Addendum to the paper On Certain Probabilities Equivalent to Wiener Measure d'après Dubins, Feldman, Smorodinsky and Tsirelson.

Walter Schachermayer

December 19, 2001

I was kindly informed by H. Buehler, that there is some confusion in the paper [S 99], as regards the notion of a "generating parametrisation" and a "standard extension" (precise definitions see below). It was claimed in [S 99, remark on p. 226] that these two concepts are "equivalent and may be mutually translated one into the other". Instead of giving a proof for this assertion, I only refered to M. Smorodinsky's paper [S 98], where these notions are indeed used synonymously.

As was pointed out by H. Buehler in his Master's Thesis, this assertion is wrong. We present an example below, showing that these two notions are indeed different.

Fortunately this confusion did not cause any damage. The above two notions are applied in [S 99] (as well as in [S 98] and [FS 00]) only, *when a quantifier is preceding them*: the relevant notions are "reverse filtrations *admitting* a generating parametrisation" and "reverse filtrations *admitting* a standard extension". The pleasant fact is that, in this form, these two notions indeed are equivalent; even more pleasantly, as is shown by J. Feldman and M. Smorodinsky in [FS 01], this equivalence immediately follows from previous results in the literature ([V 73], [DFST 96], [FS 00]). The key element is Vershik's characterisation of reverse filtrations admitting a standard extension in terms of iterated Kantorovich-Rubinstein metrics.

Summing up: although [S 99] contains the above cited erroneous statement on p. 226, this did not do any harm (just as in [S 98] and [FS 00]), as in the applications these notions were preceded by the word *admitting*. Hence — apart from taking back with apologies the claim on [S 99, p. 226] — nothing has to be changed in the presentation of [S 99] (as well as [S 98] and [FS 00]) in view of Proposition 1 below.

Fix a "reversely" filtered probability space $(\Omega, (\mathcal{F}_n)_{n=0}^\infty, \mathbf{P})$, where $(\mathcal{F}_n)_{n=0}^\infty$ is a decreasing sequence of sigma-algebras such that $\cap_{n=0}^\infty \mathcal{F}_n$ is trivial. We always assume that each $(\Omega, \mathcal{F}_n, \mathbf{P})$ is a standard Borel space. We shall call a triple $(\Omega, (\mathcal{F}_n)_{n=0}^\infty, \mathbf{P})$ satisfying these assumptions a *reverse filtration*.

A process $(X_n)_{n=0}^\infty$ taking its values in Polish spaces $(\Xi_n)_{n=0}^\infty$ will be defined via its finite dimensional distributions $law(X_0, \dots X_n)$ on $\Xi_0 \times \dots \times \Xi_n$. We associate to $(X_n)_{n=0}^\infty$ its canonical base $\Omega = \Xi = \prod_{n=0}^\infty \Xi_n$, equipped with the probability measure $\mathbf{P}$ on its Borel sets, so that the coordinate maps, still denoted by $(X_n)_{n=0}^\infty$, define a version of the process $(X_n)_{n=0}^\infty$. We then define the sigma-algebras $\mathcal{F}_n$ to be generated by $(X_k)_{k=n}^\infty$.

If $(Y_n)_{n=0}^\infty$ is an i.i.d. sequence of random variables, uniformly distributed on $[0,1]$, we denote the canonical base of $(Y_n)_{n=0}^\infty$ by $([0,1]^{\mathbb{N}}, (\mathcal{G}_n)_{n=0}^\infty, \lambda)$, and call it the *standard reverse filtration.*

Definition 1 *([DFST 96, Definitions 2.3 and 2.5]) A standard extension of the reverse filtration $(\Omega, (\mathcal{F}_n)_{n=0}^\infty, \mathbf{P})$ is a measure-preserving map $\pi : [0,1]^{\mathbb{N}} \to \Omega$ with $\pi^{-1}(\mathcal{F}_n) \subseteq \mathcal{G}_n$, and such that*

$$\mathbf{E}_\lambda[X \circ \pi | \mathcal{G}_n] = \mathbf{E}_{\mathbf{P}}[X|\mathcal{F}_n] \circ \pi, \tag{1}$$

for all $n \in \mathbb{N}$, and all bounded $\mathcal{F}_0$-measurable functions X.

Extending this notion from reverse filtrations to processes, we say that the process $(X_n)_{n=0}^\infty$, taking values in the Polish spaces $(\Xi_n)_{n=0}^\infty$, admits a standard extension, if the canonical base $(\Omega, (\mathcal{F}_n)_{n=0}^\infty, \mathbf{P})$ associated to $(X_n)_{n=0}^\infty$ does so.

Definition 2 *([S 99], compare also [S 98]) A generating parametrisation of the process $(X_n)_{n=0}^\infty$ is a process $(\widetilde{X}_n, Y_n)_{n=0}^\infty$, defined on a stochastic base $(\Omega, \mathcal{F}, \mathbf{P})$, and a sequence $(f_n)_{n=0}^\infty$ of deterministic Borel-measurable functions defined on $[0,1] \times \mathbb{R}^{\mathbb{N}}$ such that*

(i) the processes X and $\widetilde{X}$ are identical in law,

(ii) the sequence $(Y_n)_{n=0}^\infty$ is a sequence of i.i.d. random variables, uniformly distributed on $[0,1]$, and such that Y_n is independent of $(\widetilde{X}_i)_{i=n+1}^\infty$,

(iii) the equation

$$\widetilde{X}_n(\omega) = f_n(Y_n(\omega), \widetilde{X}_{n+1}(\omega), \widetilde{X}_{n+2}(\omega), \ldots) \tag{2}$$

holds true, for each $n \geq 0$ and almost each ω.

(iv) $\widetilde{X}_n$ is $\sigma(Y_n, Y_{n+1}, \ldots)$-measurable.

We shall now rephrase these concepts in a way closer to "a probabilist's mother-tongue", to quote S. Beghdadi-Sakrani and M. Emery [BE 99] (in this paper there is also a beautiful presentation of the main result of [DFST 96]).

A standard extension $\pi : [0,1]^{\mathbb{N}} \to \Xi$ of a process $(X_n)_{n=0}^\infty$ canonically defines a sequence of Borel measurable maps $g_n : [0,1]^{\mathbb{N}} \to \Xi_n$ such that the process

$$\widetilde{X}_n = g_n(Y_n, Y_{n+1}, \ldots), \qquad n \geq 0, \tag{3}$$

is a version of the process $(X_n)_{n=0}^\infty$, based on and adapted to $([0,1]^{\mathbb{N}}, (\mathcal{G}_n)_{n=0}^\infty, \lambda)$, and such that we have the following identity for the conditional distributions:

$$\mathrm{law}((\widetilde{X}_0, \ldots, \widetilde{X}_n)|Y_{n+1}, \ldots) = \mathrm{law}((\widetilde{X}_0, \ldots, \widetilde{X}_n)|\widetilde{X}_{n+1}, \ldots), \qquad n \geq 0. \tag{4}$$

Indeed, the n'th coordinate of π, denoted by $\pi_n : [0,1]^{\mathbb{N}} \to \Xi_n$, which is $\mathcal{G}_n$-measurable, canonically defines the function g_n in (3), while condition (1) is quickly seen to translate into condition (4).

In other words, a standard extension $\pi : [0,1]^{\mathbb{N}} \to \Xi$ of a process $(X_n)_{n=0}^{\infty}$ may equivalently be defined by the "parametrisation" (3) of X_n as a function of the i.i.d. sequence $(Y_k)_{k\geq n}$, such that (4) is satisfied.

Turning to the notion of a generating parametrisation: suppose now that $(X_n)_{n=0}^{\infty}$ satisfies Definition 2. As we assume that the function $f_n(Y_n, \widetilde{X}_{n+1}, \widetilde{X}_{n+2}, \ldots)$ is $\sigma(Y_n, Y_{n+1}, \ldots)$ measurable, we again may canonically define a sequence of Borel measurable maps $g_n : [0,1]^{\mathbb{N}} \to \Xi_n$ such that

$$\widetilde{X}_n = g_n(Y_n, Y_{n+1}, \ldots), \qquad n \geq 0. \tag{5}$$

Assertion (iii) of the definition of a generating parametrisation implies that the function g_n may be factored in the following way

$$\begin{aligned} \widetilde{X}_n &= f_n(Y_n, g_{n+1}(Y_{n+1}, \ldots), g_{n+2}(Y_{n+2,\ldots}), \ldots) \\ &= f_n(Y_n, \widetilde{X}_{n+1}, \widetilde{X}_{n+2}, \ldots), \qquad n \geq 0. \end{aligned} \tag{6}$$

Comparing (6) to (4), it becomes obvious that a generating parametrisation of the process $(X_n)_{n=0}^{\infty}$ canonically defines a standard extension: indeed, the functions g_n obtained from a generating parametrisation in (5) define a standard extension π via (3), as the fact that g_n may be factored in the way indicated in (6) clearly implies the validity of (4). We then say that the standard extension π is induced by the generating parametrisation. *But, and this is the point where the confusion arose,* the validity of (4) *does not* imply that g_n may be factored as in (6). In other words, a standard extension π of a process $(X_n)_{n=0}^{\infty}$ is not necessarily induced by a generating parametrisation. In fact, the arch-example in the theory of reverse filtrations shows this fact. This example is called the "phenomenon of the dying witness" by H. v. Weizsäcker (compare [ES 99] and the references given there) and is presented in Example 1 below.

First we resume the positive result from [FS 01]:

Proposition 1 *The following three properties of a reverse filtration* $(\Omega, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbf{P})$ *are equivalent:*

(i) $(\Omega, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbf{P})$ *admits a standard extension*

(ii) $(\Omega, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbf{P})$ *admits a generating parametrisation*

(iii) $(\Omega, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbf{P})$ *satisfies Vershik's criterion as defined in [ES 01, p. 284].*

Proof We have seen above that (ii) $\Rightarrow$ (i). To prove the reverse implication (i) $\Rightarrow$ (ii) one has to pass via Vershik's criterion, i.e., one has to verify (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii). As pointed out in [FS 01] these implications are explicitely proved in the previous literature. ■

We now give an example of a standard extension which is not induced by a generating parameterisation.

Example 1 Let $(Y_n)_{n=0}^{\infty}$ be an i.i.d. sequence of random variables uniformly distributed on $[0,1]$, and define

$$\begin{aligned} X_n &= Y_n \dot{+} Y_{n+1}, \\ &= g_n(Y_n, Y_{n+1}, Y_{n+2}, \ldots) \qquad n \geq 0, \end{aligned} \tag{7}$$

where $\dot{+}$ denotes addition modulo 1 on $[0,1]$.

Then $(X_n)_{n=0}^{\infty}$ again is an i.i.d. sequence of random variables, uniformly distributed on $[0,1]$, and g_n defines a standard extension π via (3). But this standard extension π is not induced by a generating parametrisation.

Proof To show that g_n indeed defines a standard extension π, we have to verify (4). To do so, it suffices to note that, for $n \geq 0$, the conditional distribution $\text{law}((X_0, \ldots, X_n)|Y_{n+1}, Y_{n+2}, \ldots)$ simply is identically equal to the uniform distribution on $[0,1]^{n+1}$.

To show that the standard extension π is not induced by a generating parametrisation, we shall prove that the function g_n defined in (7) cannot be factored in the form of (6). Indeed, the sigma-algebra generated by $(Y_n, X_{n+1}, X_{n+2}, \ldots) = (Y_n, Y_{n+1} \dot{+} Y_{n+2}, Y_{n+2} \dot{+} Y_{n+3}, \ldots)$ is independent of $X_n = Y_n \dot{+} Y_{n+1}$. ■

References

[BE 99] S. Beghdadi-Sakrani, M. Émery, (1999), *On certain probabilities equivalent to coin-tossing d'après Schachermayer.* Séminaire de Probabilités XXXIII, Springer Lecture Notes in Mathematics, Vol. 1709, pp. 240–256.

[B 01] H. Buehler, (2001), *Zur Informationsstruktur Brownscher Bewegungen.* Diploma Thesis at the Humboldt University Berlin.

[DFST 96] L. Dubins, J. Feldman, M. Smorodinsky, B. Tsirelson, (1996), *Decreasing sequences of σ-fields and a measure change for Brownian motion.* Annals of Applied Probability, Vol. 24, pp. 882–904.

[ES 99] M. Émery, W. Schachermayer, (1999), *Brownian filtrations are not stable under equivalent time-changes.* Séminaire de Probabilités XXXIII, Springer Lecture Notes in Mathematics, Vol. 1709, pp. 267–276.

[ES 01] M. Émery, W. Schachermayer, (2001), *On Vershik's Standardness Criterion and Tsirelson's Notion of Cosiness.* Séminaire de Probabilités XXXV, Springer Lecture Notes in Mathematics, Vol. 1755, pp. 265–305.

[FS 00] J. Feldman, M. Smorodinsky, (2000), *Decreasing sequences of measurable partitions: product type, standard, and prestandard.* Ergodic Theory and Dynamical Systems, Vol. 20, pp. 1079–1090.

[FS 01] J. Feldman, M. Smorodinsky, (2001), *Addendum to our paper "Decreasing sequences of measurable partitions: product type, standard, and prestandard".* Preprint, to appear in Ergodic Theory and Dynamical Systems.

[S 99] W. Schachermayer, (1999), *On Certain Probabilities Equivalent to Wiener Measure d'après Dubins, Feldman, Smorodinsky and Tsirelson.* Séminaire de Probabilités XXXIII, Springer Lecture Notes in Mathematics, Vol. 1709, pp. 221–239.

[S 98] M. Smorodinsky, (1998), *Processes with no standard extension.* Israel Journal of Mathematics, Vol. 107, pp. 327–331.

[V 73] A.M. Vershik, (1973), *Approximation in measure theory.* Doctor Thesis, Leningrad.
Expanded and updated english version:
A.M. Vershik, (1995), *The theory of decreasing sequences of measurable partitions.* St.Petersburg Mathematical Journal, Vol. 6, pp. 705–761.