**On Certain Probabilities Equivalent to Wiener Measure, d'après Dubins, Feldman, Smorodinsky and Tsirelson**

W. Schachermayer

Institut für Statistik der Universität Wien,
Brünnerstr. 72, A-1210 Wien, Austria.

Abstract. L. Dubins, J. Feldman, M. Smorodinsky and B. Tsirelson gave an example of an equivalent measure $Q$ on standard Wiener space such that each adapted $Q$-Brownian motion generates a strictly smaller filtration then the original one. The construction of this important example is complicated and technical.

We give a variant of their construction which differs in some of the technicalities but essentially follows their ideas, hoping that some readers may find our presentation easier to digest than the original papers.

# 1. Introduction

This paper grew out of the author's attempt to understand the construction of the admirable paper [DFST 96] as well as its extensions given in[FT 96] and [F 96].

Here is their main result:

**1.1 Theorem.** *(Dubins, Feldman, Smorodinsky, Tsirelson):*
*Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion defined on a stochastic base $(\Omega, \mathcal{F}, \mathbb{P})$ and its natural filtration $(\mathcal{F}_t)_{t\geq 0}$.*

*For $\varepsilon > 0$, there is a probability measure $Q$ on $\mathcal{F}$, equivalent to $\mathbb{P}$, with $1-\varepsilon \leq \frac{dQ}{d\mathbb{P}} \leq 1+\varepsilon$ and such that for every $(\mathcal{F}_t)_{t\geq 0}$-adapted process $B' = (B'_t)_{t\geq 0}$ which is a standard Brownian motion under $Q$ (relative to the filtration $(\mathcal{F}_t)_{t\geq 0}$), the process $B'$ generates a strictly smaller filtration than $(\mathcal{F}_t)_{t\geq 0}$.*

We refer to [SY 81], [RY 91], p. 336, [RY 94] p. 210 and [DFST 96] for an account on the significance of this theorem, which settled a 15-year-old question related to the Girsanov-transformation.

Let us also mention that recently B. Tsirelson [T 97] (see also [EY 98] and [BEKSY 98]) gave another example of a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, namely the space generated by a Walsh-martingale, which displays similar features as the present example $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q)$: both examples are filtered probability spaces of "instant dimension 1" and not generated by a Brownian motion; the example in [T 97] is even robust under an equivalent change of measure (while the present one, of course, is not). These two examples are, nevertheless, different in spirit: roughly speaking in the present example the argument is based on the independence of the increments of Brownian motion while the example from [T 97] is based on the difference of Walsh-martingales and Brownian motion, when these processes hit zero.

The author frankly admits that he found it quite hard to understand the construction in [DFST 96]. After having paved his own way through the construction he thought that it might be helpful to the probability community to write up his understanding of the construction in order to give a somewhat different presentation of the example. However, no claims of originality are made (we just translate the ideas from [DFST 96] into a slightly alternative language) and we are not even sure whether our construction is "simpler" (of course, it seems simpler to the author; as usual in Mathematics, everything that you know how to do, seems simple to you). There are some technical differences in the construction of the present paper, as compared to [DFST 96]: firstly, we include the strengthening of the construction obtained in [FT 96], i.e., the control on the $L^\infty$-norm rather than on the $L^2$-norm of the Radon-Nikodym-derivative $\frac{dQ}{d\mathbb{P}}$, from the very beginning into our construction (at little extra cost). This is natural, as the splitting into two steps (as in [DFST 96] and [FT 96]) apparently is only due to the way these authors gradually improved their example. Secondly we isolate a crucial step of the construction of [DFST 96] into the elementary combinatorial lemma 2.7 below, which — at least to the author — also allows for some intuitive understanding.

As regards the final strengthening by J. Feldman [F 96] we don't have any contribution: let us just note that this strengthening can be put on top of our example exactly in the same way as it was originally put on top of the example from [DFST 96] and [FT 96].

We have made an effort to keep our presentation entirely selfcontained; but, of course, we strongly recommend the reader to have a copy of [DFST 96] at hand.

My sincere thanks go to J. Feldman and M. Smorodinsky for a pleasant conversation on this topic and in particular to M. Smorodinsky for an inspiring talk in June 1997 at the Schrödinger Institute, Vienna, as well as to M. Emery and M. Yor for making me familiar with the content of the papers [T 97], [EY 98] and [BEKSY 98] and in particular to M. Emery for a lot of help and advice in the final redaction of the paper. After the completion of a first version of the present paper, M. Emery also has given a further variant of the construction of [DFST 96] as well as some more general results [BE 99].

## 2. The Example

Let $X = (X_0, X_1, \cdots)$ be a real valued stochastic process defined on a stochastic base $(\Omega, \mathcal{F}, \mathbb{P})$. We shall look at the process "in reverse order", i.e., we define the filtration $(\mathcal{F}_n)_{n=0}^\infty$ to be

$$\mathcal{F}_n = \sigma(X_n, X_{n+1}, \cdots).$$

In the present paper we always shall assume that $X$ is tail-trivial, i.e., that the sigma-algebra $\mathcal{F}_\infty = \bigcap_{n=0}^\infty \mathcal{F}_n$ only consists of sets of probability zero or one.

The subsequent definition describes the way in which the independence of Brownian increments will come into play. As we have learned from M. Smorodinsky the idea behind this definition goes back to P. Lévy (in [V 95], p. 756 it is referred to as the Lévy-Bernstein-Rosenblatt problem):

2.1 Definition (compare [S 98]). A *parametrisation* of the process $X$ is given by a two-dimensional process $(\tilde{X}, Y) = (\tilde{X}_n, Y_n)_{n=0}^{\infty}$ defined on a stochastic base $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a sequence $(f_n)_{n=1}^{\infty}$ of deterministic Borel-measurable functions defined on $[0,1] \times \mathbb{R}^{\mathbb{N}}$ such that

(i) the processes $X$ and $\tilde{X}$ are identical in law,
(ii) the sequence $(Y_n)_{n=0}^{\infty}$ is a sequence of i.i.d. random variables uniformly distributed on $[0,1]$ and such that $Y_n$ is independent of $(\tilde{X}_i)_{i=n+1}^{\infty}$,
(iii) the equation

$$\tilde{X}_{n-1}(\omega) = f_{n-1}(Y_{n-1}(\omega), \tilde{X}_n(\omega), \tilde{X}_{n+1}(\omega), \cdots)$$

holds true, for each $n \geq 1$ and almost each $\omega$.

We call the parametrisation *generating* if, in addition, for each $n$, the random variable $X_n$ is $\sigma(Y_n, Y_{n+1}, \cdots)$-measurable.

We have been somewhat pedantic in the above definition, as regards the joining of the processes $X$ and $Y$, by distinguishing between the processes $X$ and $\tilde{X}$ to have a safe ground for the subsequent, rather subtle, considerations about the sigma-algebras which are generated by $Y$ and $\tilde{X}$ rather than by $Y$ and $X$ (the latter being, strictly speaking, defined on different stochastic bases). But, if no confusion can arise, we shall follow the common habit in probability theory and write $X$ instead of $\tilde{X}$.

Assertion (iii) requires that, for each $n$, there is a deterministic rule, prescribed by the functions $f_{n-1}$, such that, for almost each $\omega$, we may determine the value $X_{n-1}(\omega)$ from the history $(X_n(\omega), X_{n+1}(\omega), \cdots)$ and the "innovation" $Y_{n-1}(\omega)$, the latter coming from a sequence of *independent* random variables. It is easy to see that any real-valued process $X$ (in fact, any process taking its values in a polish space) admits a parametrisation. The notion of a *generating parametrisation* captures the intuitive idea of a parametrisation which is chosen in such a way that we can determine (a.s.) the value of $X_n(\omega)$ by only looking at the history $(Y_n(\omega), Y_{n+1}(\omega), \cdots)$ of the "innovations".

It is rather obvious that a process $X$ admitting a generating parametrisation has to be tail-trivial: indeed, suppose to the contrary that there is a set $A \in \mathcal{F}_\infty = \bigcap_{n=0}^{\infty} \mathcal{F}_n$ with $0 < \mathbb{P}[A] < 1$ and suppose that $X$ admits a generating parametrisation: then the set $A$ is in $\sigma(X_0, X_1, \cdots)$ and is independent of $(Y_n)_{n=0}^{\infty}$ and therefore not in $\sigma(Y_0, Y_1, \cdots)$, a contradiction to the requirements of definition 2.1.

But the converse does not hold true, i.e., a tail-trivial process $X$ does not, in general, allow a generating parametrisation. This highly non-trivial and remarkable fact was first proved by A. Vershik [V 70], [V 73]. We refer to ([DFST 96], p. 885) and [S 98] for a presentation of this example.

In fact, the construction given in [DFST 96] and its presentation in the present paper is just an example displaying the phenomenon of a tail trivial $\{-1, +1\}$-valued process $X$ not admitting a generating parametrisation and such that, in addition, the process $(X_n)_{n=0}^{\infty}$ is obtained from an i.i.d. sequence $(\varepsilon_n)_{n=0}^{\infty}$ of Bernoulli-variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ by putting a slightly altered equivalent measure $Q$ on $(\Omega, \mathcal{F})$.

We start by giving an easy motivating example which should help to develop some intuition for the concept of a *generating parametrisation* (we have learned it from M. Smorodinsky and found it illuminating despite its simplicity). As the example is not needed for the sequel the reader may just as well skip it.

EXAMPLE. (compare [V 95], p. 756) Let $0 \leq \eta < \frac{1}{2}$ and define the $\{-1,+1\}$-valued Markov process $(X_n)_{n=0}^{\infty}$ via the transition probabilities

$$\begin{aligned}
\mathbb{P}\big[X_{n-1} = +1 | X_n = +1\big] &= \frac{1}{2} + \eta, \\
\mathbb{P}\big[X_{n-1} = -1 | X_n = +1\big] &= \frac{1}{2} - \eta, \\
\mathbb{P}\big[X_{n-1} = +1 | X_n = -1\big] &= \frac{1}{2} - \eta, \\
\mathbb{P}\big[X_{n-1} = -1 | X_n = -1\big] &= \frac{1}{2} + \eta,
\end{aligned}$$

for each $n$. Clearly this well-defines a stationary tail-trivial Markov process $(X_n)_{n=0}^{\infty}$.

A possible way to define a parametrisation of this process goes as follows: let $(Y_n)_{n=0}^{\infty}$ be an i.i.d. sequence of random variables uniformly distributed in $[0,1]$. Define, for $m \in \mathbb{N}$, the process $\left(X_n^{(m)}\right)_{n=0}^{m}$ by letting $X_m^{(m)} = 1$ and, for $n = 1, \ldots, m$,

$$X_{n-1}^{(m)} = f_{n-1}(Y_{n-1}, X_n^{(m)}) = \begin{cases} \mathbb{1}_{(0,\frac{1}{2}+\eta)}(Y_{n-1}) - \mathbb{1}_{(\frac{1}{2}+\eta,1)}(Y_{n-1}) \\ \qquad \text{if } X_n^{(m)} = 1 \\ \mathbb{1}_{(0,\frac{1}{2}-\eta)}(Y_{n-1}) - \mathbb{1}_{(\frac{1}{2}-\eta,1)}(Y_{n-1}) \\ \qquad \text{if } X_n^{(m)} = -1 \end{cases}$$

One easily checks that, for $n \geq 0$ fixed, the sequence of random variables $\left(X_n^{(m)}\right)_{m=n}^{\infty}$ converges almost surely to a random variable $X_n$ and the sequence $(X_n)_{n=0}^{\infty}$ satisfies the above Markov transition probabilities as well as the relations

$$X_{n-1} = f_{n-1}(Y_{n-1}, X_n) = \begin{cases} \mathbb{1}_{(0,\frac{1}{2}+\eta)}(Y_{n-1}) - \mathbb{1}_{(\frac{1}{2}+\eta,1)}(Y_{n-1}) \\ \qquad \text{if } X_n = 1 \\ \mathbb{1}_{(0,\frac{1}{2}-\eta)}(Y_{n-1}) - \mathbb{1}_{(\frac{1}{2}-\eta,1)}(Y_{n-1}) \\ \qquad \text{if } X_n = -1 \end{cases}$$

Let us verify explicitly that this parametrisation is generating: let $n \in \mathbb{N}$ and $\omega \in \Omega$ be such that $Y_n(\omega) \notin (\frac{1}{2} - \eta, \frac{1}{2} + \eta)$. In this case $Y_n(\omega)$ determines already $X_n(\omega)$, regardless of the history $(X_{n+1}(\omega), X_{n+2}(\omega), \cdots)$. But from now on we know everything about the trajectory $(X_n(\omega), X_{n-1}(\omega), \cdots, X_0(\omega))$ by only looking at $(Y_n(\omega), Y_{n-1}(\omega), \cdots, Y_0(\omega))$: the number $X_{n-1}(\omega)$ then is a deterministic function of the numbers $Y_n(\omega)$ and $Y_{n-1}(\omega)$, and so on.

More formally, for each $n$, the sigma-algebra $\sigma(Y_n, Y_{n+1}, \cdots)$ contains the sigma-algebra $\sigma(X_n)$ on the set

$$A_n = \bigcup_{i \geq n} \left\{ Y_i \notin \left( \frac{1}{2} - \eta, \frac{1}{2} + \eta \right) \right\}.$$

Noting that $\mathbb{P}[A_n] = 1$, for each $n \in \mathbb{N}$, we deduce that $X_n$ is $\sigma(Y_n, Y_{n+1}, \cdots)$-measurable, which readily shows that the above parametrisation is generating. $\square$

We now give the basic example which relates the assertion of theorem 1.1 with the notion of a generating parametrisation.

**2.2 Lemma.** *Let $(B_t)_{t\geq 0}$ be a Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with its natural filtration $(\mathcal{F}_t)_{t\geq 0}$, and let $Q$ be a probability measure on $\mathcal{F}$ equivalent to $\mathbb{P}$.*

*Fix a sequence $(t_n)_{n=0}^\infty$ strictly decreasing to zero and define the process $(X_n)_{n=0}^\infty$ by letting*

$$X_n = \begin{cases} +1 & \text{if } B_{t_n} - B_{t_{n+1}} \geq 0 \\ -1 & \text{if } B_{t_n} - B_{t_{n+1}} < 0 \end{cases}$$

*Suppose that there is an $(\mathcal{F}_t)_{t\geq 0}$-adapted process $(B'_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F})$ which is a Brownian motion under $Q$, and such that $(B'_t)_{t\geq 0}$ generates the filtration $(\mathcal{F}_t)_{t\geq 0}$.*

*Then the process $X = (X_n)_{n=0}^\infty$ under the measure $Q$ admits a generating parametrisation.*

Proof. Let $(\tilde{Y}_n)_{n=0}^\infty$ be the sequence of random variables, defined on $(\Omega, \mathcal{F}, Q)$,

$$\tilde{Y}_n = \left( B'_t - B'_{t_{n+1}} \right)_{t_{n+1} \leq t \leq t_n}, \qquad n = 0, 1, \cdots$$

where $\tilde{Y}_n$ takes its values in the polish space $C[t_{n+1}, t_n]$. As the law of $\tilde{Y}_n$ is diffuse we may find Borel-isomorphisms $(i_n)_{n=0}^\infty$ from $C[t_{n+1}, t_n]$ to $[0,1]$ such that $Y_n = i_n \circ \tilde{Y}_n$ is uniformly distributed on $[0,1]$, which furnishes an i.i.d. sequence $(Y_n)_{n=0}^\infty$ of uniformly distributed $[0,1]$-valued random variables under the measure $Q$.

Note that, for each $n \in \mathbb{N}$, the sigma-algebras $\sigma((Y_k)_{k\geq n})$ and $\sigma((B'_t)_{0\leq t\leq t_n})$ coincide, and by assumption are equal to $\mathcal{F}_{t_n}$.

It follows that, defining the random variables $\varphi_n$ and $\psi_n$ by

$$\begin{aligned} \varphi_n &= \left( (B_t - B_{t_{n+1}})_{t_{n+1} \leq t \leq t_n}, X_{n+1}, X_{n+2}, \ldots \right) \\ \psi_n &= (Y_n, X_{n+1}, X_{n+2}, \ldots) \end{aligned}$$

taking their values in $C[t_{n+1}, t_n] \times \{-1, +1\}^{\mathbb{N}}$ and $[0,1] \times \{-1,+1\}^{\mathbb{N}}$ respectively, $\varphi_n$ and $\psi_n$ generate the same sigma-algebras (up to null-sets) on $\Omega$, if we equip the respective target spaces with their Borel sigma-algebras. In particular, we may find a nullset $N$ in $\Omega$ such that, for $\omega, \omega' \in \Omega \backslash N$, we have $\varphi_n(\omega) = \varphi_n(\omega')$ iff we have $\psi_n(\omega) = \psi_n(\omega')$. We infer that we may define a Borel map $F_n = F_n(y_n, x_{n+1}, x_{n+2}, \ldots)$ from $[0,1] \times \{-1,+1\}^{\mathbb{N}}$ to $C[t_{n+1}, t_n] \times \{-1,+1\}^{\mathbb{N}}$ inducing $\varphi_n \circ \psi_n^{-1}$ (to be precise: we define $F_n$ by letting $F_n \circ \psi_n(\omega) = \varphi_n(\omega)$, for $\omega \in \Omega \backslash N$, and extend $F_n$ in a Borel-measurable (but otherwise arbitrary) way from the range $\psi_n(\Omega)$ to the entire space $[0,1]\times\{-1,+1\}^{\mathbb{N}}$). Defining $f_n$ to be the sign of the first coordinate (i.e., the $C[t_{n+1},t_n]$-coordinate) of the function $F_n$, evaluated at $t_n$, we have found the parametrisation

$$X_n = f_n(Y_n, X_{n+1}, X_{n+2}, \dots).$$

The sequence $(f_n)_{n=0}^\infty$ therefore defines a parametrisation of the process $(X_n)_{n=0}^\infty$ and it is clear that the parametrisation is generating, as by hypothesis $\sigma(Y_n, Y_{n+1}, \dots) = \mathcal{F}_{t_n}$, for each $n \in \mathbb{N}$. $\square$

REMARK. We have used the concept of *generating parametrisation*, as in [S 98], instead of the concept of *substandard processes*, i.e., processes admitting a *standard extension*, as in [DFST 96], because we find the former notion more intuitive. Both concepts are equivalent and may be mutually translated one into the other (compare also [S 98]).

The message of lemma 2.2 is that the proof of theorem 1.1 may be reduced to a coin-tossing game, indexed by the negative numbers.

**2.2a Corollary.** *In order to prove theorem 1.1 it suffices to give a proof for the subsequent assertion:*

*Denote by $\lambda$ the Haar probability measure on the Borel sigma-algebra $\mathcal{B}$ of $\mathfrak{X} = \{-1,+1\}^{\mathbb{N}}$ and by $\varepsilon_n : \mathfrak{X} \to \{-1,+1\}$ the n'th coordinate projection. For $\varepsilon > 0$, there is a probability measure $\mu$ on $\mathfrak{X}$ with $1-\varepsilon \le \frac{d\mu}{d\lambda} \le 1+\varepsilon$ and such that the process $(\varepsilon_n)_{n=1}^\infty$, as defined on $(\mathfrak{X},\mathcal{B},\mu)$, does not admit a generating parametrisation.*

PROOF. Using the notation of lemma 2.2 consider $X = (X_n)_{n=1}^\infty$ defined there as a measurable map from $(\Omega,\mathcal{F})$ to $(\mathfrak{X},\mathcal{B})$. Assuming that there is a measure $\mu$ on $\mathfrak{X}$ satisfying the above assertion define the measure $Q$ on $\mathcal{F}$ by letting

$$\frac{dQ}{d\mathbb{P}}(\omega) = \frac{d\mu}{d\lambda}(X(\omega)), \qquad \text{for } \omega \in \Omega.$$

This definition is done in such a way that the process $(\varepsilon_n)_{n=1}^\infty$, defined on $(\mathfrak{X},\mathcal{B},\mu)$ and the process $(X_n)_{n=1}^\infty$, defined on $(\Omega,\mathcal{F},Q)$ are identical in law. By assumption, $(\varepsilon_n)_{n=1}^\infty$ does not admit a generating parametrisation under $\mu$, hence $(X_n)_{n=1}^\infty$ does not admit one either under $Q$. It follows from lemma 2.2 that $(\mathcal{F}_t)_{t\ge0}$ cannot be generated by a $Q$-Brownian motion $(B'_t)_{t\ge0}$. $\square$

The remainder of the paper will be dedicated to construct a measure $\mu$ on $\mathfrak{X}$ satisfying the assertion of the above corollary.

The principal component of the construction is given in the subsequent lemma.

Let us fix some notation: for $n \in \mathbb{N}$, we denote by $\mathfrak{X}_n$ the space $\{-1,+1\}^n$ and by $\lambda_n$, or just $\lambda$, if there is no danger of confusion, the uniform probability distribution on $\mathfrak{X}_n$. By $(\varepsilon_i)_{i=1}^n$ we denote the coordinate functions on $\mathfrak{X}_n$. Note that $(\varepsilon_i)_{i=1}^n$ is an i.i.d. sequence of Bernoulli-variables if we equip $\mathfrak{X}_n$ with the probability measure $\lambda_n$.

A little notational warning: in the subsequent lemma the time $i = 1,\cdots,p$ will "run into the future" as opposed to the setting above, and — when speaking about a stopping time — we shall refer to the filtration $(\mathcal{F}_i)_{i=0}^p$, where $\mathcal{F}_i = \sigma(\varepsilon_1,\cdots,\varepsilon_i)$.

**2.3. Lemma.** *Let $p \in \mathbb{N}$, $\frac{1}{8} > \kappa > \kappa/4 > \eta > 0$, and define the density process $Z = (Z_i)_{i=0}^p$ on $(\mathfrak{X}_p, \lambda_p)$ by $Z_0 = 1$,*

$$Z_i / Z_{i-1} = 1 + \eta \varepsilon_i, \qquad\qquad i = 1, \cdots, p$$

*and the stopping time $T$ by*

$$T = \inf\big\{1 \le i \le p : Z_i \notin [1 - (\kappa - \eta(1+\kappa)), 1 + (\kappa - \eta(1+\kappa))]\big\} \wedge p.$$

*Denote by $\hat{\mu}$ and $\mu$ respectively the probability measures on $\mathfrak{X}_p$ with Radon-Nikodym derivatives*

$$\frac{d\hat{\mu}}{d\lambda} = Z_p \quad \text{and} \quad \frac{d\mu}{d\lambda} = Z_T.$$

*We then have:*

(i) $\lambda[T < p] < \frac{4p\eta^2}{\kappa^2}$,

(ii) $1 - \kappa \le \frac{d\mu}{d\lambda} \le 1 + \kappa$

(iii) *for every pair $(f_i^\mu)_{i=1}^p$ and $(f_i^\lambda)_{i=1}^p$ of parametrisations of the coordinate process $(\varepsilon_1, \cdots, \varepsilon_p)$ under the measures $\mu$ and $\lambda$ respectively we have that*

$$\mathbb{P}\big[(f_i^\mu)_{i=1}^p = (f_i^\lambda)_{i=1}^p\big] \le \Big(1 - \frac{\eta}{2}\Big)^p + \frac{4p\eta^2}{\kappa^2}$$

Before aboarding the proof we want to clarify — again somewhat pedantically — the precise meaning of assertion (iii): we equip $\Omega = \mathfrak{X}_p \times [0,1]^p = \{-1,+1\}^p \times [0,1]^p$ with measure $\mathbb{P} = \lambda \otimes m$, where $m$ denotes the p-fold product of Lebesgue-measure on $[0,1]$. We denote by $X_1, \cdots, X_p, Y_1, \cdots, Y_p$ the projections to the coordinates of $\Omega$ and by $(x_1, \cdots, x_p, y_1, \cdots, y_p)$ the elements of $\Omega$. By the parametrisations $(f_i^\mu)_{i=1}^p$ and $(f_i^\lambda)_{i=1}^p$ we mean deterministic functions $f_i^\mu(y_i, x_{i-1}, \cdots, x_1)$ and $f_i^\lambda(y_i, x_{i-1}, \cdots, x_1)$ such that the processes $(f_i^\mu(Y_i, X_{i-1}, \cdots, X_1))_{i=1}^p$ and $(f_i^\lambda(Y_i, X_{i-1}, \cdots, X_1))_{i=1}^p$ are versions of the coordinate processes $(\varepsilon_i)_{i=1}^p$ defined on $(\mathfrak{X}_p, \mu)$ and $(\mathfrak{X}_p, \lambda)$ respectively.

Proof of Lemma 2.3. (i) Writing the defining equation of $(Z_i)_{i=0}^p$ as

$$Z_{i+1} - Z_i = Z_i \eta \varepsilon_i$$

we see that $Z$ is a martingale with respect to the measure $\lambda$ and that

$$\begin{aligned} \|Z_T - 1\|_{L^2(\lambda)}^2 &= \|Z_p^T - Z_0\|_{L^2(\lambda)}^2 \\ &= \sum_{i=1}^p \|Z_i^T - Z_{i-1}^T\|_{L^2(\lambda)}^2 \\ &\le p[\eta(1 + (\kappa - \eta(1+\kappa)))]^2 \le 2p\eta^2. \end{aligned}$$

Here we denoted by $Z^T = (Z_i^T)_{i=0}^p$ the stopped process $Z^T = (Z_{i\wedge T})_{i=0}^p$. Noting that on $\{T < p\}$ we have that $|Z_T - Z_0| > \kappa - \eta(1+\kappa)$ we get

$$\lambda[T < p] \leq \frac{2p\eta^2}{(\kappa - \eta(1+\kappa))^2} \leq \frac{4p\eta^2}{\kappa^2}.$$

(ii) is rather obvious as we have defined $T$ in such a way that $Z_T$ is certain to stay within $[1-\kappa, 1+\kappa]$.

(iii) We first reason with the measure $\hat{\mu}$ instead of $\mu$ and we shall write $f_i^{\hat{\mu}}$ for a parametrisation of $(\varepsilon_i)_{i=1}^p$ under $\hat{\mu}$. Note that, for every $1 \leq i \leq p$ and every $x_1, \cdots, x_{i-1}$ we have that

$$\begin{aligned} &\mathbb{P}[f_i^{\hat{\mu}} = 1 | X_1 = x_1, \cdots, X_{i-1} = x_{i-1}] = \frac{1+\eta}{2} \\ \text{and} \quad &\mathbb{P}[f_i^{\lambda} = 1 | X_1 = x_1, \cdots, X_{i-1} = x_{i-1}] = \frac{1}{2} \end{aligned}$$

Hence, conditionally on each set$\{X_1 = x_1, \ldots, X_{i-1} = x_{i-1}\}$ the event $\{f_i^{\hat{\mu}} = f_i^{\lambda}\}$ depends only on $Y_i$ and has probability at most $1 - \frac{\eta}{2}$. Using the independence of the random variables $Y_1, \ldots, Y_i$ we therefore get

$$\mathbb{P}[f_i^{\hat{\mu}} = f_i^{\lambda} | X_1, \ldots, X_{i-1}, Y_1, \ldots, Y_{i-1}] \leq 1 - \frac{\eta}{2}$$

which gives

$$\mathbb{P}[(f_j^{\hat{\mu}})_{j=1}^i = (f_j^{\lambda})_{j=1}^i] \leq (1 - \frac{\eta}{2}) \mathbb{P}[(f_j^{\hat{\mu}})_{j=1}^{i-1} = (f_j^{\lambda})_{j=1}^{i-1}]$$

and therefore

$$\mathbb{P}[(f_i^{\hat{\mu}})_{i=1}^p = (f_i^{\lambda})_{i=1}^p] \leq (1 - \frac{\eta}{2})^p.$$

To pass from $\hat{\mu}$ to $\mu$ note that on the set $\{T = p\}$ the measures $\mu$ and $\hat{\mu}$ coincide which, using (i), readily implies the inequality

$$P\left[(f_i^{\mu})_{i=1}^n = (f_i^{\lambda})_{i=1}^n\right] \leq \left(1 - \frac{\eta}{2}\right)^p + \frac{4p\eta^2}{\kappa^2}. \quad \square$$

The message of the above lemma is quite counter-intuitive and surprising, at least to the author (when choosing the parameters such that the bounds in (i) and (iii) are close to zero and in (ii) close to one): on one hand side (ii) asserts that the random variables $(X_1, \cdots, X_p) = (\varepsilon_1, \cdots, \varepsilon_p)$ have a very similar joint distribution under $\mu$ and under $\lambda$; on the other hand (iii) implies that if we try to parameterise the process $(X_1, \cdots, X_p)$ under $\mu$ and $\lambda$ respectively then, for each parametrisation $(f_i^{\mu})_{i=1}^p, (f_i^{\lambda})_{i=1}^p$, there are only few $\omega$'s such that $f_i^{\mu}(Y_i(\omega), X_{i-1}(\omega), \cdots X_1(\omega)) = f_i^{\lambda}(Y_i(\omega), X_{i-1}(\omega), \cdots X_1(\omega))$, for $i = 1, \cdots, p$. Loosely speaking: although the result of the random variable $(X_1, \cdots, X_n)$ is likely to be the same under $\mu$ as well as under $\lambda$ we cannot materialise this probable coincidence by a sequential pathwise procedure parametrised by independent increments on the coordinates $i = 1, \cdots, p$.

A similar interpretation of the above lemma goes as follows: There is a Borel-measurable transformation $T : (\mathfrak{X}_p \times [0,1], \lambda \otimes m) \longrightarrow \mathfrak{X}_p$ such that $T(\lambda \otimes m) = \mu$ and such that

$$\mathbb{P}[T(X_1, \cdots, X_p, Y) = (x_1, \cdots, x_p) | X_1 = x_1, \cdots, X_p = x_p] \geq 1 - \kappa,$$

for each $(x_1, \cdots, x_p) \in \mathfrak{X}_p$, where $\mathbb{P}$ denotes $\lambda \otimes m$. This is just a straightforward reinterpretation of the assertion $\frac{d\mu}{d\lambda} \geq 1 - \kappa$. In particular we have

$$\mathbb{P}[T(X_1, \cdots, X_p, Y) = (X_1, \cdots, X_p)] \geq 1 - \kappa.$$

On the other hand, (iii) can be interpreted as the fact that for every Borel-measurable transformation $T : \mathfrak{X}_p \times [0,1]^p \longrightarrow \mathfrak{X}_p$ which maps $\lambda \otimes m^p$ to $\mu$ and in addition, is $\sigma(X_1, \cdots, X_{i-1}, Y_i) \longrightarrow \sigma(X_1, \cdots, X_i)$ measurable, for each $i = 1, \cdots, p$, we have

$$\mathbb{P}\left[T(X_1, \cdots, X_p, Y_1, \cdots, Y_p) = (X_1, \cdots, X_p)\right] \leq \left(1 - \frac{\eta}{2}\right)^2 + \frac{4p\eta^2}{\kappa^2}.$$

At the danger of being repetitive, let us rephrase this once more in terms of a mind experiment: suppose you are told the laws $\lambda$ and $\mu$ as above and you are given a machine which produces an i.i.d. sequence $(Y_1, \cdots, Y_p)$ of $[0,1]$-valued uniformly distributed random variables. Define (w.l.g.) the functions

$$f_i^\lambda(Y_i) = \begin{cases} +1 \text{ if } Y_i \in & [0, \frac{1}{2}] \\ -1 \text{ if } Y_i \in & ]\frac{1}{2}, 1] \end{cases}$$

so that $(X_i)_{i=1}^p = (f_i^\lambda(Y_i))_{i=1}^p$ is a fair sequence of $p$ coin tosses. Now you are asked to define a (deterministic) mechanism which associates to every outcome $(x_1, \cdots, x_p) = (X_1(\omega), \cdots, X_p(\omega))$, possibly using the information of the underlying random numbers $(y_1, \cdots, y_p) = (Y_1(\omega), \cdots, Y_p(\omega))$, a "manipulated" outcome $(\check{x}_1, \cdots, \check{x}_p) = T(x_1, \cdots, x_p, y_1, \cdots, y_p)$ such that the process $(\check{X}_1, \cdots, \check{X}_p)$ has law $\mu$ and, in addition, this application of "corriger la fortune" should only be applied rather seldomly, i.e. $\mathbb{P}[(\check{X}_1, \cdots \check{X}_p) \neq (X_1, \cdots, X_p)]$ should be small. The question is: can you do this? The answer depends on the interpretation of what we mean by "deterministic mechanism". If we are allowed to first wait until we know the entire realisation $(x_1, \cdots, x_p)$, the answer is yes, as the map $T$ constructed above, $(\check{x}_1, \cdots, \check{x}_p) = T(x_1, \cdots, x_p, y)$ satisfies $\mathbb{P}[(\check{X}_1, \cdots, \check{X}_p) \neq (X_1, \cdots, X_p)] < \kappa$ (as random source $Y$ we may, e.g., take the fractional part of the random variable $2Y_1$). But if we are confined to make our choice "in real time" (compare ([T 97], def. 1.1 and the subsequent discussion) for a precise definition of this notion), i.e., we have to decide whether we let $\check{x}_i = x_i$ or $\check{x}_i \neq x_i$ after having only seen the outcomes $x_1, \cdots, x_{i-1}$ and using the information $y_i$, then the answer is no: assertion (iii) above implies that for each such rule $(f_i(y_i, x_{i-1}, \cdots, x_i))_{i=1}^p$ producing a process $(\check{X}_i)_{i=1}^p$ under the law $\mu$, the probability that we have to change $x_i$ into $\check{x}_i \neq x_i$, for at least one $i$, is close to one.

For the proof of theorem 1.1 we shall apply the above lemma in a slightly more technical form which we describe in the next lemma.

**2.4 Lemma.** *Let $p, \kappa, \eta$ be as in lemma 2.3 above and suppose we are given in addition $0 < \alpha < 1$.*

*Let $(\tau_i)_{i=1}^p, (\tau_i')_{i=1}^p$ be two elements in $\{-1,+1\}^p$ such that*

$$\#\{i : \tau_i \neq \tau_i'\} \geq \alpha p.$$

*Define two density processes $Z, Z'$ by letting $Z_0 = Z_0' = 1$ and*

$$Z_i/Z_{i-1} = 1 + \tau_i \eta \varepsilon_i, \quad Z_i'/Z_{i-1}' = 1 + \tau_i' \eta \varepsilon_i$$

*and two stopping times $T$ and $T'$ by*

$$\begin{aligned} T &= \inf\{1 \leq i \leq p : \{Z_i \notin [1 - (\kappa - \eta(1+\kappa)), 1 + (\kappa - \eta(1+\kappa))]\} \wedge p \\ T' &= \inf\{1 \leq i \leq p : \{Z_i' \notin [1 - (\kappa - \eta(1+\kappa)), 1 + (\kappa - \eta(1+\kappa))]\} \wedge p \end{aligned}$$

*and by $\mu, \hat{\mu}, \mu', \hat{\mu}'$ the measures with densities*

$$\frac{d\hat{\mu}}{d\lambda} = Z_p, \frac{d\mu}{d\lambda} = Z_T, \quad \frac{d\hat{\mu}'}{d\lambda} = Z_p', \frac{d\mu'}{d\lambda} = Z_{T'}'.$$

*We then have*

(i) $\lambda[T < p, T' < p] < \frac{8p\eta^2}{\kappa^2}$,

(ii) $1 - \kappa \leq \frac{d\mu}{d\lambda} \leq 1 + \kappa$ *and* $1 - \kappa \leq \frac{d\mu'}{d\lambda} \leq 1 + \kappa$,

(iii) *for every pair $(f_i^\mu)_{i=1}^p$ and $(f_i^{\mu'})_{i=1}^p$ of parametrisations of the coordinate process $(\varepsilon_1, \cdots, \varepsilon_p)$ under the measures $\mu$ and $\mu'$ respectively we have that*

$$\mathbb{P}\big[(f_i^\mu)_{i=1}^p = (f_i^{\mu'})_{i=1}^p\big] \leq (1-\eta)^{\alpha p} + \frac{8p\eta^2}{\kappa^2}$$

The proof of lemma 2.4 is analogous to that of 2.3 and therefore skipped.

We now indicate for which values of the parameters $p, \kappa, \eta, \alpha$ we shall apply lemma 2.4 in our subsequent inductive construction indexed by $k = k_0, k_0 + 1, \cdots$; in the sequel we shall (almost) always remain the following relations between the integers $k, n$ and $p$:

$$\begin{aligned} n &= 2^k \\ p &= 2^{k-1} = n/2. \end{aligned}$$

This rather peculiar notation comes from the fact that we want to stick as close as possible to the notation in [DFST 96], who used the symbols $k$ and $n$ in a similar way as we do and, on the other hand, we want to avoid constant use of the notation $k - 1$ and $n/2$ for quantities which will constantly be used.

**2.5 Corollary.** *For $k \in \mathbb{N}$ we shall choose*

$$\begin{aligned} p &= p_k = 2^{k-1}, \\ \alpha &= \alpha_k = p_k^{-\frac{1}{4}}, \\ \kappa &= \kappa_k = k^{-2}, \\ \eta &= \eta_k = k^{-3} 2^{-k/2}. \end{aligned}$$

*Using these parameters in lemma 2.4 we obtain, for $k$ sufficiently large, the estimates*

(ii) $1 - k^{-2} \leq \frac{d\mu}{d\lambda} \leq 1 + k^{-2}$ *and* $1 - k^{-2} \leq \frac{d\mu'}{d\lambda} \leq 1 + k^{-2}$

(iii) $\mathbb{P}[(f_i^{\mu})_{i=1}^p = (f_i^{\mu'})_{i=1}^p] \leq 5k^{-2}$.

Proof. We have to estimate the quantities $(1-\eta)^{\alpha p}$ and $\frac{8p\eta^2}{\kappa^2}$ in assertion (iii) of lemma 2.4:

$$(1-\eta_\kappa)^{\alpha_\kappa p_\kappa} = (1-k^{-3}2^{-k/2})^{2^{k-1}\cdot 2^{-\frac{k-1}{4}}} \approx (1-2^{-k/2})^{2^{\frac{3k}{4}}} = ((1-2^{-k/2})^{2^{k/2}})^{2^{k/4}} \approx e^{-2^{k/4}}$$

and

$$\frac{8p\eta^2}{\kappa^2} = \frac{8k^{-6}2^{k-1} \cdot 2^{-k}}{k^{-4}} = 4k^{-2}. \quad \square$$

We have used in the above proof the symbol $\approx$ to describe an approximate equality and we shall freely continue to do so when it is clear that the asymptotic approximations work good enough to prove the desired estimates.

We now can formulate the result which parallels the "Fundamental Lemma" of ([DFST 96], p. 894):

**2.6 Fundamental Lemma.** *For $k$ large enough, there is a family $(\mu_j)_{j=1}^{2^{2^n}} = (\mu_j)_{j=1}^{2^{2^{k+1}}}$ of probability measures on $\mathfrak{X}_n = \mathfrak{X}_{2^k}$ such that*

$$\text{(ii)} \qquad 1 - k^{-2} \leq \frac{d\mu_j}{d\lambda} \leq 1 + k^{-2}, \qquad j = 1, \cdots, 2^{2n},$$

*(iii) for every pair $j \neq j'$ and parametrisations $(f_i^{\mu_j})_{i=1}^n$ and $(f_i^{\mu_{j'}})_{i=1}^n$ of the coordinate process $(X_1, \cdots, X_n)$ on $\mathfrak{X}_n$ under the measures $\mu_j$ and $\mu_{j'}$ respectively, we have*

$$\mathbb{P}\left[(f_i^{\mu_j})_{i=1}^n = (f_i^{\mu_{j'}})_{i=1}^n\right] \leq 6k^{-2}.$$

Of course, the idea to prove the fundamental lemma is to apply lemma 2.4 and corollary 2.5, where we let $n = p$ (in contrast to our above agreement on notation $p = n/2$; the reason why we finally have to take $p = n/2$ will soon become clear): we would like to find $2^{2n}$ many different sequences $\tau_i(j)_{i=1}^n$ taking their values in $\{-1, +1\}$, where $1 \leq j \leq 2^{2n}$, such that, for every fixed pair $j \neq j'$, we have, for at least $n^{3/4}$ many $i$'s, that $\tau_i(j) \neq \tau_i(j')$. We advise the reader to convince her- or himself that — if such a choice $(\tau_i(j))_{i=1}^n)_{j=1}^{2^{2n}}$

were indeed possible — it were straightforward to deduce the fundamental lemma from lemma 2.4.

However, life is not always as nice and easy as we would like it to be: there is no sequence $(\tau_i(\cdot))_{i=1}^n$ of $\{-1,+1\}$-valued functions on a set of cardinality $2^{2n}$ such that, for $j \neq j'$, we have $\tau_i(j) \neq \tau_i(j')$ for at least $n^{3/4}$ many $i$'s. In fact, such a sequence $(\tau_i(\cdot))_{i=1}^n$ cannot even separate the points of a set of cardinality $2^{2n}$ (it needs $2n$ functions to do this job); hence there always will be some $j \neq j'$ such that $\tau_i(j) = \tau_i(j')$, for all $i = 1, \cdots, n$.

So we have to proceed in a more sophisticated way: note that — in spite of the above sad news — for a *typical* choice of $j \neq j'$ there will be approximately $n/2$ many (i.e., much more than the required $n^{3/4}$ many) $i$'s such that $\tau_i(j) \neq \tau_i(j')$ if we take $(\tau_i)_{i=1}^n$ to be an independent sequence of functions defined on $(\mathfrak{X}_{2n}, \lambda_{2n})$ assuming the values $+1$ and $-1$ with probability $\frac{1}{2}$ (we now identify the set $\{j : 1 \leq j \leq 2^{2n}\}$ with $\mathfrak{X}_{2n}$ equipped with measure $\lambda_{2n}$). The basic idea is to consider not only one sequence $(\tau_i)_{i=1}^p$ (from now on we are generous and use only $p = \frac{n}{2}$ many functions) but a large collection $((\tau_i^r)_{i=1}^p)_{r=1}^{2^p}$ of such sequences, which we may think of as applying an i.i.d. sequence $(\tau_i)_{i=1}^p$ as above to $2^p$ many random permutations of the set $\mathfrak{X}_{2n}$. If we do this it seems quite intuitive that for *the overwhelming majority of pairs* $j \neq j'$ we have that *for most of the* $1 \leq r \leq 2^p$ we have $\tau_i^r(j) \neq \tau_i^r(j')$ for at least $n^{3/4}$ many $i$'s.

The subsequent combinatorial lemma, whose proof is based on the above ideas, shows that we even can replace the term *for the overwhelming majority of pairs* $j \neq j'$ by the term *for each pair* $j \neq j'$.

**2.7 Combinatorial Lemma.** *Letting $p = 2^{k-1}$ and $n = 2^k$, for $k$ sufficiently large, there is a family $((\tau_i^r(\cdot))_{i=1}^p)_{r=1}^{2^p}$ of $\{-1,+1\}$-valued functions defined on the set $\mathfrak{X}_{2n} = \{-1,+1\}^{2n}$ such that, for each pair $j \neq j'$ in $\mathfrak{X}_{2n}$, we have*

$$\frac{\#\{r : \#\{i : \tau_i^r(j) \neq \tau_i^r(j')\} \geq n^{3/4}\}}{2^p} \geq 1 - p^{-1/2}.$$

The proof of the lemma relies on elementary combinatorics and is somewhat lengthy. Also we suspect that there are much stronger results known in the combinatorial literature (but not known to the author). For these reasons we banned the proof of the combinatorial lemma 2.7 to the appendix.

Proof of the Fundamental Lemma 2.6. We shall define the measures $(\mu_j)_{j=1}^{2^{2n}}$ on $\mathfrak{X}_n$ by defining the density processes $(Z_i^j)_{i=1}^n$ with respect to the measure $\lambda$ on $\mathfrak{X}_n$.

For the first $p = n/2$ coordinates, we *don't do anything*! We simply let

$$Z_i^j = 1, \qquad \text{for } i = 1, \cdots, p, \quad j = 1, \cdots, 2^{2n}.$$

The first $p = n/2$ coordinates are only used to create $2^p$ many atoms in $\sigma(X_1, \cdots, X_p)$ defined by $\{X_1 = \pm 1, \cdots, X_p = \pm 1\}$, for all choices of $\pm 1$, where $X_1, \cdots, X_p$ denote the first $p$ coordinate functions on $\mathfrak{X}_n$. We enumerate these atoms by $I_1, \cdots, I_r, \cdots, I_{2^p}$.

Identifying the set $\{j : 1 \leq j \leq 2^{2n}\}$ with $\mathfrak{X}_{2n}$ apply lemma 2.7 to choose the functions $(\tau_i^r(j))_{i=1}^p$ satisfying

$$\frac{\#\{r : \#\{i : \tau_i^r(j) \neq \tau_i^r(j')\} \geq n^{3/4}\}}{2^p} \geq 1 - p^{-1/2}.$$

Now define, for $1 \leq j \leq 2^{2n}$, and $p \leq i < n$,

$$\left(Z_{i+1}^j / Z_i^j\right) \chi_{I_r} = 1 + \tau_i^r(j)\eta\varepsilon_i, \qquad r = 1, \cdots, 2^p$$

where from now on the parameters $p, \alpha, \kappa, \eta$ are understood to denote the parameters $p_k, \alpha_k, \kappa_k, \eta_k$ defined in corollary 2.5. We again stop the density processes at time

$$T_j = \inf\left\{1 \leq i \leq n : Z_i^j \notin [1 - (\kappa - \eta(1+\kappa)), 1 + (\kappa - \eta(1+\kappa))]\right\} \wedge n$$

and define

$$\frac{d\mu_j}{d\lambda} = Z_{T_j}^j.$$

Assertion (ii) of the fundamental lemma now follows from assertion (ii) of corollary 2.5.

To prove (iii) fix $j \neq j'$: on at least $(1-p^{-1/2})2^p$ many of the atoms $I_r$ the (renormalized) restrictions of $\mu_j$ and $\mu_{j'}$ to the atom $I_r$ satisfies the hypotheses of lemma 2.4 and corollary 2.5. Hence, for any pair of parametrisations $(f_i^{\mu_j})_{i=1}^n$ and $(f_i^{\mu_{j'}})_{i=1}^n$ of the coordinate process $(X_1, \cdots, X_n)$ under the measures $\mu_j$ and $\mu_{j'}$ respectively we have

$$\mathbb{P}[(f_i^{\mu_j})_{i=1}^n = (f_i^{\mu_{j'}})_{i=1}^n | (X_1, \cdots, X_p) \in I_r] \leq 5k^{-2}$$

for at least $(1 - p^{-1/2})2^p = (1 - 2^{-\frac{k-1}{2}})2^p$ many $r's$. Hence

$$\mathbb{P}[(f_i^{\mu_j})_{i=1}^n = (f_i^{\mu_{j'}})_{i=1}^n] \leq 5k^{-2} + 2^{-\frac{k-1}{2}}$$

which shows assertion (iii) and finishes the proof of the Fundamental Lemma. □

Proof of Theorem 1.1. Similarly as in [DFST 96] we only have to paste the ingredients together which are provided by the Fundamental lemma, in order to construct a probability measure $\mu$ on $\mathfrak{X} = \{-1, +1\}^{\mathbb{N}}$ satisfying the requirements of corollary 2.2a: choose $k_0$ to be large enough such that, for $k \geq k_0$, the assertions of the Fundamental Lemma hold true and such that

$$\prod_{k=k_0}^{\infty} (1 + k^{-2}) < 1 + \varepsilon \text{ and } \prod_{k=k_0}^{\infty} (1 - 6k^{-2}) > \frac{3}{4},$$

where $\frac{1}{2} > \varepsilon > 0$ is taken from the statement of Corollary 2.2a.

Let $\mathfrak{X}$ be the compact space

$$\mathfrak{X} = \prod_{k=k_0}^{\infty} \mathfrak{X}_{2^k} = \prod_{k=k_0}^{\infty} \{-1, +1\}^{2^k},$$

and define, for $k \geq k_0$, the Markov transition probabilities $(\mu_{x_{k+1}})_{x_{k+1} \in \mathfrak{X}_{2^{k+1}}}$ to be the family of probability measures on $\mathfrak{X}_{2^k}$ given by the Fundamental Lemma 2.6, where we identify the set $\{j : 1 \leq j \leq 2^{2^{k+1}}\}$ with the set $\{x_{k+1} : x_{k+1} \in \mathfrak{X}_{2^{k+1}}\}$ by an arbitrary bijection.

Denote by $V_{x_{k+1}}(x_k)$ the Radon-Nikodym derivative of $\mu_{x_{k+1}}$ with respect to Haar measure $\lambda_{2^k}$ on $\mathfrak{X}_{2^k}$, i.e.

$$V_{x_{k+1}} = \frac{d\mu_{x_{k+1}}}{d\lambda_{2^k}},$$

and by $Z$ the density function on $\mathfrak{X}$,

$$Z(x) = \prod_{k=k_0}^{\infty} V_{x_{k+1}}(x_k)$$

where $x = (x_k)_{k=k_0}^{\infty} \in \mathfrak{X}$. By assertion (ii) of the fundamental lemma 2.6 and the above choice of $k_0$ we have $\|Z - 1\|_\infty < \varepsilon$ and the measure $\mu$ on $\mathfrak{X}$ defined by

$$\frac{d\mu}{d\lambda} = Z$$

is the unique probability measure on the Borel sets of $\mathfrak{X}$ inducing the transition probabilities $(\mu_{x_{k+1}})_{x_{k+1} \in \mathfrak{X}_{2^{k+1}}}$.

We still have to show that the coordinate process on $\mathfrak{X}$, which we now denote by $X$, under the measure $\mu$ does not admit a generating parametrisation, which will finish the proof of theorem 1.1 by corollary 2.2a. So, fix a parametrisation

$$\Big(\Big(\tilde{X}_{k,i}\Big)_{i=1}^{2^k}\Big)_{k=k_0}^{\infty} = \Big(\Big(f_{k,i}(Y_{k,i}, \tilde{X}_{k,i+1}, \ldots, \tilde{X}_{k,2^k}, \tilde{X}_{k+1,1}, \ldots)\Big)_{i=1}^{2^k}\Big)_{k=k_0}^{\infty}.$$

of the process $X$, where we now are careful to write $\tilde{X}$ instead of $X$ (compare definition 2.1 and the subsequent discussion). Assuming that the parametrisation is generating let us work towards a contradiction.

To alleviate notation, we write $y_k$ and $x_k$ (resp. $Y_k$ and $X_k$ or $\tilde{X}_k$ if we refer to random variables) for the elements $y_k = (y_{k,i})_{i=1}^{2^k}$ and $x_k = (x_{k,i})_{i=1}^{2^k}$ in $\mathfrak{X}_{2^k}$, and $f_k$ for the $\mathfrak{X}_{2^k}$-valued function $f_k = (f_{k,i})_{i=1}^{2^k}$. We then may write the parametrisation as

$$(\tilde{X}_k)_{k=k_0}^{\infty} = \Big(f_k(Y_k, \tilde{X}_{k+1}, \tilde{X}_{k+2}, \ldots)\Big)_{k=k_0}^{\infty},$$

with the interpretation, that the components $(f_{k,i})_{i=1}^{2^k}$ are defined inductively (for $i = 2^k, 2^k - 1, \ldots, 1$) by the above more explicit formula, letting $x_{k,i} = f_{k,i}(y_{k,i}, x_{k,i+1}, \ldots, x_{k,2^k}, x_{k+1}, \ldots)$.

To further alleviate notation, note that by the construction of the measure $\mu$ on $\mathfrak{X}$ the random variable $\tilde{X}_k = f_k(Y_k, \tilde{X}_{k+1}, \tilde{X}_{k+2}, \ldots)$ is independent of $\tilde{X}_{k+2}, \tilde{X}_{k+3}, \ldots$, conditionally on $\tilde{X}_{k+1}$. We therefore may assume w.l.g. that the parametrisation is of the form

$$\tilde{X}_k = f_k(Y_k, \tilde{X}_{k+1}), \qquad k = k_0, k_0 + 1, \ldots.$$

We now define, similarly as in [S 98], inductively the Borel functions $(g_k)_{k=k_0}^{\infty}$ by

$$\begin{aligned}
g_{k_0}(y_{k_0}, x_{k_0+1}) &= f_{k_0}(y_{k_0}, x_{k_0+1}) \\
g_{k_0+1}(y_{k_0}, y_{k_0+1}, x_{k_0+2}) &= g_{k_0}(y_{k_0}, f_{k_0+1}(y_{k_0+1}, x_{k_0+2})) \\
&\vdots \\
g_k(y_{k_0}, \dots, y_k, x_{k+1}) &= g_{k-1}(y_{k_0}, \dots, y_{k-1}, f_k(y_k, x_{k+1}))
\end{aligned}$$

so that, for each $k$, the random variable $g_k(Y_{k_0}, \dots, Y_k, \tilde{X}_{k+1})$ equals the random variable $\tilde{X}_{k_0}$ a.s.; the function $g_k$ describes how we may determine the random variable $\tilde{X}_{k_0}$ from the "past" $\tilde{X}_{k+1}$ and the "innovations" $Y_k, Y_{k-1}, \dots, Y_{k_0}$.

**Claim.** *For $k \geq k_0$ and $x_{k+1} \neq x'_{k+1}$, where $x_{k+1}$ and $x'_{k+1}$ are fixed elements of $\mathfrak{X}_{2^{k+1}}$, we have*

$$\tilde{\mathbb{P}}\left[g_k(Y_{k_0}, \dots, Y_k, x_{k+1}) \neq g_k(Y_{k_0}, \dots, Y_k, x'_{k+1})\right] > \prod_{j=k_0}^{k} \left(1 - 6j^{-2}\right) > \frac{3}{4},$$

*where $\tilde{\mathbb{P}}$ denotes, as in definition 2.1, the probability under which $(Y_k)_{k=k_0}^{\infty}$ is an i.i.d. sequence uniformly distributed on $[0,1]$.*

To verify the claim we proceed inductively on $k = k_0, k_0+1, \dots$: for $k = k_0$ the claim follows from the construction and assertion (iii) of the Fundamental Lemma 2.6. Now suppose that the claim holds true for $k-1$; applying assertion (iii) of the Fundamental Lemma again we obtain that, for $x_{k+1} \neq x'_{k+1}$,

$$\tilde{\mathbb{P}}\left[f_k(Y_k, x_{k+1}) \neq f_k(Y_k, x'_{k+1})\right] > 1 - 6k^{-2}.$$

Applying the inductive hypothesis on all pairs $(x_k, x'_k)$, $x_k \neq x'_k$ in $\mathfrak{X}_k$ that are assumed by $(f_k(Y_k, x_{k+1}), f_k(Y_k, x'_{k+1}))$ we have proved the above claim.

Now we shall use the assumption that the parametrisation $(f_k)_{k=k_0}^{\infty}$ is generating to obtain the desired contradiction: if $\tilde{X}_{k_0}$ is $\sigma(Y_{k_0}, Y_{k_{0+1}}, \dots)$-measurable we may find $k \geq k_0$ and a Borel function $G(y_{k_0}, \dots, y_k)$ such that

$$\tilde{\mathbb{P}}\left[\tilde{X}_{k_0} = G(Y_{k_0}, \dots, Y_k)\right] > \frac{7}{8},$$

or, written differently,

$$\tilde{\mathbb{P}}\left[g_k(Y_{k_0}, \dots, Y_k, \tilde{X}_{k+1}) = G(Y_{k_0}, \dots, Y_k)\right] > \frac{7}{8}.$$

As, for each $x_{k+1} \in \mathfrak{X}_{2^{k+1}}$, we have $(1-\varepsilon) \cdot 2^{-2^{k+1}} \leq \tilde{\mathbb{P}}[\tilde{X}_{k+1} = x_{k+1}] \leq (1+\varepsilon)2^{-2^{k+1}}$, and $(Y_{k_0}, \dots, Y_k)$ is independent of $\tilde{X}_{k+1}$ under $\mathbb{P}$, it follows that there are at least two elements $x_{k+1} \neq x'_{k+1}$ in $\mathfrak{X}_{2^{k+1}}$ such that

$$\tilde{\mathbb{P}}\big[g_k(Y_{k_0},\ldots,Y_k,x_{k+1}) = G(Y_{k_0},\ldots,Y_k)\big] > \frac{3}{4},$$
$$\textit{and}\quad \tilde{\mathbb{P}}\big[g_k(Y_{k_0},\ldots,Y_k,x'_{k+1}) = G(Y_{k_0},\ldots,Y_k)\big] > \frac{3}{4},$$

which implies

$$\tilde{\mathbb{P}}\left[g_k(Y_{k_0},\ldots,Y_k,x_{k+1}) = g_k(Y_{k_0},\ldots,Y_k,x'_{k+1})\right] > \frac{1}{2}.$$

This contradiction to the above claim finishes the proof of theorem 1.1. □

## Appendix

We now prove the combinatorial lemma 2.7. We consider the space $\mathfrak{X} = \mathfrak{X}_{p2^p} = \{-1,+1\}^{p2^p} = \{-1,+1\}^{2^{k-1}2^{2^{k-1}}}$ equipped with uniform distribution $\mathbb{P} = \lambda$. We denote by $x = ((x_i^r)_{i=1}^p)_{r=1}^{2^p}$ the elements of $\mathfrak{X}$ and by $((\tau_i^r)_{i=1}^p)_{r=1}^{2^p}$ the coordinate functions.

**A.1 Lemma.** *For $k$ large enough, $p = 2^{k-1}, n = 2^k$, and fixed $x_0 \in \mathfrak{X}$, the set*

$$A = \left\{ \begin{array}{c} x \in \mathfrak{X} : \textit{there are more than } p^{-1/2}2^p \textit{many } r's \textit{ for which} \\ \textit{there are less than } n^{3/4} \textit{ many } i's \textit{ with } \tau_i^r(x_0) \neq \tau_i^r(x) \end{array} \right\}$$

*satisfies $\lambda[A] < 2^{-2^p}$.*

Proof of lemma A.1. We may assume w.l.g. that $x_0 = (1,1,\cdots,1)$ so that $\tau_i^r(x_0) \neq \tau_i^r(x)$ iff $\tau_i^r(x) = -1$.

**Claim.** *For fixed $1 \le r \le 2^p$ and*

$$A_r = \{x : \textit{for less than } n^{3/4} \textit{many } i's \textit{ we have } \tau_i^r(x) = -1\}$$

*we have*

$$\mathbb{P}[A_r] \le 2^{-p/2}.$$

To show the claim we first estimate the probability of the set

$$B_r = \{x : \text{for exactly } n^{3/4} \text{many } i's \text{ we have } \tau_i^r(x) = -1\}$$

(assuming that $n^{3/4}$ is an integer). Using the estimate $\binom{n}{k} \le n^k$ we get

$$\begin{aligned} \mathbb{P}[B_r] = \binom{p}{n^{3/4}} 2^{-p} &\le 2^{-p}(p)^{(2p)^{3/4}} \\ &= (2^{-1}p^{2^{3/4}p^{-1/4}})^p. \end{aligned}$$

Noting that the term in the bracket tends to $\frac{1}{2}$, as $p$ increases, we obtain

$$\mathbb{P}[B_r] \leq 2^{-\frac{2p}{3}}, \qquad \text{for } k \geq k_0.$$

Finally we can estimate

$$\mathbb{P}[A_r] = (2p)^{3/4}\mathbb{P}[B_r] \leq 2^{-p/2}, \qquad \text{for } k \geq k_0,$$

which proves the claim.

Using the assertion of the claim we can estimate the probability of the event

$$B = \left\{ \begin{array}{l} x \in X : \text{ there are precisely } p^{-1/2}2^p \text{ many } r's \text{ for which} \\ \text{there are less than } p^{3/4} \text{ many } i's \text{ with } \tau_i^r(x) = -1 \end{array} \right\}.$$

Applying the inequality $\binom{n}{k} \leq (\frac{en}{k})^k$ we obtain

$$\begin{aligned} \mathbb{P}[B] &\leq \binom{2^p}{p^{-1/2}2^p} \cdot \mathbb{P}[A_r]^{p^{-1/2}2^p} \\ &\leq \left(\frac{e}{p^{-1/2}}\right)^{p^{-1/2}2^p} \cdot \left(2^{-p/2}\right)^{p^{-1/2}2^p} \\ &= \left(ep^{1/2}2^{-p/2}\right)^{p^{-1/2}2^p} \approx 2^{-p^{1/2}2^p}. \end{aligned}$$

This allows us to estimate

$$\begin{aligned} \mathbb{P}[A] &\leq 2^p \cdot \mathbb{P}[B] \\ &\leq 2^p \cdot 2^{-p^{1/2}2^p} \\ &= \left((2^p)^{2^{-p}} 2^{-p^{1/2}}\right)^{2^p}. \end{aligned}$$

Noting that, for $k$ tending to infinity, the term in the outer bracket tends to zero, and therefore is eventually less than $\frac{1}{2}$, we finished the proof. $\square$

Proof of lemma 2.7. Let $\mathfrak{X} = \mathfrak{X}_{p2^p}$ and $((\tau_i^r)_{i=1}^p)_{r=1}^{2^p}$ be as above and carry out the following inductive procedure: choose an arbitrary element $x_1 \in \mathfrak{X}$ and remove from $\mathfrak{X}$ the set

$$A(x_1) = \left\{ \begin{array}{l} x \in \mathfrak{X}, x \neq x_1 : \text{there are more than } p^{-1/2}2^p \text{ many } r's \text{ for which} \\ \text{there are less than } p^{3/4} \text{ many } i's \text{ with } \tau_i^r(x_1) \neq \tau_i^r(x) \end{array} \right\}.$$

The remaining set $\mathfrak{X} \backslash A(x_1)$ has probability bigger than $1 - 2^{-2^p}$ and therefore is non-empty, so that we can choose $x_2 \in \mathfrak{X} \backslash A(x_1)$.

Now remove the set $A(x_2)$, which is defined similarly, and choose $x_3 \in \mathfrak{X} \backslash \{A(x_1) \cup A(x_2)\}$. Continuing in an obvious way we may continue the procedure to obtain $2^{2n} = 2^{4p}$ many elements $(x_j)_{j=1}^{2^{2n}}$ before this procedure stops. (In fact we could even obtain in this way $2^{2^p}$ many elements $x_j$, which shows in particular how far the assertion of lemma 2.7 is from being sharp). Identifying the points $(x_j)_{j=1}^{2^{2n}}$ with $\mathfrak{X}_{2n} = \{-1,1\}^{2n}$ and restricting the functions $((\tau_i^r)_{i=1}^p)_{r=1}^{2^p}$, to this set, the proof of lemma 2.7 now is complete. $\square$

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Institut für Statistik der Universität Wien, Brünnerstr. 72, A-1210 Wien, Austria.

*E-mail*: wschach@stat1.bwl.univie.ac.at