BROWNIAN FILTRATIONS ARE NOT STABLE UNDER EQUIVALENT TIME-CHANGES

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1. — Introduction

L. Dubins, J. Feldman, M. Smorodinsky and B. Tsirelson have shown in [DFST 96] that a small perturbation of its probability law can transform Brownian motion into a process whose natural filtration is not generated by any Brownian motion whatsoever. More precisely, they construct on Wiener space $\big(W, \mathcal{F}_\infty, \lambda, (\mathcal{F}_t)_{t\geqslant 0}\big)$ a probability μ equivalent to the Wiener measure λ, with density $d\mu/d\lambda$ arbitrarily close to 1 in L^∞-norm, but such that no process with μ-independent increments generates the canonical filtration $(\mathcal{F}_t)_{t\geqslant 0}$. In fact, the μ constructed in [DFST 96] has the stronger property of being non-cosy [BE 99]. The notion of cosiness was invented by Tsirelson [T 97] as a necessary condition for a filtration to be Brownian; non-cosiness turns out to be a most convenient tool to construct new examples of "paradoxical" filtrations.

Marc Yor raised the following question: Is there something similar to the DFST-phenomenon, with a change of time instead of a change of probability law? More precisely, does there exist on Wiener space an absolutely continuous, strictly increasing time-change such that the time-changed filtration is no longer Brownian?

This question is reasonable only for those time-changes that are absolutely continuous (with respect to dt) and strictly increasing. Indeed, if a time-change is not absolutely continuous, it transforms some non $dt\times d\mathbb{P}$-null subset of $\mathbb{R}_+\times W$ into a null one A, and the canonical Brownian motion into a martingale M such that $\int \mathbb{1}_A \, d[M,M] \neq 0$; but such a martingale cannot exist in a Brownian filtration. Similarly, if the time-change is not strictly increasing, it transforms a $dt\times d\mathbb{P}$-null set into a non null one A, and all martingales M for the new filtration verify $\int \mathbb{1}_A \, d[M,M] = 0$, so no Brownian motion can be a martingale in this filtration.

The present paper shows that the answer to Yor's question is positive; moreover, as was the case with the perturbation of measure considered in [DFST 96], the perturbation of time can be made arbitrarily small. Our main result, Theorem 4.1 below, is the existence of a family $(T_t)_{t\geqslant 0}$ of stopping times on Wiener space $\big(W, \mathcal{F}_\infty, \lambda, (\mathcal{F}_t)_{t\geqslant 0}\big)$, with the following two properties:

(i) almost surely, the function $t \mapsto T_t(\omega)$ is null at zero and differentiable, with derivative verifying $1-\varepsilon < dT_t/dt < 1+\varepsilon$;

(ii) the filtration $(\mathcal{G}_t)_{t\geqslant 0}$ defined by $\mathcal{G}_t = \mathcal{F}_{T_t}$ is not generated by any Brownian motion (more precisely, it is not cosy).

We end this introduction with an outline of the organisation of the paper: in section 2 we present the basic example 2.1 underlying the whole paper. We make an effort to present it as intuitively and non-technically as possible: we only consider sequences of finitely valued random variables which we interpret as “lotteries” and “pointers”. Also, we avoid technical concepts such as “cosy filtrations” and “immersions” (although these ideas are behind the construction). We end this section by isolating in Proposition 2.3 a seemingly innocent property of Example 2.1, which will turn out to be crucial.

In section 3 we develop the notion of “cosy filtrations” as introduced in [T 97] (see also [BE 99]). We then show that the property of Example 2.1 isolated in Proposition 2.3 is a sufficient criterion for the non-cosiness of the generated filtration. Next, we show that non-cosiness of Example 2.1 implies in particular non-substandardness in the terminology of ([DFST 96]), i.e., the filtration generated by Example 2.1 cannot be immersed into a filtration generated by a sequence of independent random variables.

Finally in section 4 we use Example 2.1 to construct a time change of Brownian motion that destroys Browniannness of the filtration, as announced in the title. This section is completely elementary and only contains the task of translating Example 2.1 into a time-change.

2. — The discrete example

2.1. Example. — We denote by $-\mathbb{N}$ the set $\{\ldots,-2,-1,0\}$ and we fix a sequence $(p_n)_{n\in-\mathbb{N}}$ of natural numbers, $p_n \geqslant 2$, such that $\sum_{n\in-\mathbb{N}} p_n^{-1} < \infty$; for example $p_n = 2^{-n+1}$ is a good choice.

Now fix a probability space $(\Omega,\mathcal{A},\mathbb{P})$ on which the following objects are defined: a family $\big((R_{n,q})_{q=1}^{p_n}\big)_{n\leqslant-1}$ of independent random variables such that $R_{n,q}$ is uniformly distributed on $\{1,\ldots,p_{n+1}\}$, and a sequence $(Q_n)_{n\leqslant 0}$ of random variables such that Q_n is uniformly distributed on $\{1,\ldots,p_n\}$, independent of $R_{m,q}$, for $m > n$, and such that

$$Q_{n+1} = R_{n,Q_n}, \qquad \text{a.s., for} \quad n \leqslant -1. \tag{2.1}$$

It is easy to see that such random variables $\big((R_{n,q})_{q=1}^{p_n}\big)_{n\leqslant-1}$ and $(Q_n)_{n\leqslant0}$ can indeed be defined on a suitable stochastic basis $(\Omega,\mathcal{A},\mathbb{P})$ (first consider only $n \geqslant n_0$, then take a projective limit) and that the above properties properties already characterize the joint law of the random variables $\Big(\big((R_{n,q})_{q=1}^{p_n}\big)_{n\leqslant-1}, (Q_n)_{n\leqslant0}\Big)$.

Instead of giving a formal proof of these assertions we give an intuitive explanation of the situation: for fixed n, we interpret the random variables $(R_{n,q})_{q=1}^{p_n}$ as p_n successive “lotteries” yielding random results uniformly distributed in $\{1,\ldots,p_{n+1}\}$. The random variable Q_n, taking its value in a uniformly distributed way in $\{1,\ldots,p_n\}$, will be interpreted as the “pointer” which tells us, which of these lotteries (which are drawn independently of Q_n) is relevant for us: if $Q_n(\omega) = q_n$ for some $1 \leqslant q_n \leqslant p_n$, we look at the lottery R_{n,q_n} (and ignore all the other lotteries $(R_{n,q})_{q\neq q_n}$); the outcome $R_{n,q_n}(\omega) = R_{n,Q_n(\omega)}(\omega)$ of this lottery defines by (2.1) the value of the next pointer Q_{n+1}, which tells us which lottery among $(R_{n+1,q})_{q=1}^{p_{n+1}}$ is relevant for us at time $n+1$ and in turn determines the pointer Q_{n+2} via (2.1), and so on.

The basic feature of the example is as follows: if we know the value of the pointer Q_{n_0}, for some $n_0 \in -\mathbb{N}$ which we should think of as lying in the remote past, then we can determine the values of $Q_{n_0+1}, Q_{n_0+2}, \ldots, Q_0$ by only observing the results of lotteries $(R_{n_0,q})_{q=1}^{p_{n_0}}, (R_{n_0+1,q})_{q=1}^{p_{n_0+1}}, \ldots, (R_{-1,q})_{q=1}^{p_{-1}}$. On the other hand, if we only know the results for all the lotteries $\big((R_{n,q})_{q=1}^{p_n}\big)_{n\leqslant -1}$ (without additional knowledge of some Q_{n_0}), then we do not know enough to determine Q_0. This should be rather obvious on an intuitive level (provided $(p_n)_{n\leqslant -1}$ tends fast enough to infinity) and will be proved below. Hence the random variables $(Q_n)_{n\leqslant 0}$ contain some additional information which is not provided by the random variables $\big((R_{n,q})_{q=1}^{p_n}\big)_{n\leqslant -1}$.

But although, for any $n_0 \in -\mathbb{N}$, the information provided by Q_{n_0} in conjunction with $\big((R_{n,q})_{q=1}^{p_n}\big)_{n\leqslant -1}$ determines the value of Q_0, we shall see that the intersection of the sigma-algebras $\mathcal{G}_{n_0} = \sigma(Q_n : n \leqslant n_0)$ is trivial, i.e., $\bigcap_{n_0\in-\mathbb{N}} \mathcal{G}_{n_0}$ consists only of sets of measure zero or one, and therefore contains no information.

So far, we have only reencountered a well-known pathology of decreasing filtrations (see Exercise 4.12 of [W 91] for a particularly easy example, pointed out by M. Barlow and E. Perkins, also displaying the above described phenomenon; see [vW 83] for a detailed study). The present example has—in contrast to the Barlow-Perkins example—the additional feature that it gives rise to a filtration that is not standard, thus displaying the same (additional) phenomenon as an example due to A. Vershik [V 73].

We now proceed to prove the above assertions.

We consider the two-dimensional process $(X_i)_{i\in I} := ((R_{n,q}, Q_n)_{q=1}^{p_n})_{n\leqslant -1}$, where the index set $I = \{(n,q) : 1 \leqslant q \leqslant p_n,\ n \leqslant -1\}$ is ordered lexicographically. We denote by $(\mathcal{F}_i)_{i\in I} = ((\mathcal{F}_{n,q})_{q=1}^{p_n})_{n\leqslant -1}$ the filtration generated by the process X; we shall give below an intuitive explanation of the following fact: for $1 \leqslant q \leqslant p_n$, $n \leqslant -1$ and arbitrary $n_0 \leqslant n$

$$\begin{aligned} \mathcal{F}_{n,q} &= \sigma(R_{m,r}, Q_\ell : (m,r) \leqslant (n,q) \text{ and } \ell \leqslant n) = \\ &= \sigma(R_{m,r}, Q_\ell : (m,r) \leqslant (n,q) \text{ and } \ell \leqslant n_0). \end{aligned} \tag{2.2}$$

Formula (2.2) implies in particular that, for $1 \leqslant q \leqslant p_n$ and $n \leqslant -1$,

$$\mathcal{F}_{n,q} = \sigma(R_{n,q}) \vee \sigma(\mathcal{F}_{m,r} : (m,r) < (n,q)),$$

i.e., the information gained, by passing to (n,q) from its predecessor (which is $(n, q-1)$ for $q > 1$ and $(n-1, p_{n-1})$ for $q = 1$), is given by $R_{n,q}$.

Here is the intuitive explanation of the above formulae (2.2): at time (n,q) the sigma-algebra $\mathcal{F}_{n,q}$ contains, by definition, all the information of the previous lotteries $(R_{m,r})_{(m,r)\leqslant(n,q)}$ as well as the information of all the previous pointers $(Q_\ell)_{\ell\leqslant n}$. If instead we only know the positions of the pointers $(Q_\ell)_{\ell\leqslant n_0}$ for some $n_0 \leqslant n$, then we don't lose any information as the knowledge of Q_{n_0} in conjunction with the knowledge of $(R_{m,r})_{(m,r)\leqslant(n,q)}$ allows us to reconstruct via (2.1) the positions of the pointers $Q_{n_0+1}, \ldots, Q_n$.

Lemma 2.2. — *The intersection*

$$\mathcal{F}_{-\infty} := \bigcap_{(n,q)\in I} \mathcal{F}_{n,q}$$

is trivial, i.e., consists only of sets of measure zero or one.

PROOF. — We start with an observation, which is notable in its own right: $(Q_n)_{n\leqslant 0}$ is an *independent* sequence of random variables. It is instructive to convince oneself on an intuitive level of this property: although, for $n \leqslant -1$ fixed, Q_n determines *which of the lotteries* $(R_{n,q})_{q=1}^{p_n}$ is chosen to define the value of Q_{n+1} via (2.1), we nonetheless have that the *result* Q_{n+1} of the lottery is independent of Q_n as all the random variables $(R_{n,q})_{q=1}^{p_n}$ are independent of Q_n and have the same law. The independence of the whole sequence $(Q_n)_{n\leqslant 0}$ now follows easily.

Next we observe the "skip-independence" of the process $(X_i)_{i\in I}$: for $n_0 \leqslant -1$ the family $((R_{n,q}, Q_n)_{q=1}^{p_n})_{n\geqslant n_0}$ is independent of $((R_{n,q}, Q_n)_{q=1}^{p_n})_{n\leqslant n_0-2}$, which again is rather obvious.

Hence, any event measurable with respect to the sigma-algebra generated by $((R_{n,q}, Q_n)_{q=1}^{p_n})_{n\geqslant n_0}$ for some $n_0 \in -\mathbb{N}$ is independent of $\mathcal{F}_{-\infty}$, which implies the triviality of $\mathcal{F}_{-\infty}$. ∎

PROPOSITION 2.3. — *Let* $(\bar{\Omega}, \bar{\mathcal{A}}, ((\bar{\mathcal{F}}_{n,q})_{q=1}^{p_n})_{n\leqslant -1}, \bar{\mathbb{P}})$ *be a filtered probability space and suppose that two processes* $((R'_{n,q}, Q'_n)_{q=1}^{p_n})_{n\leqslant -1}$ *and* $((R''_{n,q}, Q''_n)_{q=1}^{p_n})_{n\leqslant -1}$ *are defined on* $\bar{\Omega}$, *such that*

(i) $((R'_{n,q}, Q'_n)_{q=1}^{p_n})_{n\leqslant -1}$ *and* $((R''_{n,q}, Q''_n)_{q=1}^{p_n})_{n\leqslant -1}$

are adapted to the filtration $((\bar{\mathcal{F}}_{n,q})_{q=1}^{p_n})_{n\leqslant -1}$ *and*

(ii) *the processes* $((R'_{n,q}, Q'_n)_{q=1}^{p_n})_{n\leqslant -1}$ *and* $((R''_{n,q}, Q''_n)_{q=1}^{p_n})_{n\leqslant -1}$

both have the law of the process defined in Example 2.1 and, for each (n,q), *the random variables* $(R'_{m,r})_{(m,r)>(n,q)}$ *and* $(R''_{m,r})_{(m,r)>(n,q)}$ *are independent of the sigma-algebra* $\bar{\mathcal{F}}_{n,q}$.

Then, for $n<0$, *we have*

$$\bar{\mathbb{P}}[Q'_{n+1} \neq Q''_{n+1} | \bar{\mathcal{F}}_{n,1}] = 1 - p_{n+1}^{-1} \quad \text{on the event } \{Q'_n \neq Q''_n\}\,. \tag{2.3}$$

PROOF. — It suffices to show (2.3) on the event $\{Q'_n = q'_n, Q''_n = q''_n\}$, where q'_n and q''_n are such that $1 \leqslant q'_n \leqslant p_n$, $1 \leqslant q''_n \leqslant p_n$ and $q'_n \neq q''_n$. So fix such q'_n and q''_n and assume w.l.g. that $q'_n < q''_n$. We shall show, more precisely, that

$$\bar{\mathbb{P}}[Q'_{n+1} \neq Q''_{n+1} | \bar{\mathcal{F}}_{n,q''_n-1}] = 1 - p_{n+1}^{-1} \text{ on the event } \{Q'_n = q'_n, Q''_n = q''_n\}\,. \tag{2.4}$$

Indeed, for each fixed $1 \leqslant q'_{n+1} \leqslant p_{n+1}$, use assumption (i) to conclude that the event $\{Q'_{n+1} = q'_{n+1}\} = \{R'_{n,Q'_n} = q'_{n+1}\}$ is in $\mathcal{F}_{n,q'_n}$ and therefore in $\mathcal{F}_{n,q''_n-1}$. By assumption (ii) the random variable R''_{n,q''_n} is independent of $\mathcal{F}_{n,q''_n-1}$ and uniformly distributed on $\{1,\ldots,p_{n+1}\}$. Hence the $\bar{\mathcal{F}}_{n,q''_n-1}$-conditional probability for R''_{n,q''_n} to be different from q'_{n+1} identically equals $1-p_{n+1}^{-1}$. This proves the validity of (2.4) on the event $\{Q'_n = q'_n, Q''_n = q''_n, Q'_{n+1} = q'_{n+1}\}$ and therefore (2.4) and (2.3). ∎

As the next section will show, Proposition 2.3 implies that *the filtration* $(\mathcal{F}_i)_{i\in I}$ *is not generated by any independent sequence of random variables.* This will be proved by Proposition 3.2, which relies on two hypotheses. In the case of the above example, the first hypothesis is just Property (2.3) together with the convergence of the series $\sum p_n^{-1}$; the second hypothesis is the fact that the sequence $Q = (Q_n)_{n\leqslant 0}$ has a diffuse law. And indeed, by independence of $(Q_n)_{n\leqslant 0}$ (seen in the proof of Lemma 2.2), for any deterministic sequence $q = (q_n)_{n\leqslant 0}$ one has

$$\mathbb{P}[Q=q] = \prod_{n\in -\mathbb{N}} \mathbb{P}[Q_n = q_n] = \prod_{n\in -\mathbb{N}} \frac{1}{p_n} \leqslant \prod_{n\in -\mathbb{N}} \frac{1}{2} = 0\,.$$

3. — Cosiness

This section is borrowed, almost verbatim, from [BE 99], to which we refer for details, comments, and complements.

We shall consider filtrations $(\mathcal{F}_t)_{t\in\mathcal{T}}$, where $\mathcal{T}$ is totally ordered; this includes the discrete filtrations considered in the previous section, as well as the continuous case $\mathcal{T}=\mathbb{R}_+$. We denote by $\mathcal{F}_\infty$ the σ-field $\bigvee_{t\in\mathcal{T}}\mathcal{F}_t$ generated by the field $\bigcup_{t\in\mathcal{T}}\mathcal{F}_t$ (note that $\mathcal{F}_\infty=\mathcal{F}_0$ when $\mathcal{T}=-\mathbb{N}$).

DEFINITION. — *An* embedding *of a probability space* $(\Omega,\mathcal{A},\mathbb{P})$ *into another one* $(\bar\Omega,\bar{\mathcal{A}},\bar{\mathbb{P}})$ *is a mapping* Ψ *from* $\mathrm{L}^0(\Omega,\mathcal{A},\mathbb{P})$ *to* $\mathrm{L}^0(\bar\Omega,\bar{\mathcal{A}},\bar{\mathbb{P}})$ *that commutes with Borel operations on finitely many r.v.'s:*

$$\Psi\bigl(f(X_1,\dots,X_n)\bigr)=f\bigl(\Psi(X_1),\dots,\Psi(X_n)\bigr)\qquad\textit{for every Borel } f$$

and preserves the probability laws:

$$\bar{\mathbb{P}}\bigl[\Psi(X)\in E\bigr]=\mathbb{P}[X\in E]\qquad\textit{for every Borel } E.$$

An embedding is always injective and transfers not only random variables, but also sub-σ-fields, filtrations, processes, etc. It is called an *isomorphism* if it is surjective; it then has an inverse. An embedding Ψ of $(\Omega,\mathcal{A},\mathbb{P})$ into $(\bar\Omega,\bar{\mathcal{A}},\bar{\mathbb{P}},)$ is always an isomorphism between $(\Omega,\mathcal{A},\mathbb{P})$ and $\bigl(\bar\Omega,\Psi(\mathcal{A}),\bar{\mathbb{P}}\bigr)$.

DEFINITIONS. — *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be two filtrations on a probability space* $(\Omega,\mathcal{A},\mathbb{P})$.

The filtration $\mathcal{F}$ *is* immersed *in* $\mathcal{G}$ *if every* $\mathcal{F}$*-martingale is a* $\mathcal{G}$*-martingale.* (Note that this implies in particular $\mathcal{F}_t\subset\mathcal{G}_t$ for each $t\in\mathcal{T}$.)

The filtrations $\mathcal{F}$ *and* $\mathcal{G}$ *are* separate *if* $\mathbb{P}[F=G]=0$ *for all random variables* $F\in\mathrm{L}^0(\mathcal{F}_\infty)$ *and* $G\in\mathrm{L}^0(\mathcal{G}_\infty)$ *with diffuse laws.*

DEFINITION. — *A filtered probability space* $(\Omega,\mathcal{A},\mathbb{P},\mathcal{F})$ *is* cosy, *if there exist a filtered probability space* $(\bar\Omega,\bar{\mathcal{A}},\bar{\mathbb{P}},\bar{\mathcal{F}})$ *and a sequence* $(\Psi^n)_{n\in\mathbb{N}\cup\{\infty\}}$ *of embeddings of* $(\Omega,\mathcal{F}_\infty,\mathbb{P})$ *into* $(\bar\Omega,\bar{\mathcal{A}},\bar{\mathbb{P}})$ *such that*

(i) *for each* $n\leqslant\infty$, *the filtration* $\Psi^n(\mathcal{F})$ *is immersed in* $\bar{\mathcal{F}}$;

(ii) *for each finite* n, *the filtrations* $\Psi^n(\mathcal{F})$ *and* $\Psi^\infty(\mathcal{F})$ *are separate;*

(iii) *for each* $U\in\mathrm{L}^0(\Omega,\mathcal{F}_\infty,\mathbb{P})$, *the r.v.'s* $\Psi^n(U)\in\mathrm{L}^0(\bar\Omega,\bar{\mathcal{A}},\bar{\mathbb{P}})$ *converge in probability to* $\Psi^\infty(U)$.

Instead of saying that $(\Omega,\mathcal{A},\mathbb{P},\mathcal{F})$, is cosy, we shall often simply say that the filtration $\mathcal{F}$ is cosy. But it should be remembered that cosiness does depend on the probability $\mathbb{P}$.

As mentioned in the introduction, the notion of cosiness is due to Tsirelson [T 97]. We have slightly modified his definition: our separabililty condition is not equivalent to his. (His definition was intended only for filtrations where all martingales are continuous; the sufficient condition for non-cosiness given by Proposition 3.2 works simultaneously for the discrete example of section 2 and for the Brownian time-changes in section 4.)

PROPOSITION 3.1. — *Let* $\big(\Omega,\mathcal{A},\mathbb{P},(\mathcal{F}_t)_{t\in\mathcal{T}}\big)$ *be a filtered probability space and* $U\in\mathrm{L}^0(\mathcal{F}_\infty)$ *a random variable assuming only finitely many values. Fix* $\gamma>0$.

Suppose that for any filtered probability space $(\bar\Omega,\bar{\mathcal{A}},\bar{\mathbb{P}},\bar{\mathcal{F}})$ *and for any two filtrations* $\mathcal{F}'$ *and* $\mathcal{F}''$ *isomorphic to* $\mathcal{F}$, *immersed in* $\bar{\mathcal{F}}$ *and separate, one has* $\bar{\mathbb{P}}[U'\neq U'']\geqslant\gamma$ *(where* U' *and* U'' *are the copies of* U *in the* σ*-fields* $\mathcal{F}'_\infty$ *and* $\mathcal{F}''_\infty$*).*

Then $\mathcal{F}$ *is not cosy.*

PROOF. — Let $(\Omega,\mathcal{A},\mathbb{P},\mathcal{F})$, U and γ satisfy the hypothesis of this proposition. Suppose we have some filtered probability space $(\bar\Omega,\bar{\mathcal{A}},\bar{\mathbb{P}},\bar{\mathcal{F}})$ and some sequence $(\Psi^n)_{n\in\mathbb{N}\cup\{\infty\}}$ of embeddings of $(\Omega,\mathcal{F}_\infty,\mathbb{P})$ into $(\bar\Omega,\bar{\mathcal{A}},\bar{\mathbb{P}})$, fulfilling the first two conditions (i) and (ii) in the definition of cosiness. For every finite n, our hypothesis can be applied to the filtrations $\mathcal{F}'=\Psi^n(\mathcal{F})$ and $\mathcal{F}''=\Psi^\infty(\mathcal{F})$; this gives $\bar{\mathbb{P}}[\Psi^n(U)\neq\Psi^\infty(U)]\geqslant\gamma$. As U takes finitely many values, the third condition in the definition of cosiness is not satisfied. Consequently, $\mathcal{F}$ cannot be cosy. ∎

PROPOSITION 3.2. — *Let* $\big(\Omega,\mathcal{A},\mathbb{P},(\mathcal{F}_t)_{t\in\mathcal{T}}\big)$ *be a filtered probability space. Suppose given a strictly increasing sequence* $(t_n)_{n\leqslant 0}$ *in* $\mathcal{T}$ *(that is,* $t_{n-1}<t_n$*), a sequence* $(\varepsilon_n)_{n<0}$ *in* $\mathcal{T}$ *such that* $\sum_n\varepsilon_n<\infty$, *and an* $\mathbb{R}^{-\mathbb{N}}$*-valued random vector* $(U_n)_{n\leqslant 0}$, *with diffuse law, such that* U_n *is* $\mathcal{F}_{t_n}$*-measurable for each* n *and* U_0 *takes only finitely many values.*

Assume that for any filtered probability space $(\bar\Omega,\bar{\mathcal{A}},\bar{\mathbb{P}},\bar{\mathcal{F}})$ *and for any two filtrations* $\mathcal{F}'$ *and* $\mathcal{F}''$ *isomorphic to* $\mathcal{F}$ *and immersed in* $\bar{\mathcal{F}}$, *one has for each* $n<0$

$$\bar{\mathbb{P}}[U'_{n+1}=U''_{n+1}|\bar{\mathcal{F}}_{t_n}]\leqslant\varepsilon_n\quad\textit{on the event }\{U'_n\neq U''_n\}$$

(where U'_n *and* U''_n *denote the copies of* U_n *in the* σ*-fields* $\mathcal{F}'_\infty$ *and* $\mathcal{F}''_\infty$*).*

Then $\mathcal{F}$ *is not cosy.*

PROOF. — If $\mathcal{F}'$ and $\mathcal{F}''$ are isomorphic to $\mathcal{F}$ and immersed in $\bar{\mathcal{F}}$, we know that

$$\mathbb{1}_{\{U'_n\neq U''_n\}}\,\bar{\mathbb{P}}[U'_{n+1}\neq U''_{n+1}|\bar{\mathcal{F}}_{t_n}]\geqslant\mathbb{1}_{\{U'_n\neq U''_n\}}\,(1-\varepsilon_n)\,.$$

by induction on n, this implies

$$\mathbb{1}_{\{U'_n\neq U''_n\}}\,\bar{\mathbb{P}}[U'_{n+1}\neq U''_{n+1},\ldots,U'_0\neq U''_0|\bar{\mathcal{F}}_{t_n}]\geqslant\mathbb{1}_{\{U'_n\neq U''_n\}}\,(1-\varepsilon_{-1})\ldots(1-\varepsilon_n)$$

and a fortiori

$$\bar{\mathbb{P}}[U'_0\neq U''_0|\bar{\mathcal{F}}_{t_n}]\geqslant\gamma\quad\text{on the event }\{U'_n\neq U''_n\}\,,\tag{3.1}$$

where $\gamma>0$ denotes the value of the convergent infinite product $\prod\limits_{n<0}(1-\varepsilon_n)$.

To establish non-cosiness, we shall apply Proposition 3.1 with $U=U_0$. So suppose $\mathcal{F}'$ and $\mathcal{F}''$ are two filtrations isomorphic to $\mathcal{F}$, separate and immersed in some $\bar{\mathcal{F}}$. As the law of $(U_n)_{n\leqslant 0}$ is diffuse, the separation assumption gives $\bar{\mathbb{P}}[U'_n\neq U''_n$ for some $n\leqslant 0]=1$, and there exists an $m\leqslant 0$ such that

$$\bar{\mathbb{P}}[U'_n\neq U''_n\text{ for some }n\in\{m,m{+}1,\ldots,0\}]\geqslant\tfrac12\,.$$

Call N the smallest n in $\{m,m{+}1,\ldots,0\}$ such that $U'_n\neq U''_n$ (if there is one). The random variable T equal to t_N if N exists and to $+\infty$ else, is an $\bar{\mathcal{F}}$-stopping time, that verifies $\bar{\mathbb{P}}[T<\infty]\geqslant\frac12$ and $U'_n\neq U''_n$ on $\{T=t_n\}$. The minoration (3.1) gives

$$\bar{\mathbb{P}}[U'_0\neq U''_0|\bar{\mathcal{F}}_T]\geqslant\gamma\quad\text{on }\{T<\infty\}\,,$$

whence $\bar{\mathbb{P}}[U'_0\neq U''_0]\geqslant\bar{\mathbb{P}}[U'_0\neq U''_0,T<\infty]\geqslant\frac12\,\gamma$; and Proposition 3.1 applies. ∎

As shown by Tsirelson [T 97], a Brownian filtration is always cosy. His proof works just as well with our definition, and shows more generally that the filtration generated by a Gaussian process is always cosy (see [BE 99]). It is easy to verify that a filtration immersed into a cosy filtration is itself cosy.

DEFINITIONS. — *A filtration* $(\mathcal{F}_t)_{t\in-\mathbb{N}}$ *is* standard *if it is generated by an independent sequence* $(V_t)_{t\in-\mathbb{N}}$ *of random variables with diffuse laws.*

A filtration is substandard *if it is isomorphic to a filtration immersed in a standard filtration.*

For instance, a filtration generated by an independent sequence $(V'_t)_{t\in-\mathbb{N}}$ of random variables is always substandard; indeed, up to isomorphism, it is possible to consider V'_t as given by $V'_t = f_t(V_t)$, where V_t are independent and diffuse; and in this case, the natural filtration of V' is immersed in that of V.

Clearly, all standard filtrations are isomorphic to each other. A standard filtration is generated by an independent sequence of Gaussian random variables, so it is always cosy; consequently, all substandard filtrations are cosy too.

Proposition 2.3 shows that the filtration generated by Example 2.1 satisfies the criterion for non-cosiness formulated in Proposition 3.2. So it is not cosy, and a fortiori not substandard.

4. — A continuous time-change for Brownian motion

THEOREM 4.1. — *On some* $(\Omega,\mathcal{A},\mathbb{P})$*, let* B *be a Brownian motion and* $\mathcal{F}$ *its natural filtration. Given any* $\varepsilon>0$*, there exists a family* $(T_t)_{t\geqslant 0}$ *of stopping times such that*
(i) $T_0=0$*;*
(ii) *almost all functions* $t\mapsto T_t$ *are smooth, increasing, with derivative* $\dfrac{dT_t}{dt}$ *verifying*

$$\left|\frac{dT_t}{dt}-1\right|\leqslant\varepsilon\ ;$$

(iii) *the time-changed filtration* $\mathcal{G}$ *defined by* $\mathcal{G}_t=\mathcal{F}_{T_t}$ *is not cosy.*

We start proving the theorem. From now on, ε, Ω, $\mathcal{A}$, $\mathbb{P}$, B and $\mathcal{F}$ are fixed. The construction of the time-change T_t will use a small lemma, notationally complicated but actually quite elementary.

LEMMA 4.2. — *The data are an integer* $p\geqslant 2$ *and an interval* $I=[a,b]$ *(with* $a<b$*). There exist* p *increasing bijections* $\sigma_1,\ldots,\sigma_p$ *from* I *onto itself,* p *numbers* $s_1<\ldots<s_p$ *in* I*, and an interval* $J\subset I$ *with non-empty interior, such that*
(i) *each* σ_q *is smooth, is identity in a neighbourhood of* a *and* b*, and its derivative satisfies*

$$\left|\frac{d\sigma_q}{dt}-1\right|\leqslant\varepsilon\ ;$$

(ii) *for* $1\leqslant r\leqslant q\leqslant p$*,* $\sup J\leqslant\sigma_r(s_q)$*; for* $1\leqslant q<r\leqslant p$*,* $\inf J\geqslant\sigma_r(s_q)$*.*

PROOF OF LEMMA 4.2. — Let ϕ be a C^∞ function equal to 1 on the interval $[\frac{3a+b}{4},\frac{a+3b}{4}]$ and with compact support included in the open interval (a,b). Setting $\alpha=\varepsilon/(p\sup|\phi'|)$, the functions $\sigma_q(t)=t-\alpha q\,\phi(t)$ clearly satisfy (i).

Put $s_q=\frac{a+b}{2}+q\alpha$. Using $\sup|\phi'|\geqslant 4/(b-a)$ and $\varepsilon<1$, one easily sees that $\frac{a+b}{2}<s_1<\ldots<s_p<\frac{a+3b}{4}$, implying $\phi(s_q)=1$ and $\sigma_r(s_q)=\frac{a+b}{2}+(q-r)\alpha$. So (ii) holds with $J=[\frac{a+b}{2}-\alpha,\frac{a+b}{2}]$. ∎

The next lemma describes the elementary bricks to be used in the construction. If J is an interval $[s,t]$ with $0 \leqslant s < t$, B^J will denote the normalized Brownian increment $(B_t - B_s)/\sqrt{t-s}$, which is $N(0,1)$-distributed.

LEMMA 4.3. — *Given an interval $I = [a,b]$ with $0 < a < b$, an integer $p \geqslant 2$, and a bounded, Borel function f defined on $\mathbb{R}$, let $\sigma_1, \ldots, \sigma_p$, $s_1, \ldots, s_p$ and J be as in Lemma 4.2; let Q be an $\mathcal{F}_a$-measurable r.v. with values in $\{1, \ldots, p\}$; set $T_t = \sigma_Q(t)$ and $R = f(B^J)$.*

(i) *For each $t \in I$, the random variable T_t is an $\mathcal{F}$-stopping time; the function $t \mapsto T_t$ from I to I is smooth, with derivative ε-close to 1.*

(ii) *For each $q \in \{1, \ldots, p\}$, there exists an $\mathcal{F}_{T_{s_q}}$-measurable r.v. R_q, equal to R on the event $\{Q \leqslant q\}$.*

(iii) *For each $q \in \{1, \ldots, p\}$ and every bounded, Borel ϕ, on the event $\{Q > q\}$ one has $\mathbb{E}[\phi \circ R | \mathcal{F}_{T_{s_q}}] = \mathbb{E}[\phi \circ R]$.*

The meaning of (ii) and (iii) is that, at time T_{s_q}, R is already known if $Q \leqslant q$, but still completely unknown if $Q > q$.

PROOF OF LEMMA 4.3. — (i) Since Q is $\mathcal{F}_a$-measurable, so is T_t too; as $T_t \geqslant a$, it is a stopping time. For fixed ω, the function $t \mapsto T_t(\omega)$ is one of the σ_q's constructed in Lemma 4.2, so its derivative is close to 1.

(ii) On the event $\{Q \leqslant q\}$, Lemma 4.2 (ii) gives $\sup J \leqslant \sigma_Q(s_q) = T_{s_q}$, so R_q can be defined by

$$R_q = \begin{cases} f(B^J) & \text{if } \sup J \leqslant T_{s_q}; \\ 0 & \text{if } \sup J > T_{s_q}. \end{cases}$$

(iii) On the event $\{Q > q\}$, Lemma 4.2 (ii) gives $\inf J \geqslant \sigma_Q(s_q) = T_{s_q}$, so on this event J is equal to the random interval

$$K = \begin{cases} J & \text{if } T_{s_q} \leqslant \inf J \\ [T_{s_q}, T_{s_q}+1] & \text{if } T_{s_q} > \inf J \end{cases}$$

and R to the random variable $S = f(B^K)$. But the Markov property at time T_{s_q} implies that B^K is independent of $\mathcal{F}_{T_{s_q}}$, with law $N(0,1)$; so, on the $\mathcal{F}_{T_{s_q}}$-event $\{Q > q\}$, one can write $\mathbb{E}[\phi \circ R \,|\mathcal{F}_{T_{s_q}}] = \mathbb{E}[\phi \circ S \,|\mathcal{F}_{T_{s_q}}] = \mathbb{E}[\phi \circ R]$. ∎

PROOF OF THE THEOREM. — Put $I_n = [2^n, 2^{n+1}]$; when n ranges over $\mathbb{Z}$, the intervals I_n form a subdivision of $(0,\infty)$. Choose a sequence $(p_n)_{n\in\mathbb{Z}}$ of integers such that $p_n \geqslant 2$ and $\sum_{n\leqslant 0} 1/p_n < \infty$. For each $n \in \mathbb{Z}$, Lemma 4.2 applied to I_n and p_n gives p_n bijections σ_q^n from I_n to itself, p_n numbers $s_q^n \in I_n$ and a sub-interval $J_n \subset I_n$. Choose some functions $f_n : \mathbb{R} \to \{1, \ldots, p_{n+1}\}$ such that the image of $N(0,1)$ by f_n is the uniform law on $\{1, \ldots, p_{n+1}\}$, and define $Q_n = f_n(B^{J_n})$; this random variable is $\mathcal{F}_{2^{n+1}}$-measurable and uniformly distributed on $\{1, \ldots, p_{n+1}\}$.

Set $T_0 = 0$ and for $t \in I_n$ let

$$T_t = \sigma^n_{Q_{n-1}}(t) .$$

At this point, it is worth interrupting the proof for a minute, to compare this formula with the discrete formula (2.1). The pointer Q_{n-1} depends only on the behaviour of B in the interval J_{n-1}; it tells us which of the σ_q will be used to time-change the interval I_n. According to Lemma 4.2, the image $\sigma_q^{-1}(J_n)$ of J_n by the chosen time-change will be included in one of the intervals $[s_r, s_{r+1}]$, and this interval is also completely determined by the pointer Q_{n-1}. The rôle of the lotteries $R_{n,q}$ is played by the behaviour of B on those intervals $[s_r, s_{r+1}]$; as they are disjoint intervals, the lotteries are independent. And the definition $Q_n = f_n(B^{J_n})$ says that the choice of the next pointer depends only on the result of the current lottery.

We resume proving the theorem. Since Q_{n-1} is $\mathcal{F}_{2^n}$-measurable, Lemma 4.3 (i) tells us that T_t is a stopping time, depends smoothly upon t, and that its derivative dT_t/dt is ε-close to 1. It remains to prove that the time-changed filtration $\mathcal{G}_t = \mathcal{F}_{T_t}$ is not cosy; this will be done by applying Proposition 3.2 to $\mathcal{G}$, with $t_n = 2^n$ and $U_n = Q_{n-1}$.

As $\mathcal{G}_{2^n} = \mathcal{F}_{2^n}$, Q_{n-1} is $\mathcal{G}_{2^n}$-measurable. As the J_n's are disjoint, the Q_n's are independent, and, for a deterministic sequence $(q_n)_{n\leqslant 0}$, the estimation

$$\mathbb{P}\big[(Q_n)_{n\leqslant 0} = (q_n)_{n\leqslant 0}\big] = \prod_{n\leqslant 0} \frac{1}{p_{n+1}} \leqslant \prod_{n\leqslant 0} \frac{1}{2} = 0$$

shows that the law of $(Q_n)_{n\leqslant 0}$ is diffuse.

To obtain non-cosiness, we shall show that, for any filtered probability space $(\bar\Omega, \bar{\mathcal{A}}, \bar{\mathbb{P}}, \mathcal{H})$ and any two filtrations $\mathcal{G}'$ and $\mathcal{G}''$ isomorphic to $\mathcal{G}$ and immersed in $\mathcal{H}$, one has for every $n \in \mathbb{Z}$

$$(*) \qquad \bar{\mathbb{P}}[Q'_n = Q''_n \,|\, \mathcal{H}_{2^n}] = \frac{1}{p_{n+1}} \qquad \text{on the event } \{Q'_{n-1} \neq Q''_{n-1}\};$$

since $\sum_{n\leqslant 0} 1/p_n$ converges, Proposition 3.2 will then apply, with $\varepsilon_n = 1/p_n$.

So n is now fixed, and, to simplify the notations, we shall write I, J, p, f, s_q instead of I_n, J_n, p_n, f_n, s^n_q. We shall also set $a = 2^n$ and substitute Q for Q_{n-1} and R for Q_n; so $T_t = \sigma^n_{Q_{n-1}}(t)$ becomes $T_t = \sigma_Q(t)$, $Q_n = f_n(B^{J_n})$ becomes $R = f(B^J)$, and we may freely use Lemma 4.3.

Supposing $\mathcal{G}'$ and $\mathcal{G}''$ are two filtrations isomorphic to $\mathcal{G}$ and immersed in some $\mathcal{H}$, Assertion $(*)$ can now be written

$$\bar{\mathbb{P}}[R' = R'' \,|\, \mathcal{H}_a] = \frac{1}{p_{n+1}} \qquad \text{on the event } \{Q' \neq Q''\}.$$

As $\mathcal{G}'$ and $\mathcal{G}''$ play the same role, it suffices by symmetry to establish this equality on the event $\{Q' < Q''\}$. So we may fix q in $\{1, \ldots, p\}$ and work on the event $A = \{Q' = q\,,\, Q'' > q\}$. The event $\{Q' = q\}$ is in $\mathcal{G}'_a$, hence also in $\mathcal{H}_a$; similarly $\{Q'' > q\}$ is in $\mathcal{H}_a$, and their intersection A is in $\mathcal{H}_a$ too. By isomorphic transfer, the following two facts are obtained from Lemma 4.3 (ii) and (iii):

a) There exists a $\mathcal{G}'_{s_q}$-measurable r.v. R'_q equal to R' on $\{Q' \leqslant q\}$; a fortiori, R'_q is $\mathcal{H}_{s_q}$-measurable and equal to R' on A.

b) For $1 \leqslant r \leqslant p_{n+1}$, $\bar{\mathbb{P}}[R'' = r \,|\, \mathcal{G}''_{s_q}] = \bar{\mathbb{P}}[R'' = r]$ on the event $\{Q'' > q\}$; since R'' is $\mathcal{G}''_\infty$-measurable, $\mathcal{G}''$ immersed in $\mathcal{H}$, and R uniformly distributed, this implies $\bar{\mathbb{P}}[R'' = r \,|\, \mathcal{H}_{s_q}] = 1/p_{n+1}$ on $\{Q'' > q\}$, and a fortiori on A.

For $r \in \{1, \ldots, p_{n+1}\}$, we may write

$$\begin{aligned} \mathbb{1}_A\, \bar{\mathbb{P}}[R' = R'' = r \,|\, \mathcal{H}_{s_q}] &= \mathbb{1}_A\, \bar{\mathbb{P}}[A\,,\, R' = R'' = r \,|\, \mathcal{H}_{s_q}] \\ &= \mathbb{1}_A\, \bar{\mathbb{P}}[A\,,\, R'_q = R'' = r \,|\, \mathcal{H}_{s_q}] = \mathbb{1}_A\, \mathbb{1}_{\{R'_q = r\}}\, \bar{\mathbb{P}}[R'' = r \,|\, \mathcal{H}_{s_q}] \\ &= \mathbb{1}_A\, \mathbb{1}_{\{R' = r\}}\, \frac{1}{p_{n+1}}\,. \end{aligned}$$

Summing over all r's from 1 to p_{n+1} gives $\mathbb{1}_A\, \bar{\mathbb{P}}[R' = R'' \,|\, \mathcal{H}_{s_q}] = \mathbb{1}_A\, \dfrac{1}{p_{n+1}}$; and, as $A \in \mathcal{H}_a$, applying $\bar{\mathbb{E}}[\;\;|\mathcal{H}_a]$ to both sides establishes the claim. ∎

REMARK. — Theorem 4.1 can be restated in terms of laws of martingales. Recall that every continuous martingale M with $M_0 = 0$ and $[M,M]_\infty = \infty$ can be written as a time-changed Brownian motion: $M = B_{[M,M]}$, where B is some Brownian motion and $[M,M]$ the quadratic variation of M. The sigma-field $\sigma(M)$ generated by M always contains $\sigma(B)$; when $\sigma(M) = \sigma(B)$, each $[M,M]_t$ is a stopping time for the filtration of B, and one says that M is *pure;* whether M is pure or not depends only on its law. (For more on pure martingales, see for instance Section V.4 of [RY 91].)

Call M *very pure* if it is pure and if the time-change $t \mapsto [M,M]_t$ that makes it Brownian is absolutely continuous and strictly increasing. If B, $(T_t)_{t\geqslant 0}$ and $\mathcal{G}$ are as in Theorem 4.1, the martingale $M_t = B_{T_t}$ is a very pure martingale, whith a non-cosy (and a fortiori non-Brownian) filtration $\mathcal{G}$.

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