mu_0 > 0$ (to be chosen later), we can and will assume by passing again to a subsequence and invoking (10), that

$$z_{U_{n-1}} \in]0, \mu_0[\cup]2-\mu_0, 2[\tag{12}$$

on an $\mathcal{F}_{U_{n-1}}$-measurable subset of E_{n-1} of Q-probability at least $\frac{\beta}{2}$, which we again denote by E_{n-1}.

We then define a sequence of stopping times $(V_n)_{n\in\mathbb{N}}$ by

$$V_n = \inf\left\{t : U_{n-1} \le t, h_t - h_{U_{n-1}} \ge \gamma\right\},$$

so that (11) implies

$$\mathbb{E}\left[\mathbb{1}_{\{V_n<U_n\}}|\mathcal{F}_{U_{n-1}}\right] < \gamma \quad Q\text{-a.s. on } E_{n-1}. \tag{13}$$

The definition of $(V_n)_{n\in\mathbb{N}}$ ensures, that

$$h_{V_n} - h_{U_{n-1}} < 2 \quad Q\text{-a.s.}, \tag{14}$$

since the jumps of h are bounded by 1 (see, e. g. [JS], IV 1.30).

In order to get grip of h, we employ the Doob-Meyer decomposition of the supermartingale Y given by (1)

$$Y = M - Y_- \bullet h$$

where h is the Hellinger process of order $\frac{1}{2}$ in the strict sense and M is a uniformly integrable martingale (cf. [JS], IV 1.18). Taking the difference $Y_t - Y_{U_{n-1}}$, and dividing by $Y_{U_{n-1}}$, we formally arrive at

$$\frac{Y_t}{Y_{U_{n-1}}} = 1 + \frac{1}{Y_{U_{n-1}}}\left(M_t - M_{U_{n-1}}\right) - \left[\left(\frac{Y_-}{Y_{U_{n-1}}} \bullet h\right)_t - \left(\frac{Y_-}{Y_{U_{n-1}}} \bullet h\right)_{U_{n-1}}\right].$$

This can be looked upon as the Doob-Meyer decomposition of the supermartingale

$$Y_t^{(n)} := \mathbb{1}_{\{t\le U_{n-1}\}} + \mathbb{1}_{\{t>U_{n-1}\}}\frac{Y_t}{Y_{U_{n-1}}}$$

on the set $\{t > U_{n-1}\}$, which we rewrite as

$$Y^{(n)} = M^{(n)} - Y_-^{(n)} \bullet h^{(n)},$$

with $M_t^{(n)} := 1 + \frac{1}{Y_{U_{n-1}}}\left(M_t - M_{U_{n-1}}\right)\mathbb{1}_{\{t>U_{n-1}\}}$ and $h_t^{(n)} := \left(h_t - h_{U_{n-1}}\right)\mathbb{1}_{\{t>U_{n-1}\}}$. The martingale $M^{(n)}$ is again uniformly integrable and starts at 1. In the sequel all expectations are taken with respect to Q. We are going to derive a contradiction in computing expectations at time $U_n \wedge V_n$ conditional on $\mathcal{F}_{U_{n-1}}$:

$$\mathbb{E}\left[Y_{U_n\wedge V_n}^{(n)}|\mathcal{F}_{U_{n-1}}\right] = \mathbb{E}\left[M_{U_n\wedge V_n}^{(n)}|\mathcal{F}_{U_{n-1}}\right] - \mathbb{E}\left[Y_-^{(n)} \bullet h_{U_n\wedge V_n}^{(n)}|\mathcal{F}_{U_{n-1}}\right].$$

From corollary 2.1 below (which is a kind of "uniformly strict" Jensen inequality for the concave function $f(x) = \sqrt{x(2-x)}$) it follows that for properly chosen μ_0 (cf. (12)) there exists $\epsilon > 0$ such that on the set E_{n-1} we have Q-a.s.

$$\mathbb{E}\left[Y_{U_n}^{(n)}|\mathcal{F}_{U_{n-1}}\right] \le 1 - \epsilon + \epsilon Q\left[U_n = \infty|\mathcal{F}_{U_{n-1}}\right].$$

This yields

$$\begin{aligned}\mathbb{E}\left[Y_{U_n\wedge V_n}^{(n)}|\mathcal{F}_{U_{n-1}}\right] &= \mathbb{E}\left[Y_{U_n}^{(n)}|\mathcal{F}_{U_{n-1}}\right] - \mathbb{E}\left[Y_{U_n}^{(n)}\mathbb{1}_{\{V_n<U_n\}}|\mathcal{F}_{U_{n-1}}\right] + \mathbb{E}\left[Y_{V_n}^{(n)}\mathbb{1}_{\{V_n<U_n\}}|\mathcal{F}_{U_{n-1}}\right] \\ &\le \left[1 - \epsilon + \epsilon Q\left[U_n = \infty|\mathcal{F}_{U_{n-1}}\right]\right] + 2\gamma,\end{aligned}$$

where the estimate of the third term employs (8) and (13).

Furthermore (8), (13) and (14) yield

$$\mathbb{E}\left[\left(Y_-^{(n)}\bullet h\right)_{U_n\wedge V_n}-\left(Y_-^{(n)}\bullet h\right)_{U_{n-1}}\Big|\mathcal{F}_{U_{n-1}}\right]\leq 2\mathbb{E}\left[(h_{U_n\wedge V_n}-h_{U_{n-1}})|\mathcal{F}_{U_{n-1}}\right]$$

$$=2\mathbb{E}\left[(h_{V_n}-h_{U_{n-1}})\mathbb{1}_{\{V_n<U_n\}}|\mathcal{F}_{U_{n-1}}\right]+2\mathbb{E}\left[(h_{U_n}-h_{U_{n-1}})\mathbb{1}_{\{V_n\geq U_n\}}|\mathcal{F}_{U_{n-1}}\right]\leq 4\gamma+2\gamma$$

also Q-a.s. on the set E_{n-1}. Thus we obtain Q-a.s. on the set E_{n-1}

$$1-\epsilon+\epsilon Q\left[U_n=\infty|\mathcal{F}_{U_{n-1}}\right]\geq 1-8\gamma,$$

which, if γ is small enough ($\gamma=\frac{\epsilon}{10}$ will do), can only be true for finitely many n, since $Q\left[U_n=\infty,U_{n-1}<\infty|\mathcal{F}_{U_{n-1}}\right]\to 0$ in Q-probability. This is the desired contradiction. □

In order to have a means of discussing the problem of replacing h in Theorem 1.5 by certain Hellinger-like processes (we will address this problem in more detail in the remark after corollary 2.1), we introduce some classes of concave functions, and prove a lemma, which is a "uniformly strict" version of Jensen's inequality for one of these classes of functions.

Definition 2.1. *Let $\mathbb{F}_0$ be the class of concave functions $f:[0,2]\to\mathbb{R}^+$, satisfying $f(0)=0$ and*

$$1-\frac{a}{2}-2^{-b}>0,\qquad 1+b-2^a>0, \tag{15}$$

where we denote

$$a=\limsup_{x\to 0}\frac{xf'(x)}{f(x)},\qquad b=\liminf_{x\to 0}\frac{xf'(x)}{f(x)}. \tag{16}$$

Let moreover $\mathbb{F}_2=\{f(\cdot):\ f(2-\cdot)\in\mathbb{F}_0\}$ and $\mathbb{F}=\mathbb{F}_0\cap\mathbb{F}_2$.

Note, that this classes are not empty. The set of admissible values of (a,b) in (15) is in fact a subset of $]0,1[^2$ containing the set $\{(a,b):\ 0<a=b<1\}$. Practical members of $\mathbb{F}$ are the functions $f(x)=x^\alpha(2-x)^\beta$, where $0<\alpha,\beta<1$.

Lemma 2.3. *Let $f\in\mathbb{F}_0$. Then there exist $\epsilon_f>0$ and $\mu_f>0$ such that for all $\mu\in]0,\mu_f]$ the inequality*

$$\frac{\mathbb{E}\left[f(X)\right]}{f(\mu)}\leq 1-\epsilon_f p$$

holds for any random variable X ranging in $[0,2]$ and satisfying

$$\mathbb{E}X=\mu,\quad \mathbb{P}\left[X\in D(\mu)\right]=p>0,$$

where $D(\mu)=]0,2[\setminus]\frac{\mu}{2},2\mu[$.

Proof. Since f is concave, the affine function $\ell(x)=f(\mu)+f'(\mu)(x-\mu)$ satisfies $f(x)\leq\ell(x)$ and $f(\mu)=\ell(\mu)$. On the set $D(\mu)$ we even have $f(x)\leq\ell(x)-m(\mu)$ with

$$m(\mu)=\min\left(\ell(\tfrac{\mu}{2})-f(\tfrac{\mu}{2}),\ell(2\mu)-f(2\mu)\right).$$

Therefore

$$\mathbb{E}\left[f(X)\right]\leq\mathbb{E}\left[\ell(X)-m(\mu)\mathbb{1}_{\{X\in D(\mu)\}}\right]=\ell(\mu)-m(\mu)\mathbb{P}\left[X\in D(\mu)\right]=f(\mu)-m(\mu)p.$$

It remains to show that $\frac{m(\mu)}{f(\mu)}$ is bounded away from 0 uniformly as $\mu\to 0$: By (16) we have $\mu f'(\mu)\leq af(\mu)\left(1+o(1)\right)$ and $\frac{f'(\mu)}{f(\mu)}\geq\frac{b}{\mu}\left(1+o(1)\right)$. Integrating the latter inequality from $\frac{\mu}{2}$ to μ yields $\ln\frac{f(\mu)}{f(\frac{\mu}{2})}\geq b\ln 2\left(1+o(1)\right)$ and thus $f(\frac{\mu}{2})\leq 2^{-b}f(\mu)\left(1+o(1)\right)$. Therefore, as $\mu\to 0$,

$$\ell(\frac{\mu}{2})-f(\frac{\mu}{2})=f(\mu)-\frac{\mu}{2}f'(\mu)-f(\frac{\mu}{2})\geq\left(1-\frac{a}{2}-2^{-b}+o(1)\right)f(\mu),$$

and similarely $\ell(2\mu)-f(2\mu)\geq\left(1+b-2^a+o(1)\right)f(\mu)$. We have derived

$$\frac{m(\mu)}{f(\mu)}\geq\min(1-\frac{a}{2}-2^{-b},\ 1+b-2^a)+o(1).$$

Choosing now $\epsilon_f=\frac{1}{2}\min(1-\frac{a}{2}-2^{-b},\ 1+b-2^a)$ and μ_f such, that the function $g(x)$ implied by the symbol $o(1)$ in the last inequality satisfies $|g(x)|\leq\epsilon_f$ for $x\in]0,\mu_f]$ makes the proof complete. □

Corollary 2.1. *There exist $\epsilon > 0$ and $\mu_0 > 0$ such that for all $\mu \in]0, \mu_0] \cup [2-\mu_0, 2[$ and for any random variable X ranging in $[0,2]$ and satisfying*

$$\mathbb{E}X = \mu, \quad \mathbb{P}[X \in C(\mu)] = p > 0,$$

where $C(\mu)$ is given by (9), we have

$$\mathbb{E}\left[\sqrt{\frac{X(2-X)}{\mu(2-\mu)}}\right] \leq 1 - \epsilon p.$$

Proof. This follows from Lemma 2.3 and the fact that $f(2-x) = f(x)$.

Remark. For given concave function f we can define a supermartingale $Y_t = f(z_t) = M_t + A_t$, and ask, if the increasing process H, defined via $Y = M - Y_- \bullet H$ and required to be 0 at 0 and constant after S (cf.(2)), can replace the Hellinger process h of order $\frac{1}{2}$ in Theorem 1.5, i.e., satisfies

$$\{Y_{S-} = 0\} = \{H_S = \infty\} \quad Q\text{–a.s.} \tag{17}$$

for any Q–martingale $(z_t)_{t=0}^{\infty}$ with $z_0 = 1$ and $0 \leq z_t \leq 2$. Note that the Hellinger process $h(\frac{1}{2})$ (resp. $h(\alpha)$, for $0 < \alpha < 1$) corresponds to the choice $f(z) = z^\alpha(2-z)^{1-\alpha}$.

One obvious condition, f must satisfy, is $f(0) = f(2) = 0$. Otherwise H_t, given by $\int_0^t \frac{dA_s}{Y_{s-}}$, would be finite Q–a.s. on at least one of the sets $\{z_{S-} = 0\}$ resp. $\{z_{S-} = 2\}$.

Another necessary condition for (17) is $f'(0) = f'(2) = \infty$. To see this, take $z_t = 1 + B_{t\wedge S}$, where (B_t) is a standard Brownian motion with respect to Q and $S = \inf\{t \geq 0 : |B_t| = 1\}$, and take $f(x) = 1 - |x-1|$, which satisfies $f'(0) = 1$. The Tanaka formula then reveals that $H_t = L_{t\wedge S}$, where L denotes the local time at 0 of (B_t). Now $S < \infty$ Q-a.s. and therefore also $H_S < \infty$ Q-a.s., but $Y_{S-} = f(z_{S-}) = 0$ Q-a.s.

The methods of our paper suffice to prove the $\subseteq$-part of (17) for concave functions f belonging to the class $\mathbb{F}$. In particular equations (8) and (14) remain true for $f \in \mathbb{F}$. The $\supseteq$-part of (17) for $f \in \mathbb{F}$ can be proved as in ([JS], lemma IV 2.12 a). Thus, in (17), H can in particular be one of the Hellinger processes of order α, for $0 < \alpha < 1$. However, there are concave functions satisfying $f(0) = f(2) = 0$ and $f'(0) = f'(2) = \infty$, but not contained in $\mathbb{F}$, such as $f(x) = x(2-x)\ln(\frac{x}{e})\ln(\frac{2-x}{e})$ and $f(x) = \left(\ln(\frac{x}{2e})\ln(\frac{2-x}{2e})\right)^{-1}$.

We do not know whether for general concave functions f the conditions

$$f(0) = f(2) = 0, \quad f'(0) = f'(2) = \infty$$

are also sufficient for (17) to hold, and leave this question for future research.

3. Example 1.4

Here we write down the example refered to in 1.4. Let $(\Omega, (\mathcal{F}_t)_{t=0}^2, \mathcal{F})$ and two probability measures P, P' be given by

$$\begin{aligned}
\Omega =& \{\omega_1, \omega_2, \omega_3, \omega_4\}, \\
\mathcal{F}_0 =& \{\emptyset, \Omega\}, \\
\mathcal{F}_1 =& \sigma(\{\omega_1\}, \{\omega_2\}, \{\omega_3, \omega_4\}), \\
\mathcal{F}_2 =& \mathcal{F} = \sigma(\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}), \\
P(\omega_1) =& 2P(\omega_3) = \tfrac{2}{3}, \\
P'(\omega_2) =& 2P'(\omega_4) = \tfrac{2}{3}.
\end{aligned}$$

In this case we have $S = \mathbb{1}_{\{\omega_1, \omega_2\}} + 2 \cdot \mathbb{1}_{\{\omega_3, \omega_4\}}$.

Claim. *There is no $[0,\infty]$-valued predictable process H such that, for every stopping time T,*

$$P'_T \perp P_T \Leftrightarrow P'(H_T = \infty) = 1.$$

Proof. Assume that there is. Then $H_1 \equiv \text{const} < \infty$, P'-a.s., since $\mathcal{F}_0$ is trivial and $P'_1 \not\perp P_1$. On the other hand $H_S \equiv \infty$, P'-a.s., since $P_S \perp P'_S$. Now on the set $\{S = 1\}$, which has positive P'-measure, we have conflicting definitions of H_1. This contradicts our assumption. $\square$

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