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The Limitations of No-Arbitrage Arguments for Real Options

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Abstract

We consider an option c which is contingent on an underlying $\tilde{S}$ that is not a traded asset. This situation typically arises in the context of real options. We investigate the situation when there is a "surrogate" traded asset S whose price process is highly correlated with that of $\tilde{S}$. An illustration would be the cases where S and $\tilde{S}$ model two different brands of crude oil. The main result of the paper shows that in this case one cannot draw any non-trivial conclusions on the price of the option by only using no arbitrage arguments.

In a second step we try to isolate hedging strategies on the traded asset S which minimize the variance of the hedging error. We show in particular, that the naive strategy of simply replacing $\tilde{S}$ by S fails to be optimal and we are able to quantify how far it is from being optimal.

1 Introduction

The success of the celebrated Black-Scholes formula in the context of pricing and hedging derivative securities in financial markets is largely due to an important feature of this model: the prices obtained in the Black-Scholes model do not depend on preferences but only on a no-arbitrage argument. This pleasant fact corresponds to a basic feature of this model: any derivative security (e.g., a European call option just as well as a more complicated path-dependent option) can be *perfectly replicated* by an appropriate trading strategy on the underlying asset.

The methodology of the treatment of options in financial markets was extended to the context of real options [2]; the value of a real option is typically not derived from a traded asset as in the classical case, but rather from a random variable which is not traded in a liquid market. Nonetheless one tries to apply similar arguments as in the case of derivatives contingent on traded assets, sometimes referring to the possibility of hedging with *surrogate assets*; by this we mean a traded asset whose price process is closely related (but not identical) to the price process of the underlying of the real option, which typically is not a traded asset. The usual argument in favor of the use of "surrogate trading strategies" is that a sufficiently good "surrogate asset" should do just as well, or maybe almost just as well, as the underlying asset itself (if it were a traded asset).

In this note we critically analyze this common belief. In Section 2 below we formalize the setting of a real option on an underlying $\tilde{S}$, which is not a traded asset, but such that there is a traded assets S which is close to $\tilde{S}$. We do this by modeling S and $\tilde{S}$ as geometric Brownian motions with drift, correlated by a correlation coefficient ρ which is close to one (but not equal to one!). For an intuitive illustration one might think of the following situation: An option is written on the price of some brand XYZ of crude oil. We assume that there is no liquid market for this brand of crude oil (or, more realistically, for futures contracts on this brand), but there is some other brand UVW of crude oil for which a liquid futures market is available, allowing for (almost) frictionless trading. The idea is that the price process of these two brands should be sufficiently similar to justify the use of UVW as a "surrogate asset" for XYZ.

The main result of this note shows that for a European Call option written on the underlying $\tilde{S}$ we cannot conclude *anything* on its price by using only no-arbitrage arguments: if only trading in the surrogate asset S is allowed, then any number in $(0, \infty)$ is a possible price for this option without violating the no-arbitrage principle.

From an economic point of view this result is, of course, absurd. To fix ideas think of a European at-the-money call option on $\tilde{S}$. Nobody will be willing to sell the option at a price of, say, one thousandth of the price of $\tilde{S}$, or — vice versa — to buy the option at a price of, say, ten times the price of $\tilde{S}$. What the theorem states is that such absurd prices are not ruled out by no-arbitrage considerations (under the assumption that we are not allowed to trade in the asset $\tilde{S}$). The message of this theorem is not that applying the Black-Scholes methodology to real options is not correct, but: whenever one applies the Black-Scholes methodology to real options whose underling is not a traded asset one must be very careful and one has to be aware that preferences, subjective probabilities etc. have to come into the play. Relying on pure no-arbitrage arguments does not lead anywhere.

Having seen that prefect replication of an option written on the underlying $\tilde{S}$ is not possible by only trading in S, it is natural to ask how small the hedging error can be made. One way to quantify the hedging error is by considering the variance of this random variable which subsequently is to be minimized. This leads to the well-known concept of the variance-optimal martingale measure (see [10] and [12, 13]) which is closely related to the notion of the minimal martingale measure (see [4] and [3]) and gives rise to a hedging strategy using the Galtchouk-Kunita-Watanabe projection in an appropriate Hilbert space. Following [11, Example 4.3] we explicitly calculate this strategy and the variance of the corresponding hedging error and analyze its asymptotic behavior as the correlation coefficient ρ tends to one. We also compare this variance-optimal strategy with other — more naive — strategies. These latter considerations also bear some practical relevance as they clearly show that naive strategies, such as simply replacing S by $\tilde{S}$, are not optimal.

2 The Main Results

We consider a probability space $(\Omega, \mathcal{F}, P)$ that supports two independent standard Brownian motions W and $W^{\perp}$. Our time horizon is T so that we may write W as $(W_t)_{0 \le t \le T}$, $W^{\perp}$ as $(W_t^{\perp})_{0 \le t \le T}$, and the filtration $(\mathcal{F}_t)_{0 \le t \le T}$ is defined as the natural (right-continuous, saturated) filtration generated by W and $W^{\perp}$. We also assume $W_0 = W_0^{\perp} = 0$ which implies in particular that $\mathcal{F}_0$ is trivial. Another technical assumption which we make without loss of generality is that $\mathcal{F} = \mathcal{F}_T$.

Fix a constant $\rho \in (-1, 1)$ and define the Brownian motion $\tilde{W}$ by

$$\tilde{W} = \rho W + \sqrt{1 - \rho^2} W^{\perp}. \tag{1}$$

$\tilde{W}$ again is a standard Brownian motion whose correlation to W equals ρ, which we should think of as being close to one. We now fix real numbers μ, $\tilde{\mu}$, and r, and strictly positive numbers σ, $\tilde{\sigma}$ to define the asset price processes

$$dS = \mu S dt + \sigma S dW, \qquad d\tilde{S} = \tilde{\mu}\tilde{S} dt + \tilde{\sigma}\tilde{S} d\tilde{W} \tag{2}$$

where the initial values S_0 and $\tilde{S}_0$ are positive constants. Using Itô's formula, it follows that the random variables S_T and $\tilde{S}_T$ modeling the terminal values of the corresponding assets are given by the familiar expression

$$S_T = S_0 \exp(X_T), \quad \tilde{S}_T = \tilde{S}_0 \exp(\tilde{X}_T) \tag{3}$$

with

$$X_T = (\mu - \sigma^2/2)T + \sigma W_T, \quad \tilde{X}_T = (\tilde{\mu} - \tilde{\sigma}^2/2)T + \tilde{\sigma}\tilde{W}_T \tag{4}$$

two correlated Gaussian random variables. The bond price process is given by $B_t = B_0 e^{rt}$.

The financial market is given by the traded assets $(S_t)_{0 \le t \le T}$ and $(B_t)_{0 \le t \le T}$ and we assume – as usual in the Black-Scholes world — that frictionless trading in continuous time is possible in these assets. In contrast, we assume that the asset $\tilde{S}$ cannot be traded at all; but it is the underlying for a European call option C. The value of C at time T is given by the random variable

$$C_T = (\tilde{S}_T - K)_+, \tag{5}$$

where $K > 0$ is the strike price.

Our goal is to investigate what can be said about the pricing and hedging of C if we only can trade in the bond and the "surrogate asset" S.

The basic theme in the theory of pricing and hedging derivative securities is to determine the price C_0 of this option at time $t = 0$ and — if possible — to justify this price by replicating the option using an appropriate trading strategy on the available traded assets. In mathematical terms this amounts to writing the random variable C_T as the sum of a constant c_0, a stochastic integral $\int_0^T H_t dS_t$ and an integral $\int_0^T G_t dB_t$, where the predictable processes $(H_t)_{0 \le t \le T}$ and $(G_{0 \le t \le T})$ represent the investments at time t in stock and bond respectively. If such a replication of an option C is possible, then the usual no-arbitrage argument allows to conclude that c_0 is the unique arbitrage-free price at time $t = 0$ for the option C.

As announced in the introduction in the present setting we are far away from this situation and we only obtain trivial bounds for the possible arbitrage-free prices c_0.

Theorem 1 *Under the above assumptions, for any number $c_0 \in (0, \infty)$ the price $C_0 = c_0$ of the option at time zero is compatible with the no-arbitrage principle.*

More precisely, for every $c_0 \in (0, \infty)$, there is a probability measure Q on $\mathcal{F}$, equivalent to P, such that the discounted traded asset $(S_t e^{-rt})_{0 \le t \le T}$ is a Q-martingale and such that the pricing rule

$$C_0 := e^{-rT} E_Q[(\tilde{S}_T - K)_+] \tag{6}$$

yields the value $C_0 = c_0$.

Hence the financial market consisting of the traded assets $(S_t)_{0 \le t \le T}$, $(B_t)_{0 \le t \le T}$ and $(C_t)_{0 \le t \le T}$, where the option price process C is defined by

$$C_t = e^{-r(T-t)} E_Q[(\tilde{S}_T - K)_+ | \mathcal{F}_t] \tag{7}$$

admits an equivalent martingale measure and is therfore free of arbitrage.

Proof: The setting of the theorem actually is a special case of a by now well-known topic, namely the theme of hedging under convex constraints. The assertion of the theorem can be derived from [8], Exercise 4.5.6, p.86 and Example 4.1.2, p.75.

Since the general results of [8], Chapter 4, require a more complicated machinery and since the subsequent direct argument also allows for some economic interpretation and understanding we give a self-contained proof of the theorem.

The basic issue is to determine the set $\mathcal{M}^e(S)$ of all probability measures Q on $\mathcal{F}$, equivalent to P under which the discounted price process $S/B = (S_t e^{-rt})_{0 \le t \le T}$ of the traded asset S is a martingale. The basic insight of the seminal papers [5] and [6] was that – under some regularity conditions — the measures $Q \in \mathcal{M}^e(S)$ are in one-to-one correspondence to the consistent, i.e., arbitrage-free, pricing rules via formula (6). We refer to [1] for a general version of theses issues which are dealt with in full mathematical rigor and where — among other technicalities — one has to pass to the concept of local martingales. But in the present context these rather subtle considerations are not needed.

What are the equivalent martingale measures $Q \in \mathcal{M}^e(S)$ in the present situation? They are precisely those probability measures Q on $\mathcal{F}$, such that the logarithmic returns $(X_t)_{0 \le t \le T}$ as defined in (2) have drift rate $r - \sigma^2/2$ under the measure Q. Indeed, for an equivalent probability measure Q, this latter assertion is equivalent to the Q-martingale property of the discounted traded asset S/B.

Note that this requirement does not imply any restrictions on the drift of the process $(W_t^\perp)_{0\le t\le T}$ under Q, as the two Brownian motions W and $W^\perp$ are assumed to be independent under P, hence strongly orthogonal. This allows us to define for *arbitrary* $\nu \in \mathbb{R}$ a probability measure Q_ν, such that the drift rate of $(X_t)_{0\le t\le T}$ is $r - \sigma^2/2$, but the drift rate of $(\tilde{X}_t)_{0\le t\le T}$ is ν.

Fix $\nu \in \mathbb{R}$. By elementary algebra we can write

$$\begin{aligned} X_t &= \left(r - \frac{\sigma^2}{2}\right) t + \sigma(W_t - \lambda t) \\ \tilde{X}_t &= \nu t + \tilde{\sigma}\left[\rho(W_t - \lambda t) + \sqrt{1-\rho^2}(W_t^\perp - \lambda^\perp t)\right] \end{aligned} \tag{8}$$

with

$$\lambda = \frac{r-\mu}{\sigma} - \frac{\sigma}{2}, \quad \lambda^\perp = \frac{1}{\sqrt{1-\rho^2}}\left[\frac{\nu - \tilde{\mu}}{\tilde{\sigma}} - \rho\left(\frac{r-\mu}{\sigma} - \frac{\sigma}{2}\right)\right]. \tag{9}$$

We define Q_ν by $dQ_\nu/dP = Z_T$, where

$$Z_t = \exp\left[\lambda W_t + \lambda^\perp W_t^\perp - \frac{1}{2}\left(\lambda^2 + (\lambda^\perp)^2\right) t\right], \qquad 0 \le t \le T, \tag{10}$$

is a uniformly integrable martingale on $[0,T]$. Girsanov's Theorem [9, Chapter 3.5] tells us, that $(W_t - \lambda t)_{0\le t\le T}$ and $(W_t^\perp - \lambda_t^\perp)_{0\le t\le T}$ are two independent standard Brownian motions under Q_ν.

In particular, the random variable

$$U_T^{(\nu)} = \rho(W_t - \lambda t) + \sqrt{1-\rho^2}(W_t^\perp - \lambda^\perp t) \tag{11}$$

is Gaussian with mean zero and variance T under Q_ν. Now we let ν vary in $\mathbb{R}$: from

$$(\tilde{S}_T - K)_+ = (\tilde{S}_0 e^{\tilde{X}_T} - K)_+ = (S_0 e^{\nu T + \tilde{\sigma} U_T^{(\nu)}} - K)_+ \tag{12}$$

we immediately see that

$$\lim_{\nu\to-\infty} (\tilde{S}_0 e^{\nu T + \tilde{\sigma} U_T^{(\nu)}} - K)_+ = 0, \quad \lim_{\nu\to+\infty} (\tilde{S}_0 e^{\nu T + \tilde{\sigma} U_T^{(\nu)}} - K)_+ = \infty \tag{13}$$

almost surely. As the expectations, involving lognormal random variables are all finite, one easily verifies by the monotone convergence theorem that these limiting results also hold true for the corresponding expectations under Q_ν, i.e.,

$$\lim_{\nu\to-\infty} E_{Q_\nu}[(S_0 e^{\nu T + \tilde{\sigma} U_T^{(\nu)}} - K)_+] = 0, \quad \lim_{\nu\to+\infty} E_{Q_\nu}[(S_0 e^{\nu T + \tilde{\sigma} U_T^{(\nu)}} - K)_+] = \infty. \tag{14}$$

Summing up in less formal terms what we have done so far: For arbitrary $\nu \in \mathbb{R}$, we have constructed a probability measure Q_ν, equivalent to the original measure P, such that under Q_ν the process of logarithmic returns $(X_t)_{0\le t\le T}$ of the traded asset $(S_t)_{0\le t\le T}$ has the correct drift, namely $r - \sigma^2/2$, to make S/B a martingale. On the other hand Q_ν was fabricated in such a way that the process of logarithmic returns $(\tilde{X}_t)_{0\le t\le T}$ on the non-traded asset $\tilde{S}$ has a drift coefficient equal to ν. Speaking economically, for a given value of ν close to $+\infty$, the choice of the probability distribution Q_ν corresponds to the pricing rule applied by an agent believing that the asset $\tilde{S}$ will perform very well in the average; on the other hand, a given value of ν close to $-\infty$ corresponds to the pricing rule applied by an agent believing that $\tilde{S}$ will perform very poorly. Not too surprisingly — from an economic point of view — the above calculations reveal that in the former case an agent will price a call option on $\tilde{S}$ very high while in the latter case she will only be willing to pay a very low price for it.

This proves the first and the second assertion of the theorem; the third assertion now follows from the general theory as developed in [5] and [6]. $\square$

Remark: We have stated and proved the above theorem under the assumptions of constant coefficients $\rho, \sigma, \tilde{\sigma}, \mu, \tilde{\mu}$. But the proof easily carries over to the case, when we assume these quantities to be optional processes. We do not carry this out in detail, but only state the result in one important special case: we assume $\sigma, \tilde{\sigma}, \mu, \tilde{\mu}$ still to be constant, while $\rho = (\rho_t)_{0 \le t \le T}$ now is assumed to be an optional process taking values in $[-1, +1]$. Defining $(\tilde{W}_t)_{0 \le t \le T}$ via

$$d\tilde{W}_t = \rho_t dW_t + \sqrt{1 - \rho_t^2} dW_t^\perp \tag{15}$$

we are in an analogous situation as in the above theorem.

The conclusion — generalizing the above theorem — now reads as follows: either ρ attains its values almost everywhere (with respect to $P \otimes \lambda$, where λ denotes Lebesgue-measure on $[0, T]$) in $\{-1, +1\}$, in which case we are in the classical situation of a complete market and a European call option on the underlying $\tilde{S}$ can be perfectly replicated by trading in the asset S and therefore has a unique arbitrage-free price; or ρ attains values in $(-1, +1)$ with strictly positive $P \otimes \lambda$-measure, in which case again all prices in $(0, \infty)$ are possible arbitrage-free prices for a European call option on $\tilde{S}$, if only trading in S is permitted. The argument is the same as in the above proof with some minor technical modifications.

3 Trading strategies related to minimizing the variance of the hedging error

We have seen that in the setting of the previous section it is impossible to obtain a perfect replication of an option on $\tilde{S}$ by trading on the asset S. Hence we have to lower the stakes and look for trading strategies on the asset S such that the outcome is close to the random variable $(\tilde{S} - K)_+$. The concept of "close" will be interpreted in terms of the variance of the hedging error, which we shall minimize.

For the sake of simplicity and in order not to overload the presentation by too many constants, (we are afraid, the reader had already a sufficient dose in the above proof), we shall assume throughout this section that $S_0 = \tilde{S}_0 = K = 1$, $r = \mu = \tilde{\mu} = 0$, and $\sigma = \tilde{\sigma} = 1$. We observe that the assumptions $S_0 = \tilde{S}_0 = 1$, $r = 0$, and $\sigma = \tilde{\sigma} = 1$ are essentially just normalizing assumptions and do not really restrict the generality (as regards the assumption $r = 0$ one may always take the bond as numeraire to reduce to this case). On the other hand, the assumptions $\mu = \tilde{\mu} = 0$ indeed reduce the generality of presentation as this implies that S and $\tilde{S}$ are already martingales with respect to the original measure P. The more realistic case $\mu \neq 0$ and/or $\tilde{\mu} \neq 0$ requires more involved arguments and will be treated elsewhere.

The classical Black-Scholes theory applied to the financial market

$$(\tilde{S}_t)_{0 \le t \le T} = (\exp(\tilde{W}_t - t/2))_{0 \le t \le T}, \quad \text{and} \quad B_t \equiv 1,$$

yields the representation

$$(\tilde{S}_T - K)_+ = c + \int_0^T f(S_t, t) d\tilde{S}_t \tag{16}$$

where

$$c = \tilde{S}_0 N(d_1) - KN(d_2), \quad f(S_t, t) = N(d_1) \tag{17}$$

with

$$d_{1,2} = \frac{\ln(S_t/K) \pm T/2}{\sqrt{T}}. \tag{18}$$

As discussed above, this corresponds to the setting where $\tilde{S}$ is a traded asset, in which case perfect replication of the option $(\tilde{S}_T - K)_+$ by a trading strategy in the asset $\tilde{S}$ is possible. But our question is: what is the wisest thing to do, if we only can trade in the asset S ?

To formalize this question, we call a predictable process $(h_t)_{0\le t\le T}$ an *admissible trading strategy* for the asset S if the stochastic integral $(\int_0^t h_u dS_u)_{0\le t\le T}$ is an L^2-bounded martingale. As in view of our assumptions on the constants μ and σ we have

$$dS_t = S_t dW_t, \tag{19}$$

we also may write the above considered stochastic integral as

$$\int_0^t h_u dS_u = \int_0^t h_u S_u \frac{dS_u}{S_u} = \int_0^t h_u S_u dW_u. \tag{20}$$

From basic facts of stochastic integration we infer that a predictable process is an admissible trading strategy iff $E[\int_0^T h_t^2 S_t^2 dt] < \infty$.

We now can formalize our problem of minimizing the variance of the hedging error

$$\mathrm{Var}\left[(\tilde{S}_T - K)_+ - \left(x + \int_0^T h_t dS_t\right)\right] \to \min! \tag{21}$$

with minimization over all $x \in \mathbb{R}$, and all admissible trading strategies h. The term $x + \int_0^T h_t dS_t$ denotes the result of a trading strategy starting with an initial investment x at time $t = 0$ and subsequently holding h_t units of the asset S at time t. (As we assumed for simplicity, that $B_t \equiv 1$ here is no term corresponding to trading on the bond B.) Our aim is to determine the pair $(\hat{x}, \hat{h})$ such that $\hat{x} + \int_0^T \hat{h}_t dS_t$ minimizes (21).

Proposition 1 *The optimal solution to the optimization problem (21) is given by the following pair* $(\hat{x}, \hat{h})$*:*

1. $\hat{x} = c = \tilde{S}_0 N(d_1) - KN(d_2)$, *i.e.,* $\hat{x}$ *is just the Black-Scholes price of the option* $(\tilde{S}_T - K)_+$
2. $\hat{h} = \rho f(S_t, t)\frac{\tilde{S}_t}{S_t}$, *i.e., the optimal strategy for trading on* S *simply equals the* ρ*-fold of the optimal strategy for trading on* $\tilde{S}$ *(if this were permitted) times the fraction* $\tilde{S}_t/S_t$.
3. *The variance of the hedging error defined in (21) is given by* $(1-\rho^2)\mathit{Var}[(\tilde{S}_T - K)_+]$.

Proof: Setting $\tilde{h}_t = f(\tilde{S}_t, t)$ we have

$$\begin{aligned}(\tilde{S}_T - K)_+ &= \hat{x} + \int_0^T \tilde{h}_t d\tilde{S}_t = \hat{x} + \int_0^T \tilde{h}_t \tilde{S}_t d\tilde{W}_t \\ &= \hat{x} + \rho \int_0^T \tilde{h}_t \tilde{S}_t dW_t + \sqrt{1-\rho^2} \int_0^T \tilde{h}_t \tilde{S}_t dW_t^{\perp}. \end{aligned} \tag{22}$$

Obviously $E[(\tilde{S}_T - K)_+] = \hat{x}$ and

$$\mathrm{Var}[(\tilde{S}_T - K)_+] = E\left[\left(\int_0^T \tilde{h}_t \tilde{S}_t d\tilde{W}_t\right)^2\right] = E\left[\int_0^T \tilde{h}_t^2 \tilde{S}_t^2 dt\right]. \tag{23}$$

The hedging error for starting with capital $\hat{x}$ and trading according to the strategy $\hat{h}_t = \rho \tilde{h}_t \tilde{S}_t / S_t$ is

$$L_T = (\tilde{S}_T - K)_+ - \left(\int_0^T \hat{h}_t dS_t\right) = \sqrt{1-\rho^2} \int_0^T \tilde{h}_t \tilde{S}_t dW_t^{\perp}, \tag{24}$$

and so $E[L_T] = 0$ and

$$\mathrm{Var}[L_T] = (1-\rho^2)E\left[\int_0^T \tilde{h}_t^2 \tilde{S}_t^2 dt\right] = (1-\rho^2)\mathrm{Var}[(\tilde{S}_T - K)_+]. \tag{25}$$

To prove that $\hat{h}$ is in fact optimal, it suffices to show, that the hedging error L_T is orthogonal to the result of any admissible trading on S. Let

$$G_T = x + \int_0^T h_t dS_t = x + \int_0^T h_t S_t dW_t \tag{26}$$

with arbitrary $x \in \mathbb{R}$ and arbitrary admissible h. Then

$$E[G_T L_T] = E\left[\sqrt{1-\rho^2}\left(x + \int_0^T h_t S_t dW_t\right) \cdot \int_0^T \tilde{h}_t \tilde{S}_t dW_t^{\perp}\right] = 0, \tag{27}$$

since W and $W^{\perp}$ are orthogonal martingales. □

Remark: The variance of the hedging error as a function of the correlation ρ is a parabola; it is $1-\rho^2$ times the constant $\mathrm{Var}[(\tilde{S}_T - K)_+]$. As $\rho \to 1$ we can write $(1-\rho^2) = (1-\rho)(1+\rho) \sim 2(1-\rho)$, thus the variance of the hedging error tends linearly to zero in $1-\rho$.

It is not difficult to compute explicitly an expression for the constant $\mathrm{Var}[(\tilde{S}_T - K)_+] = E[(\tilde{S}_T - K)_+^2] - E[(\tilde{S}_T - K)_+]^2$, namely

$$\begin{aligned} \mathrm{Var}[(\tilde{S}_T - K)_+] &= e^T S_0^2 N(d_0) - 2S_0 K N(d_1) + K^2 N(d_2) \\ &\quad - [S_0 N(d_1) - K N(d_2)]^2 \end{aligned} \tag{28}$$

with

$$d_0 = \frac{\ln(S_0/K) + 3T/2}{\sqrt{T}} \tag{29}$$

and d_1, d_2 as above.

4 Comparison with a naive strategy

We recall once more that, if trading in $\tilde{S}$ were allowed, we could replicate the option $(\tilde{S}_T - K)_+$ perfectly,

$$(\tilde{S}_T - K)_+ = c(\tilde{S}_0, T) + \int_0^T f(\tilde{S}_t, t) d\tilde{S}_t \tag{30}$$

with c the Black-Scholes price and f the corresponding delta-hedging strategy. Therefore we could try — somewhat naively — to simply replace $\tilde{S}$ by the traded asset S. We will call this strategy the *imitation strategy*.

What is the result of this "naive" strategy: it simply replicates the option on S, i.e.,

$$(S_T - K)_+ = c(S_0, T) + \int_0^T f(S_t, t) dS_t. \tag{31}$$

The hedging error $\tilde{L}_T$ is given by the random variable

$$\tilde{L}_T = (\tilde{S}_T - K)_+ - (S_T - K)_+. \tag{32}$$

Again for (mainly notational) simplicity we only discuss the case

$$\mu = 0, \sigma = 1, \tilde{\mu} = 0, \tilde{\sigma} = 1, r = 0, S_0 = 1, \tilde{S}_0 = 1, K = 1, T = 1.$$

Then

$$\tilde{L}_T = (e^{\tilde{U}} - 1)_+ - (e^U - 1)_+ \tag{33}$$

with $(U, \tilde{U})$ denoting a bivariate normal random variable with mean $-1/2$, variance 1, and correlation ρ. To state the following result we need the bivariate standard normal distribution function, see [7, Appendix 11B].

Proposition 2 *The variance of the imitation strategy is*

$$\mathit{Var}[\tilde{L}_T] = 2\left(F_2 - F_{11}(\rho)\right), \tag{34}$$

where

$$F_2 = eN(\tfrac{3}{2}) - 3N(\tfrac{1}{2}) + 1 \tag{35}$$

$$F_{11}(\rho) = e^{\rho} M(\rho + \tfrac{1}{2}, \rho + \tfrac{1}{2}, \rho) - 2M(\tfrac{1}{2}, \rho - \tfrac{1}{2}, \rho) + M(-\tfrac{1}{2}, -\tfrac{1}{2}, \rho). \tag{36}$$

Here N denotes the univariate standard normal distribution function and M the bivariate standard normal distribution function.

Asymptotically, as $\rho \to 1$, we have

$$\mathit{Var}[\tilde{L}_T] \sim 2eN(\tfrac{3}{2}) \cdot (1 - \rho), \tag{37}$$

with coefficient $2eN(\frac{3}{2}) \approx 5.0734$.

The technical proof of this result is given in the appendix. Let us compare this with the optimal solution of the previous section: The minimal variance of the hedging error is

$$\mathrm{Var}[L_T] = \left(eN(\tfrac{3}{2}) + N(\tfrac{1}{2}) - 4N(\tfrac{1}{2})^2\right) \cdot (1 - \rho^2), \tag{38}$$

which asymptotically yields

$$\mathrm{Var}[L_T] = 2\left(eN(\tfrac{3}{2}) + N(\tfrac{1}{2}) - 4N(\tfrac{1}{2})^2\right) \cdot (1 - \rho) \tag{39}$$

with coefficient $2\left(eN(\frac{3}{2}) + N(\frac{1}{2}) - 4N(\frac{1}{2})^2\right) \approx 2.6313$.

This shows that the variance of the hedging error of the imitation strategy tends to zero as $\rho \to 1$ with the same order as the minimal variance, but the leading coefficient is almost twice as high. Hence, roughly speaking, the price of being naive is that one ends up with a variance of the hedging error which is twice as high as it can be by acting a little wiser.

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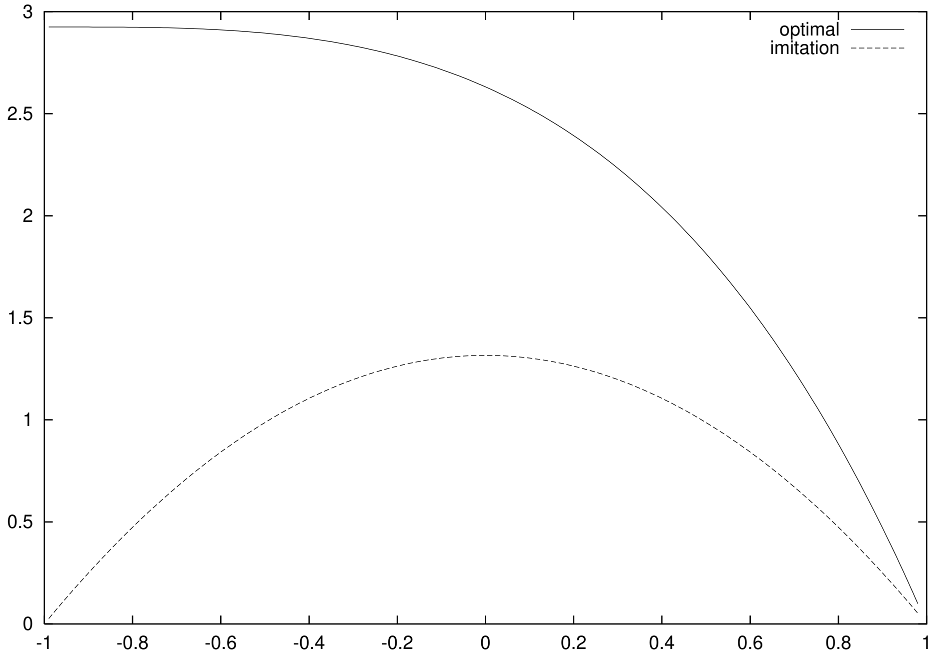

Figure 1: Comparison of variance of hedging error for optimal and imitation strategy as function of the correlation coefficient ρ.

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A Appendix

We start with the following elementary formulas:

$$\mathrm{Var}[\tilde{L}_T] = 2\mathrm{Var}[(e^U - 1)_+] - 2\mathrm{Cov}[(e^{\tilde{U}} - 1)_+, (e^U - 1)_+] \tag{40}$$

$$\mathrm{Var}[(e^U - 1)_+] = E[(e^U - 1)_+^2] - E[(e^U - 1)_+]^2 \tag{41}$$

$$\mathrm{Cov}[(e^{\tilde{U}} - 1)_+, (e^U - 1)_+] = E[(e^{\tilde{U}} - 1)_+(e^U - 1)_+] - E[(e^U - 1)_+]^2. \tag{42}$$

Let $A = \{U > 0, \tilde{U} > 0\}$. Completing the squares in the exponent (or more sophisticated) by using the bivariate Esscher transform, we obtain

$$E[(e^{\tilde{U}} - 1)_+(e^U - 1)_+] = E[e^{\tilde{U}+U}; A] - 2E[e^U; A] + P[A] \tag{43}$$
$$= e^{\rho} M(\rho + \tfrac{1}{2}, \rho + \tfrac{1}{2}, \rho) - 2M(\tfrac{1}{2}, \rho - \tfrac{1}{2}, \rho) + M(-\tfrac{1}{2}, -\tfrac{1}{2}, \rho) \tag{44}$$

where M is the standard bivariate normal distribution function,

$$M(x_1, x_2, \rho) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} m(z_1, z_2, \rho) dz_1 dz_2, \tag{45}$$

and

$$m(z_1, z_2, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{z_1^2 - 2\rho z_1 z_2 + z_2^2}{2(1-\rho^2)}\right]. \tag{46}$$

the standard bivariate normal density. We are interested in $\rho \to 1$ from below. Since $m(z_1, z_2, \rho)$ behaves rather irregularly as $(z_1, z_2, \rho) \to (0, 0, 1)$ it is not clear that or how a simple trivariate Taylor expansion can be applied, so we need the following detour. Let $M(x, \rho) := M(x, x, \rho)$, $M_1(x, y, \rho) := \partial M(x, y, \rho)/\partial x$ and $M_1(x, \rho) := M_1(x, x, \rho)$.

Lemma 1 *Using the above notation we have the following asymptotic relations:*

$$M(\rho + \tfrac{1}{2}, \rho + \tfrac{1}{2}, \rho) = M(\tfrac{3}{2}, \tfrac{3}{2}, \rho) - 2M_1(\tfrac{3}{2}, \tfrac{3}{2}, \rho)(1-\rho) + O(1-\rho)^2 \tag{47}$$
$$M(\tfrac{1}{2}, \rho - \tfrac{1}{2}, \rho) = M(\tfrac{1}{2}, \tfrac{1}{2}, \rho) - M_1(\tfrac{1}{2}, \tfrac{1}{2}, \rho)(1-\rho) + O(1-\rho)^{3/2} \tag{48}$$

Proof: A careful look at the bivariate Taylor expansion of $M(x, y, \rho)$ with respect to x and y shows that the remainder term is uniformly $O(1-\rho)^2$. This is not true for M_1, but there appears $(1-\rho)^{-1/2}$, which reduces the quadratic error term to $O(1-\rho)^{3/2}$. □

Next we look at $M(x, \rho)$ as $\rho \to 1$. By conditioning we obtain

$$M(x_1, x_2, \rho) = \int_{-\infty}^{x_1} N\left(\frac{x_2 - \rho z_1}{\sqrt{1-\rho^2}}\right) n(z_1) dz_1 \tag{49}$$

in particular

$$M(x, x, \rho) = \int_{-\infty}^{x} N\left(\frac{x - \rho z}{\sqrt{1-\rho^2}}\right) n(z) dz. \tag{50}$$

Since $N'''(x) = (x^2 - 1)n(x)$ is bounded we get from Taylor's Theorem for fixed $x \in \mathbb{R}$

$$N\left(\frac{x - \rho z}{\sqrt{1-\rho^2}}\right) = N\left(\frac{x - z}{\sqrt{1-\rho^2}}\right) + N'\left(\frac{x - z}{\sqrt{1-\rho^2}}\right) z\sqrt{\frac{1-\rho}{1+\rho}} \tag{51}$$
$$+\frac{1}{2}N''\left(\frac{x - z}{\sqrt{1-\rho^2}}\right) z^2 \frac{1-\rho}{1+\rho} + O\left(z^3(1-\rho)^{3/2}\right) \tag{52}$$

uniformly in z as $\rho \to 1$. Plugging this expression into (50) yields

$$M(x, \rho) = I_0(x, \rho) + I_1(x, \rho)\sqrt{\frac{1-\rho}{1+\rho}} \tag{53}$$
$$+\frac{1}{2}I_2(x, \rho)\frac{1-\rho}{1+\rho} + O\left((1-\rho)^{3/2}\right) \tag{54}$$

with

$$I_k(x, \rho) = \int_{-\infty}^{x} z^k N^{(k)}\left(\frac{x - z}{\sqrt{1-\rho^2}}\right) n(z) dz. \tag{55}$$

With a simple substitution and partial integration we rewrite

$$I_0(x,\rho) = \frac{1}{2}N(x) + \frac{1}{\sqrt{2\pi(1-\rho^2)}}\int_0^\infty n\left(\frac{u}{\sqrt{1-\rho^2}}\right)N(x-u)du \tag{56}$$

Watson's Lemma tells us now

$$I_0 = N(x) - \frac{n(x)}{\sqrt{2\pi}}\sqrt{1-\rho^2} + \frac{n'(x)}{4}(1-\rho^2) + O(1-\rho)^{3/2}. \tag{57}$$

The integrals I_1 and I_2 can be computed explicitly, and after painful elementary calculations we see

$$I_1 = \frac{xn(x)}{\sqrt{2}}\sqrt{1-\rho} + O(1-\rho) \tag{58}$$

and

$$I_2 = O\left(\sqrt{1-\rho}\right). \tag{59}$$

Combining these results yields

$$M(x,\rho) = N(x) - \frac{n(x)}{\sqrt{\pi}}\sqrt{1-\rho} + O(1-\rho)^{3/2} \tag{60}$$

Also

$$M_1(x,\rho) = \frac{n(x)}{2} + \frac{xn(x)}{\sqrt{4\pi}}\sqrt{1-\rho} + O(1-\rho). \tag{61}$$

Using this expansion in (47) we obtain (37). □