No Arbitrage: On the Work of David Kreps

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Abstract

Since the seminal papers by Black, Scholes and Merton on the pricing of options (Nobel Prize for Economics, 1997), the theory of No Arbitrage plays a central role in Mathematical Finance. Pioneering work on the relation between no arbitrage arguments and martingale theory has been done in the late seventies by M. Harrison, D. Kreps and S. Pliska.

In the present note we give a brief survey on the relation of the theory of No-Arbitrage to coherent pricing of derivative securities. We focus on a seminal paper published by D. Kreps in 1981, and give a solution to an open problem posed in this paper.

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1 Introduction: "No arbitrage" and "No Free Lunch"

The *principle of no arbitrage* formalizes a very convincing economic argument: in a financial market it should not be possible to make a profit with zero net investment and without bearing any risk. It is surprising, how much can be deduced from this primitive principle. For example, in the celebrated model used by Black, Scholes [BS 73], and Merton [M 73], which is based on geometric Brownian motion, the price of any derivative security (e.g., a European call option) is already determined by this principle.

In addition, the *fundamental theorem of asset pricing*, as isolated in the work of Harrison, Kreps and Pliska ([HK 79], [HP 81], [K 81]) allows to relate the no arbitrage arguments with martingale theory.

Here is the mathematical formulation of the concept of no arbitrage as formalized in [K 81]: we start with an ordered topological vector space X, equipped with a locally convex topology τ, and its positive cone K (with the origin deleted). A typical situation is $X = L^p(\Omega, \mathcal{F}, \mathbb{P})$, equipped with its natural order structure and topology, where $1 \leq p \leq \infty$, and $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space modeling the "possible states of the world" ω at some fixed time horizon T. In this setting the elements $x = x(\omega) \in X$ are random variables modeling contingent claims at time T (denoted in terms of the single consumption good under consideration, which we choose as numéraire). The arch-example of a contigent claim x is a European call option on a stock modeled by a stochastic process $(S_t)_{0 \leq t \leq T}$. In this case $x = (S_T - k)_+$, where k is a positive real number (the striking price of the option).

Returning to the general framework of [K 81]: in addition to the ordered locally convex space (X, τ, K), we are given a subspace $M \subseteq X$ and a linear functional $\pi : M \to \mathbb{R}$. The interpretation is that M consists of those contingent claims, which are traded on the market at a price given by π. The notion of "traded claims" typically is understood in a broad sense: in the setting of the pricing theory of contigent claims on a stock modeled by the stochastic process $S = (S_t)_{0 \leq t \leq T}$, one usually defines M as the set spanned by the constants and all random variables which are the outcome $(H \cdot S)_T = \int_0^T H_t dS_t$ of a *predictable trading strategy* H, where, in general, some additional care is needed, to rule out, e.g., "doubling strategies" (see [HP 81]). One therefore has to restrict to a convenietly chosen sub-class of the predictable trading strategies. We don't elaborate on the details here (see, e.g., [HP 81] or [DS 94]), and simply note that one is naturally led to a subspace (or, more generally, subcone) M of X, consisting of "marketed claims", and a linear pricing functional $\pi : M \to \mathbb{R}$.

What does it mean that a market modeled by (X, τ, K, M, π) does not allow arbitrage? Letting $M_0 = \{x \in M : \pi(x) = 0\}$ and C the convex cone $M_0 - K = \{y \in X : \exists x \in M_0 \text{ s.t. } y \leq x\}$, we now may formally define this concept:

Definition 1.1 *The market model (X, τ, K, M, π) satisfies the condition (NA) of* No Arbitrage, *if $M_0 \cap K = \emptyset$ or, equivalently, $C \cap K = \emptyset$.*

The interpretation of this definition is rather obvious: an arbitrage opportunity is a positive element $x \in K$, which is marketed at price $\pi(x) = 0$. The model (X, τ, K, M, π) is free of arbitrage, if no such opportunity exists.

The basic question consists in the possibility of extending π from M to the entire space X to obtain a *coherent pricing system*, i.e. a strictly positive, continuous linear functional $\psi : X \to \mathbb{R}$ extending π. We then say that (X, τ, K, M, π) has the *extension property* [K 81]. Again, the interpretation of a coherent pricing system ψ is rather straight-forward: this pricing rule assigns to each contingent claim $x \in X$ — whether marketed or not — a price $\psi(x)$ in a continuous and linear way, and such that $x \in K$ implies $\psi(x) > 0$. Of course, ψ and π should agree on M, the space of marketed contingent claims. We then say [K 81] that (X, τ, K, M, π) is *viable*.

It is obvious, that the condition of no arbitrage is *necessary* for viability; the interesting issue is, whether it is also *sufficient*. This is the theme of the so-called "Fundamental Theorem of Asset Pricing". Before passing to the mathematical theorems in this direction, we discuss the economic meaning of such an implication: the theorems below give precise statements for the following intuitive idea: if the pricing system π, defined on M, is not obviously incoherent (in the sense, that it does not allow for arbitrage), then one may extend it to a coherent price system ψ, defined on all of X.

The most basic version of the fundamental theorem of asset pricing is due to Harrison and Pliska [HP 81] (compare also [HK 79], and [K 81], p. 16 and 17). Suppose that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ is a *finite* filtered probability space and that (X, τ, K) equals $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ equipped with its natural topology and order structure. Note that there is no ambiquity in the notion of natural topology, as X now is finite-dimensional.

Let $S = (S_t)_{t=0}^T$ be a (say) real-valued stochastic process, defined on and adapted to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$, and define M_0 as the subspace of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ formed by the random variables

$$f = (H \cdot S)_T = \sum_{t=1}^{T} H_t(S_t - S_{t-1}), \tag{1}$$

where $H = (H_t)_{t=1}^T$ runs throught the predictable real-valued processes defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$. Supposing that the constant function $\mathbf{1}$ is not in M_0 (otherwise the no arbitrage condition is violated in an obvious way), define M as the linear space spanned by of M_0 and $\mathbf{1}$, and π by $\pi_{|M_0} = 0$ and $\pi(\mathbf{1}) = 1$.

We now have assembled all the ingredients (X, τ, K, M, π) of the setting of [K 81] and may formulate the basic theorem.

Theorem 1.2 ([HP 81]) *Let (X, τ, K, M, π) be defined over the finite filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ as indicated above.*

Then (X, τ, K, M, π) is arbitrage-free iff it has the extension property.

Proof By the definition no arbitrage we have the equality $M_0 \cap K = \emptyset$. In the present case of a finite-dimensional space $X = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ it is rather straight-forward (but not trivial!) to deduce from Hahn-Banach that we may find a hyperplane H, containing M_0, and such that we still have $H \cap K = \emptyset$. We refer, e.g., to the survey paper [S 01] for a thorough discussion of this argument.

Once we have found the hyperpane H, it suffices to define the linear functional ψ by $\psi|_H = 0$ and $\psi(\mathbf{1}) = 1$.

This takes care of the "only if" implication, while the "if" implication is trivial. ■

In the context of the above theorem the positive, normalized, linear functional $\psi \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ may be identified with a probability measure Q on $\mathcal{F}$ via $\frac{dQ}{d\mathbb{P}} = \psi$. One easily verifies that the linear functional ψ is strictly positive iff the probability measure Q is equivalent to $\mathbb{P}$. In addition, ψ vanishes on M_0 iff S is a martingale under Q. Therefore the above theorem of Harrison-Pliska may be rephrased in the following way: the market model (X, τ, K, M, π) is arbitrage-free iff there is an equivalent martingale measure for S.

The central topic of Kreps' paper [K 81] is the issue, how the above theorem extends to the infinite-dimensional case. His fundamental insight was, that the — purely algebraic — notion of no arbitrage has to be strengthened by a topological notion. We restate the basic concept of [K 81] in a slightly different (but equivalent) wording.

Definition 1.3 *The model (X, τ, K, M, π) satisfies the condition of* no free lunch, *if the τ-closure $\overline{C}$ of $C = M_0 - K$ satisfies $\overline{C} \cap K = \emptyset$.*

There are two issues worth noting in the above definition: the passage to the τ-closure is rather natural when we try to strengthen the no arbitrage condition, in order to extend the above theorem to the infinite-dimensional setting. What is less obvious is the passage from M_0 to $C = M_0 - K$. Economically speaking, this is the passage from the contingent claims, marketed at price 0, which constitute M_0, to those contingent claims, where the agents also are allowed to pass from boundles $x \in M_0$ to boundles of the form $y = x - h$, where h is in the positive cone K. In many applications this may be interpreted as the possibility of the agents to "throw away money". It turns out, that this — at first glance somewhat ackward — possibility of "free disposal" is in fact the key feature to make the extension of the above theorem work. For a counterexample, showing that the condition, that the τ-closure $\overline{M_0}$ of M_0 satisfies $\overline{M_0} \cap K = \emptyset$, does not allow, in general, for an extension of the above theorem, we refer to [S 94].

The central result ([K 81], Theorem 3) states that, under some mild separability assumption, the condition of "no free lunch" is equivalent to the extension property of (X, τ, K, M, π). We now formulate Theorem 3 of [K 81] in a version specializing to the case of $L^p(\Omega, \mathcal{F}, \mathbb{P})$, where it turns out that we need not to bother about any seperability assumption.

Theorem 1.4 *Let (X, τ, K) equal $L^p(\Omega, \mathcal{F}, \mathbb{P})$, where $1 \leq p \leq \infty$, K its positive orthant (with the origin deleted) and, for $1 \leq p < \infty$, τ equals the norm-topology of L^p, while, for $p = \infty$, τ equals the weak-star topology $\sigma(L^\infty, L^1)$.*

Let C denote a τ-closed, convex cone, containing $-K$, and such that $C \cap K = \emptyset$. Letting $\frac{1}{q} + \frac{1}{p} = 1$, there is $g \in L^q(\Omega, \mathcal{F}, \mathbb{P})$, $g > 0$ almost surely, such that $g_{|C} \leq 0$.

The above theorem implies that Theorem 1.2 above extends to the case where (X, τ, K) equals $L^p(\Omega, \mathcal{F}, \mathbb{P})$ for an arbitrary probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if the notion of "no arbitrage" is replaced by the notion of "no free lunch".

Let us give a brief review on the development following Kreps' paper: Duffie and Huang [DH 86] applied Kreps' theorem in its original form to formulate and prove the above theorem, assuming the separability of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$; it was noticed by Stricker [St 90] that, using a theorem of Yan [Y 80], this assumption is superfluous (see, e.g., [S 94] for an exposition of the proof and additional references): loosely speaking, the countability argument, needed in the proof of theorem 3 of [K 81], can be replaced by an exhaustion argument using the finiteness of the underlying measure space $(\Omega, \mathcal{F}, \mathbb{P})$. Hence we do not need the separability of the Banach space $L^p(\Omega, \mathcal{F}, \mathbb{P})$. It was noted in [C 93], that one may use the classical Halmos-Savage theorem [HS 49] to formalize this exhaustion argument (For extensions of the Halmos-Savage theorem and its relations to asymptotic arbitrage we also refer to [KS 96].)

We again refer, e.g., to [S 01] for a more extensive account on the sharpenings of Theorem 1.4 above, which have been obtained after [K 81].

2 Some remarks on Kreps' paper

In this section we shall focus on some of the technical issues of Kreps' paper: firstly we give a counterexample to the "enticing conjecture" raised in ([K 81], p. 29). Briefly, the conclusion of our example is that we only can expect "nice results" in the context of Theorem 1.4 above, where (X, τ, K) now is arbitrary, if we allow the commodity space (X, τ, K) not to be too pathological: the Counterexample 2.1 below considers a very peculiar topology τ on $X = l^\infty$.

Secondly, we address the question treated in [K 81, p. 30] in the context of Theorem 4 of [K 81], where, for a bundle $x \in X$, prices $\underline{\pi}(x)$ and $\overline{\pi}(x)$ are defined, which are the lower and upper bound of the arbitrage-free prices for x. If $\underline{\pi}(x) = \overline{\pi}(x)$ then the situation is clear, as in this case this is the unique arbitrage-free price for x. But if $\underline{\pi}(x) < \overline{\pi}(x)$ the situation is more delicate. Let us quote from ([K 81], p. 30):

"This leaves the question: If $\underline{\pi}(x) < \overline{\pi}(x)$, are $\overline{\pi}(x)$ and $\underline{\pi}(x)$ prices for x that are consistent with (M, π)? In economies with finite dimensional commodity spaces the answer is always no. But in general, the answer may be yes or no in either or both cases. (Examples are easy to construct.)"

Some remarks seem in order in view of the work which has been done since Kreps' paper: firstly, for the convenience of the reader we provide an easy example below substantiating Kreps' claim. Secondly, and more importantly, we want to point out that since Kreps' paper several theorems have been proved which allow — under additional assumptions — more precise information on this question than the general "anything goes" statement quoted above. For example, take the important case of a semi-martingale $S = (S_t)_{0 \le t \le T}$ with continuous paths (taking values in a finite or infinite dimensional space) satisfying the condition of "no free lunch". To relate this setting to the abstract framework of [K 81] we again let (X, τ, K) be the space $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the weak-star topology τ induced by L^1, and K the positive cone of L^∞. For M we choose the subspace of elements $x = c + \int_0^T H_t dS_t$ in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, where $c \in \mathbb{R}$ and $(H_t)_{0 \le t \le T}$ ranges through the predictable trading strategies, such that the stochastic intergral $(\int_0^t H_u dS_u)_{0 \le t \le T}$ is uniformly bounded; for π we take the linear functional on M defined by $\pi(c + \int_0^T H_t dS_t) = c$. In this setting it was shown by F. Delbaen [D 92] that, for $x \in L^\infty$ verifying $\underline{\pi}(x) < \overline{\pi}(x)$, the set of prices for x consistent with (M, π) equals the *open interval* $]\underline{\pi}(x), \overline{\pi}(x)[$.

On the other hand, Ph. Artzner and D. Heath [AH 95] gave an example of a process S (taking values in an infinite dimensional space and failing to have continuous paths) such that the set of equivalent martingale measures for S forms a compact set which is not reduced to a singleton. This implies that, for every $x \in X$, the set of consistent prices for x forms a compact interval; in addition, there are $x \in X$, for which this interval is not reduced to a singleton.

We refer to [D 92], [J 92], [S 93], [AS 94], [DS 94], [KQ 95], [AH 95] and [DS 95] for further results on this issue, as well as to [S 01] for a more detailed survey.

For further examples in the context of Kreps' paper we also refer to [C 00].

We now present the two examples announced above. We assume that the reader is familiar with Kreps' paper [K 81] and use its terminology without further reference.

Example 2.1 *(answering negatively the conjecture raised in [K 81, 4. Remark (6)]):*
There is an economy (X, τ, K, M, π) *such that*

1. Ψ *is non-empty;*
2. (M, π) *fails to be* (X, τ, K)*-viable;*
3. *there are no free lunches.*

Proof The construction will be a modification of [K 81, Example 4.3].

Let $X = l^\infty$. To define the dual space Y, we consider two elements y_1, y_2 in l^1: denoting by $(e_n)_{n=1}^\infty$ the n'th unit vector of l^1 let

$$y_1 = \sum_{n=1}^{\infty} 2^{-n} e_n, \qquad y_2 = \sum_{n=1}^{\infty} 2^{-n} e_{2n-1}.$$

Let Y_0 be the linear subspace of l^1 spanned (algebraically) by the unit vectors $(e_n)_{n=1}^\infty$ and define Y to be the linear space generated by Y_0, y_1 and y_2. We shall consider the dual pair $\langle X, Y \rangle$ equipped with its natural scalar product $\langle \cdot, \cdot \rangle$ and we define on X the weak topology $\tau = \sigma(X, Y)$ generated by the space Y (for a definition of the weak topology defined for a dual pair of vector spaces see, e.g., [Sch 66]).

Observe that this topology is — just slightly — finer than the product topology on l^∞ which was used by D. Kreps in [K 81, Example 4.3]: indeed, the product topology on l^∞ equals the weak topology $\sigma(X, Y_0)$; also note that Y_0 is a subspace of codimension 2 of Y.

The cone K will be the positive orthant of l^∞ with the origin deleted, i.e.

$$K = \{x = (x^n)_{n=1}^\infty, x^n \geq 0, x \not\equiv 0\}.$$

Note that we have constructed (X, τ, K) in such a way that Ψ is non-empty: for example, y_1 defines an element of Ψ, as it is a τ-continuous linear functional on X, which is strictly positive on K. This proves assertion (i).

We now turn to the definition of M and π. Let

$$M = \{x = (x^n)_{n=1}^\infty : x^{2i-1} = x^{2i}, \text{ for } i \in \mathbb{N}\}$$

and let π be the linear functional defined on M by restricting y_2 to M, i.e.,

$$\pi(x) = \sum_{n=1}^{\infty} 2^{-n} \cdot x^{2n-1}, \text{ for } x = (x^1, x^2, \dots) \in M.$$

To show (ii), we have to prove that there is no $\psi \in \Psi$, which extends π from M to X. Supposing that such a ψ exists we shall work towards a contradiction.

We may write ψ as

$$\psi = y_0 + \ \lambda_1 y_1 + \ \lambda_2 y_2$$

where $y_0 \in Y_0$. As we assumed ψ to be strictly positive on K we necessarily have $\lambda_1 > 0$.

Let n be sufficiently large such that y_0 vanishes on the two elements z_n, z_{n+1} of M, where

$$z_n = e_{2n-1} + e_{2n} \text{ and } z_{n+1} = e_{2n+1} + e_{2n+2}.$$

As we assumed that ψ extends π we must have that $\psi(z_n) = \pi(z_n)$ and $\psi(z_{n+1}) = \pi(z_{n+1})$, i.e.,

$$\langle z_n, \ \lambda_1 y_1 + \ \lambda_2 y_2 \rangle = \langle z_n, y_2 \rangle$$

and

$$\langle z_{n+1}, \ \lambda_1 y_1 + \ \lambda_2 y_2 \rangle = \langle z_{n+1}, y_2 \rangle.$$

This yields the two linear equations

$$\begin{array}{ccccccc} (2^{-(2n-1)} + 2^{-2n}) & \lambda_1 & + & 2^{-n} & \lambda_2 & = & 2^{-n}, \\ (2^{-(2n+1)} + 2^{-(2n+2)}) & \lambda_1 & + & 2^{-(n+1)} & \lambda_2 & = & 2^{-(n+1)} \end{array}$$

with the unique solution $\lambda_1 = 0, \ \lambda_2 = 1$, a contradiction to the assumption $\lambda_1 > 0$ finishing the proof of (ii).

(iii): Let us now assume that there is a free lunch and again work towards a contradiction: let $\{(m_\alpha, x_\alpha); \alpha \in A\} \subseteq M \times X$ be a net such that $m_\alpha - x_\alpha \in K \cup \{0\}$, and let $k \in K$ such that $x_\alpha \xrightarrow{\tau} k$ and $\lim_\alpha \pi(m_\alpha) \leq 0$.

If $k = (k^n)_{n=1}^\infty$, has a strictly positive entry for an odd coordinate n we are finished, as then $\langle k, y_2 \rangle > 0$ while $\limsup_\alpha \langle x_\alpha, y_2 \rangle \leq \lim_\alpha \pi(m_\alpha) \leq 0$, in contradiction to $x_\alpha \xrightarrow{\tau} k$.

If all the odd coordinates of k are zero, we need a little extra argument: in this case there is an even number $2i$ such that $k^{2i} > 0$. Define $\widetilde{x}_\alpha$ to equal x_α for all coordinates $n \neq 2i$ and $\widetilde{x}_\alpha^{2i} = x_\alpha^{2i-1}$. As $m_\alpha^{2i} = m_\alpha^{2i-1}$ we still have $m_\alpha - \widetilde{x}_\alpha \in K \cup \{0\}$ and a moment's reflection reveals that $(\widetilde{x}_\alpha)_{\alpha \in a}$ still is τ-convergent, the limit now being $\widetilde{k}$, which is obtained from k by leaving all coordinates $n \neq 2i$ unchanged and setting $\widetilde{k}^{2i} = k^{2i-1}$. Hence we have reduced the situation to the one considered in the above paragraph and thus have arrived at the desired contradiction.

The proof of properties (i), (ii) and (iii) now is complete. ■

Example 2.2 *There is an economy (X, τ, K, M, π) verifying the assumptions of [K 81, Theorem 4] and bundles x_1, x_2, x_3, x_4 in X such that $\underline{\pi}(x_i) < \overline{\pi}(x_i)$, for $i = 1, \ldots, 4$, and the consistent prices for these bundles equal $[\underline{\pi}(x_1), \overline{\pi}(x_1)]$, $]\underline{\pi}(x_2), \overline{\pi}(x_2)]$, $[\underline{\pi}(x_3), \overline{\pi}(x_3)[$ and $]\underline{\pi}(x_4), \overline{\pi}(x_4)[$ respectively.*

In other words, all possibilities of closed, half-open or open intervals of consistent prices may occur.

Proof Let $(\Omega, \mathcal{F}, \mathbb{P})$ equal $([0,1], \text{Borel}([0,1]), \lambda)$, with λ denoting Lebesgue-measure and $(X, \tau, K) = (L^2[0,1], \tau_{\|.\|_2}, L^2_+ \backslash \{0\})$ and choose three elements g_1, g_2, g_3 in $L^\infty[0,1]$ with disjoint support and expectation equal to zero, such that g_1 does not attain its supremum and its infimum at a set of positive measure, g_2 attains its supremum on a set of positive measure but not its infimum, while g_3 attains the supremum as well as the infimum on a set of positive measure.

For example, we may take, with ω ranging in $[0,1]$,

$$\begin{aligned} g_1(\omega) &= \left(\omega - \frac{1}{6}\right)\chi_{[0,\frac{1}{3}]} \\ g_2(\omega) &= \left(\omega - \frac{1}{2}\right)\chi_{[\frac{1}{3},\frac{1}{2}]} + \frac{1}{12}\chi_{[\frac{1}{2},\frac{2}{3}]} \\ g_3(\omega) &= -\chi_{[\frac{2}{3},\frac{5}{6}]} + \chi_{[\frac{5}{6},1]}. \end{aligned} \tag{2}$$

Let M be the subspace of X of codimension 3 defined by

$$M = \{x \in L^2[0,1] : (x, g_1) = (x, g_2) = (x, g_3) = 0\},$$

and let π be the linear functional on M defined by taking expectation

$$\pi(x) = E[x] = (x, \mathbf{1}), \qquad \text{for } x \in M.$$

In this case we can explicitly determine the set Ψ of linear extensions of π to M which are strictly positive on K;

$$\begin{aligned} \Psi &= \{g = \mathbf{1} + \mu_1 g_1 + \mu_2 g_2 + \mu_3 g_3 : \mathbb{P}[g > 0] = 1\} \\ &= \{g = \mathbf{1} + \mu_1 g_1 + \mu_2 g_2 + \mu_3 g_3 : \mu_1 \in [-6, 6], \\ &\qquad \mu_2 \in]-12, 6], \mu_3 \in]-1, 1[\} \end{aligned} \tag{3}$$

To find a bundle $x_1 \in L^2[0,1]$ such that the set of consistent prices is a closed interval, it suffices to choose an element x_1 supported by $[0, \frac{1}{3}]$ such that $(x_1, g_1) > 0$; in this case the set of consistent prices for x_1 equals the non-degenerate closed interval $[\underline{\pi}(x_1), \overline{\pi}(x_1)] = [(x_1, \mathbf{1} - 6g_1), (x_1, \mathbf{1} + 6g_1)]$.

To find x_2 (resp. x_3) we choose x_2 (resp. x_3) in $L^2[0,1]$ supported by $[\frac{1}{3}, \frac{2}{3}]$ such that $(x_2, g_2) > 0$ (resp. $(x_2, g_2) < 0$), so that the set of consistent prices equals $](x_2, \mathbf{1} - 12g_2), (x_2, \mathbf{1} + 6g_2)]$ (resp. $[(x_3, \mathbf{1} + 6g_2), (x_3, \mathbf{1} - 12g_2)[$).

Finally, if we choose x_4 to be supported by $[\frac{2}{3}, 1]$ and such that $(x_4, g_3) \neq 0$, we obtain an open interval of consistent prices.

The proof of the assertions of Example 2.1 thus is finished. ■

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