Pricing, No-arbitrage Bounds and Robust Hedging of Installment Options*

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Abstract

An installment option is a European option in which the premium, instead of being paid up-front, is paid in a series of installments. If all installments are paid the holder receives the exercise value, but the holder has the right to terminate payments on any payment date, in which case the option lapses with no further payments on either side. We discuss pricing and risk management for these options, in particular the use of static hedges, and also study a continuous-time limit in which premium is paid at a certain rate per unit time.

Key words: Option Pricing, Exotic Options, Stable Hedging, Replicating Portfolios, No-arbitrage bounds.
JEL classification: C 15, G 13

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1 Introduction

In a conventional option contract the buyer pays the premium up front and acquires the right, but not the obligation, to exercise the option at a fixed time T in the future (for European-style exercise) or at any time at or before T (for American-style exercise). In this paper we consider an alternative form of contract in which the buyer pays a smaller up-front premium and then a sequence of "installments", i.e. further premium payments at — generally — equally spaced time intervals before the maturity time T. A typical case would be a $T = 1$-year option in which four installments are paid at 0, 3, 6 and 9 months. If all installments are paid the buyer can exercise the option, European-style, at time T. Crucially, though, the buyer has the right to 'walk away': if any installment is not paid then the contract terminates with no further payments on either side. We argue that this structure is attractive from several points of view. From the buyer's side he can enter the option at low initial cost and has a great deal of 'optionality' in the form of the right to cancel at each installment date. From the writers perspective, hedging is simple. We will show that there is a very effective static hedge that largely immunizes the writer against volatility and model risk (for typical market situations).

The case of two installments is the compound option (an option on an option), previously considered by Geske [8] and Selby and Hodges [17]. Let $C(t, T, S, K)$ denote the Black-Scholes value at time t of a European call option with strike K maturing at time T when the current underlying price is S (all other model parameters are constant). Installments p_0, p_1 are paid at times t_0, t_1 and final exercise is at time $T > t_1$. At time t_1 the holder can either pay the premium p_1 and continue to hold the option, or walk away, so the value at t_1 is $\max(C(t_1, T, S(t_1), K) - p_1, 0)$. The holder will pay the premium p_1 if this is less than the value of the call option. The value of this contract at t_0 is thus the value of a call on C with 'strike' p_1.

Another way of looking at it, that will be useful later, is this: the holder buys the underlying call at time t_0 for a premium $p = p_0 + e^{-r(t_1-t_0)}p_1$ (the NPV of the two premium payments where r denotes the riskless interest rate) but has the right to sell the option at time t_1 for price p_1. The compound call is thus equivalent to the underlying call option plus a put on the call with exercise at time t_1 and strike price p_1. The value p is thus greater than the Black-Scholes value $C(t_0, T, S(t_0), K)$, the difference being the value of the put on the call.

A similar analysis applies to installment options with premium payments[1] $p_0, p_1, \ldots, p_k$ at times $t_0, \ldots, t_k < T$. The NPV of the premium payments is $p = \sum_{i=0}^{k} p_i e^{-r(t_i-t_0)}$ and the installment option is equivalent to paying p at time t_0 and acquiring the underlying option plus the right to sell it at time t_j, $1 \le j \le k$ at a price $q_j = \sum_{i=j}^{k} p_i e^{-r(t_i-t_j)}$ (all subsequent premiums are 'refunded' when the right to sell is exercised). The installment option is thus equivalent to the underlying option plus a Bermuda put on the underlying option with time-varying strike q_i.

There is very little literature on installment options, the only paper we know

[1] Invariably $p_1 = p_2 = \ldots = p_k$ but it may be the case that $p_0 \neq p_1$, i.e. the up-front payment is not the same as subsequent installments.

of being an article by Karsenty and Sikorav [15] for a popular publication. Pricing models for installment options are included in some option pricing software packages, for example Monis.

Installment options are currently the most actively traded warrant offered at the Australian Stock Exchange. The most popular type is a two-payment installment option, which allows the buyer to pay 1/2 of the stock price now and subsequently pay the remainder to own the share. In this guise, this product is simply a compound call on a European call. Recently, Deutsche Bank offered a 10-year warrant with 9 annual payments. This product has been extremely successful and is closer to the installment options examined here.

While the concept of compound options can be generalised to allow the underlying option to be non-standard (for example, American or an exotic option), we will restrict our analysis here to standard European options as the underlying. This is done as most of these products assume an underlying European option and most extensions of compound option methodology to other areas of finance make similar assumptions. In addition, we will concentrate on installment call options. While some discussion will examine compound (and installment) put options, the most common of these products used by investors is the compound (or installment) call options and pricing relationships follow directly from parity relationships.

This paper is laid out as follows. In the next section we discuss pricing in the Black-Scholes framework. As for American options we cannot provide an analytic formula for the price and a finite-difference algorithm must be used. In section 3, simply-stated 'no-arbitrage' bounds on the price are derived valid for very general price process models. As will be seen, these depend on comparison with other options and suggest possible classes of hedging strategies. In section 4 we introduce and analyse static hedges for installment options. The concluding section 5 gives further remarks and applications.

2 Pricing in the Black-Scholes framework

Consider an asset whose price process S_t is the conventional log-normal diffusion

$$dS_t = rS_t\,dt + \sigma S_t\,dw_t, \tag{1}$$

where r is the riskless rate and w_t a standard Brownian motion; thus (1) is the price process in the risk-neutral measure. The volatility $\sigma > 0$ will presently be assumed to be constant while in the subsequent sections we shall also allow for stochastic volatility. We consider a European call option on S_t with exercise time T and payoff

$$[S_T - K]^+ = \max(S_T - K, 0). \tag{2}$$

The Black-Scholes value of this option at time 0 is of course

$$p_{\mathrm{BS}} = Ee^{-rT}[S_T - K]^+. \tag{3}$$

p_{BS} is the unique arbitrage-free price for the option, to be paid at time 0. As an illustrative example we will take $T = 1$ year, $r = 0$, $K = 100$, $S_0 = 100$ and $\sigma = 25.132\%$, giving $p_{\mathrm{BS}} = 10.00$.

In an installment option we choose times $0 = t_0 < t_1 < \cdots < t_n = T$ (generally $t_i = iT/n$ to a close approximation). We pay an upfront premium p_0 at t_0 and pay an 'installment' of p_1 at each of the $n-1$ times $t_1, \ldots, t_{n-1}$. We also have the right to walk away from the deal at each time t_i: if the installment due at t_i is not paid then the deal is terminated with no further payments on either side. The pricing problem is to determine what is the no arbitrage value of the premium p_1 for a given value of p_0. The present value of premium payments – assuming they are all paid – is

$$p_0 + p_1 \sum_{i=1}^{n-1} e^{-rt_i}, \tag{4}$$

so in view of the extra optionality we certainly expect that

$$p_1 > \frac{1}{\sum_1^{n-1} e^{-rt_i}} \left(p_{\mathrm{BS}} - p_0\right). \tag{5}$$

Computing the exact value is straightforward in principle. Let $V_i(S)$ denote the net value of the deal to the holder at time t_i when the asset price is $S_{t_i} = S$. In particular

$$V_n(S) = [S - K]^+. \tag{6}$$

At time t_i we can either walk away, or pay p_1 to continue, the continuation value being

$$E_{t_i,S(t_i)}[e^{-r(t_{i+1}-t_i)} V_{i+1}(S_{t_{i+1}})]. \tag{7}$$

Thus

$$V_i(S) = \max(0, E_{t_i,S}[e^{-r(t_{i+1}-t_i)} V_{i+1}(S_{t_{i+1}})] - p_1). \tag{8}$$

In particular, V_{n-1} is just the maximum of 0 and $\mathrm{BS} - p_1$, where BS denotes the Black-Scholes value of the option at time t_{n-1}. The unique arbitrage free value of the initial premium is then

$$p_0 = V_0^+(S_0, p_1) := E_{t_0,S_0}\left[e^{-r(t_1-t_0)} V_1(S_{t_1})\right].$$

For fixed p_1, $V_0^+(S_0, p_1)$ is easily evaluated using a binomial or trinomial tree and this determines the up-front payment p_0. If we want to go the other way round, pre-specifying p_0, then we need a simple one-dimensional search to solve the equation $p_0 = V_0^+(S_0, p_1)$ for p_1. A similar search solves the equation $\widehat{p} = V_0^+(S_0, \widehat{p})$ giving the installment value $\widehat{p}$ when all installments, including the initial one, are the same.

Figure 1 shows the price $\widehat{p}$ at time 0 for our standard example with 4 equal installments. For comparison, one quarter of the Black-Scholes value is also shown. At $S_0 = 100$, $\widehat{p} = 3.284$, which is 31% greater than one quarter of the Black-Scholes value. At higher values of S_0 there is less of an increase, not surprisingly since when the option is well into the money there is high probability of paying all the installments and collecting the exercise value. The value of the walkaway optionality in this case is small. For example when $S_0 = 120$ the Black-Scholes price is 23.75 and $\widehat{p} = 6.90$ which is 16% greater then 23.75/4. Figure 2 shows $V_1(S)$, the value of the option at time t_1 as defined by (8), with $p_1 = 3.284$. It has positive value when $S(t_1) > 98.28$.

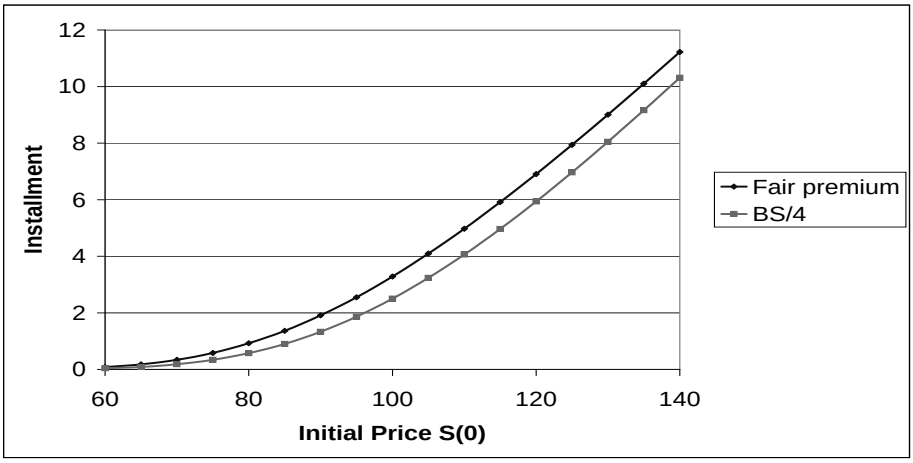

Figure 1: *Fair installment value and Black-Scholes value*

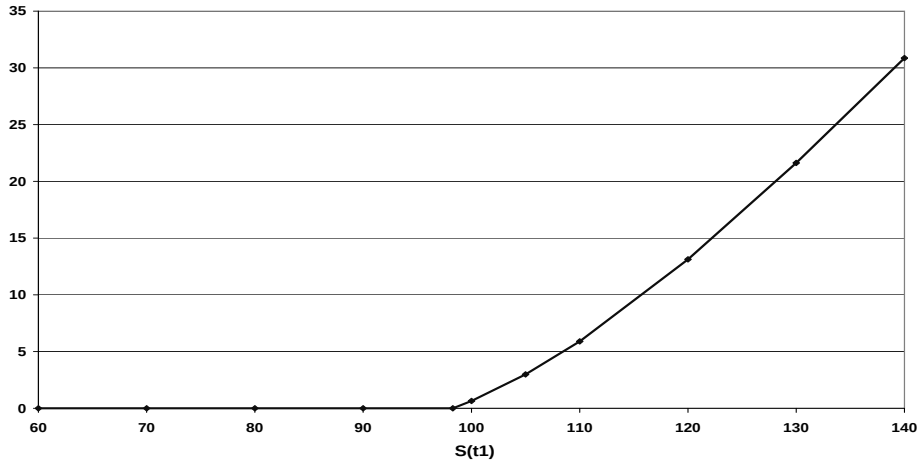

Figure 2: *Value of installment option at time t_1 as function of price $S(t_1)$*

3 No-arbitrage bounds derived from static hedges

The pricing model of the previous section makes the standard Black-Scholes assumptions: log-normal price process, constant volatility. By considering static super-replicating portfolios, however, we can determine easily computable bounds on the price valid for essentially arbitrary price models. We need only assume that for any $s \in [t_0, T]$ there is a liquid market for European calls with maturities $t \in [s, T]$, the price being given by[2]

$$C(s,t,K) = \mathbb{E}_Q \left[e^{-r(t-s)} (S_t - K)^+ \middle| \mathcal{F}_s \right] \tag{9}$$

where Q is a martingale measure for the process S and $\mathcal{F}_s$ denotes the information available at time s. By put-call parity this also determines the value of put options $P(s,t,K)$. We know today's prices $C(t_0,t,K)$ and that is *all* we know about the

[2]We use the compressed notation $C(s,t,K)$ here, denoting the option price as an $\mathcal{F}_s$-measurable random variable. It may depend not only on the current price $S(s)$ but also on some additional information contained in the sigma-algebra $\mathcal{F}_s$.

process S and the measure Q. We ignore interest rate volatility, assuming for notational convenience that the riskless rate is a constant, r, in continuously compounding terms. We also assume that no dividends are paid.

Let us first consider a 2-installment, i.e. compound, option, with premiums p_0, p_1 paid at t_0, t_1 for an underlying option with strike K maturing at $T = t_2$.

The subsequent result provides no-arbitrage bounds on the prices p_0, p_1 which are independent of the special choice of the model S and the equivalent martingale measure Q.

Proposition 1 *For the compound option described above, there is an arbitrage opportunity if p_0, p_1 do not satisfy the inequalities*

$$C\left(t_0, T, K + e^{r(T-t_1)}p_1\right) \geq p_0 \geq C(t_0, T, K) - e^{-r(t_1-t_0)}p_1 + P(t_0, t_1, p_1). \quad (10)$$

Proof Denote $K' = K + e^{r(T-t_1)}p_1$ and suppose we sell the compound option with agreed premium payments p_0, p_1 such that $p_0 \geq C(t_0, T, K')$. We then buy the call with strike K' and place $x = p_0 - C(t_0, T, K') \geq 0$ in the riskless account. If the second installment is not paid, the value of our position at time t_1 is $xe^{r(t_1-t_0)} + C(t_1, T, K') \geq 0$, whereas if the second installment is paid we add it to the cash account, and the value at time T is then $xe^{r(T-t_0)} + p_1e^{r(T-t_1)} + C(T, T, K') - C(T, T, K) \geq 0$. This is an arbitrage opportunity, giving the left-hand inequality in (10).

Now suppose the compound option is available at p_0, p_1 satisfying

$$p_0 + e^{-r(t_1-t_0)}p_1 \leq C(t_0, T, K) + P(t_0, t_1, p_1). \quad (11)$$

We buy it, i.e. pay p_0, and sell the two options on the right (call them $\widehat{C}$, $\widehat{P}$), so our cash position is $\widehat{C} + \widehat{P} - p_0 \geq e^{-r(t_1-t_0)}p_1$. At time t_1 the cash position is therefore $x \geq p_1$, and we have the right to pay p_1 and receive the call option. We exercise this right if $C(t_1, T, K) \geq p_1$. Then our cash position is $x - p_1 \geq 0$, $\widehat{C}$ is covered and $\widehat{P}$ will not be exercised because $p_1 < C(t_1, T, K) \leq S(t_1)$ (the call option value is never greater than the value of the underlying asset). On the other hand, $\max(p_1 - C(t_1, T, K), 0) \geq \max(p_1 - S(t_1), 0)$, so if $p_1 > C(t_1, T, K)$ we do not pay the second installment and still have enough cash to cover $\widehat{C}$ and $\widehat{P}$. Thus there is an arbitrage opportunity when the right-hand inequality in (10) is violated. ■

How tight are the no-arbitrage bounds given by the proposition in practice? To examine this we consider the illustrative example of section 2, but with just two installments. Figure 3 shows the Black-Scholes value of p_0 as a function of the second installment p_1, and the upper and lower bounds given by (10). Looking at the 3 lines in the middle of the figure they all start at the value $p_0 = 10$ on the left end of the graph; this is just the Black-Scholes price of the option when there is no second installment. Now vary the payment p_1 of the second installment along the horizontal axis and look at the resulting initial payment p_0: the line in the middle indicates the Black-Scholes price as derived in section 2 while the upper and the lower lines indicate the upper and the lower bounds provided by proposition 1. For example, if $p_1 = 3$, the Black-Scholes value for the up-front

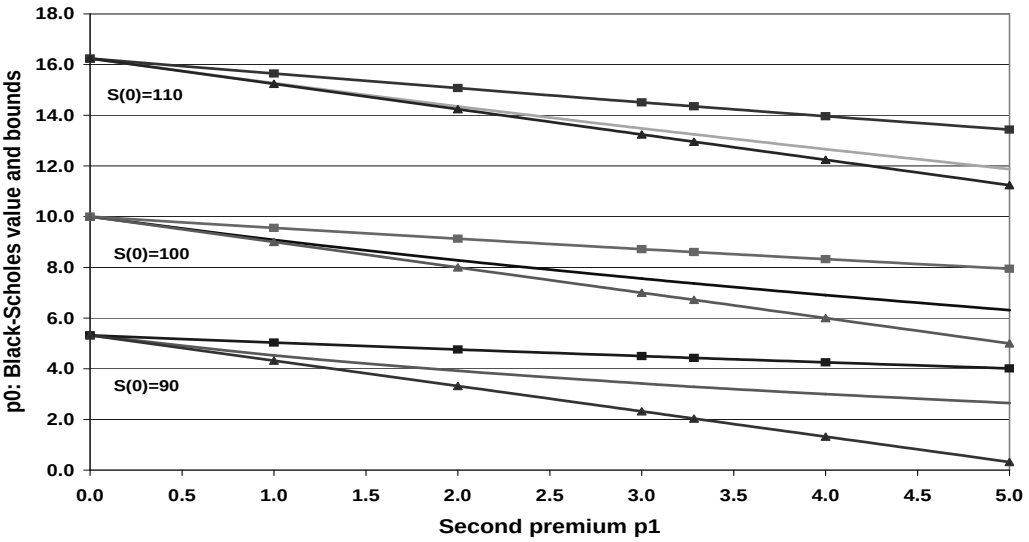

Figure 3: *"Standard example" with 2 installments: No-arbitrage bounds for the initial premium p_0 as a function of the second installment p_1.*

payment equals $p_0 = 7.556$; the upper bound ($p_0 = 8.720$) is 15.4% higher, while the lower bound ($p_0 = 7.000$) is 7.3% lower than the Black-Scholes price.

When moving to higher values of p_1 the tightness of the bounds decreases: for example, for $p_1 = 5$ the upper (resp. lower) bound is 25.8% (resp. 20.8%) higher (resp. lower) than the Black-Scholes price $p_0 = 6.311$.

While the three lines in the middle indicate the situation for at-the-money options (recall that we had $S_0 = K = 100$ in our standard example) the three lines on the top resp. bottom of the figure indicate the situation of in-the-money ($S_0 = 110$, $K = 100$) resp. out-of-the-money ($S_0 = 90$, $K = 100$) options. We observe that in the former case the Black-Scholes price is closer to the lower bound while in the latter case it is closer to the upper bound. This reflects the fact that for in-the-money options the installment p_1 is paid with high probability while this probability decreases when we shift to the out-of-the-money situation.

How can these no-arbitrage bounds be used in practice? To expect that one can use them to effectively make arbitrage profits is probably too bold a hope, at least in liquid markets. Rather one should read them as a recipe to get a limit on the maximal losses by adopting a static hedge.

Taking the point of view of the writer of a compound option she can obtain a static hedge limiting the loss to the difference of the terms in the left hand inequality of (10), namely $C\left(t_0, T, K + e^{r(T-t_1)}p_1\right) - p_0$. As argued in the first part of the above proof the writer of the option can buy a European call with strike $K' = K + e^{r(T-t_1)}p_1$ by adding to the received up-front premium p_0 the difference $C(t_0, T, K') - p_0$ from her own cash account. The remaining portfolio can only generate gains thus limiting the overall loss to this difference. If the difference is non-positive she was lucky enough to encounter an arbitrage opportunity; if the difference is positive but not too big, she still might be interested in this trading opportunity.

The appealing feature of this estimate is its robustness: it does not depend on the choice of the model and/or the martingale measure.

Let us give an interpretation of the difference $C(t_0, T, K') - p_0$ of the left hand side inequality as the premium of a contingent claim which may be interpreted as the advantage of "being clairvoyant" at time t_1: What is the difference of the discounted pay-off function of the European option which can be purchased at price $C(t_0, T, K')$ and the pay-off function of the compound option which can be purchased at price p_0? When comparing these two random variables we assume that the holder of the compound option behaves rationally, i.e. paying the second installment p_1 iff $C(t_1, T, K) \geq p_1$. The discounted pay-off function of the difference of these two contingent claims is given by

$$Cl = \begin{cases} e^{-r(t_1-t_0)} \left(\left(p_1 - (S_T - K)e^{-r(T-t_1)}\right)_+ \wedge p_1 \right) & \text{if } C(t_1, T, K) \geq p_1 \\ e^{-r(t_1-t_0)} \left(\left((S_T - K)e^{-r(T-t_1)} - p_1\right)_+ \right) & \text{if } C(t_1, T, K) < p_1. \end{cases} \tag{12}$$

Indeed the random variable Cl is the pay-off of a "clair-voyant" agent disposing at time t_1 of the information S_T as opposed to an agent disposing only of the information $\mathcal{F}_{t_1}$ and thus paying the installment p_1 iff $C(t_1, T, K) \geq p_1$. In this case the latter agent will at time T regret her decision of having paid p_1 if it finally turns out that $(S_T - K)e^{-r(T-t_1)}$ is less than p_1 while in the opposite case when $C(t_1, T, K) < p_1$ she will regret not having paid p_1 at time t_1 when $(S_T - K)e^{-r(T-t_1)}$ is bigger than p_1.

Quantifying the degree of regret quickly yields formula (12). Hence by no-arbitrage the difference between $C(t_0, T, K)$ and p_0 is just the price for the contingent claim Cl, in other words

$$C(t_0, T, K') - p_0 = \mathbb{E}_Q \left[Cl \,|\mathcal{F}_{t_0}\right]. \tag{13}$$

Lwt us now pass to the right hand side of inequality (10): it pertains to the situation of the buyer of a compound option who looks for a static hedge to limit the maximal loss as is explained in the second half of the above proof.

Of course, for practical purpose $P(t_0, t_1, p_1)$ will be a negligible quantity (and there will be no liquid market as typically $p_1 \ll S_0$) but for obtaining theoretically sharp bounds one must not forget this term.

In fact, the above inequalities are sharp: it is not hard to construct examples of arbitrage-free markets such that the differences in the left (resp. right) inequality in (10) become arbitrarily small.

Finally let us interpret the right hand side of inequality (10) by using the interpretation of the compound option given in the introductory section: the net present value $p_0 + e^{-r(t_1-t_0)}p_1$ of the payment for the compound must equal — by no-arbitrage — the price $C(t_0, T, K)$ of the corresponding European option plus a put option to sell this call option at time t_1 at price p_1. Denoting the latter security by Put(Call) we obtain the no-arbitrage equality.

$$p_0 = C(t_0, T, K) - e^{-r(t_1-t_0)}p_1 + \text{Put(Call)}. \tag{14}$$

In the proof of proposition 1 we have (trivially) estimated this Put on the Call from below by the corresponding Put $P(t_0, t_1, p_1)$ on the underlying S. We now see that the difference in the right hand inequality of (10) is precisely equal to the difference Put(Call) $- P(t_0, t_1, p_1)$ in this estimation.

Similar arguments as for the compound options apply for n installments. As we just argued for the 2-installments case, holding the installment option is equivalent to holding the underlying option plus the right to sell this option at any installment date at a price equal to the NPV of all future installments. The value of the Bermuda option on the option is greater than the equivalent option on the stock. This gives us the following result.

Proposition 2 *For the n-installment call option with premium payment p_0 at time t_0 and p_1 at times $t_1, \ldots, t_{n-1}$ there is an arbitrage opportunity if p_0, p_1 do not satisfy*

$$C(t_0, T, K + \widehat{p}_1) > p_0 > \left[C(t_0, T, K) - e^{r(T-t_0)}\widehat{p}_1 + P_{\text{Ber}}(t_0)\right]_+ . \tag{15}$$

Here

$$\widehat{p}_1 = p_1 \sum_{i=1}^{n-1} e^{r(T-t_i)} \tag{16}$$

and $P_{\text{Ber}}(t_0)$ denotes the price at time t_0 of a Bermuda put option on the underlying $S(t)$ with exercise times $t_1, \ldots, t_{n-1}$ and strike price K_i at time t_i, where

$$K_i = p_1 \sum_{j=i}^{n-1} e^{-r(t_j - t_i)}. \tag{17}$$

4 Dynamic and Static Hedging

A current theme in the literature has been the implementation of static as opposed to dynamic hedging strategies for exotic options. Recent papers, which propose static hedging (using standard European options) for exotic option, include Derman, Ergener and Kani [7, 6], Carr, Ellis and Gupta [3], Chou and Georgiev [5] and Carr and Pichon [4]. For most of these papers, the emphasis has been on the hedging of barrier and lookback options. Thomsen [19] has compared the relative benefits of such static strategies to traditional dynamic approaches for barrier options and Tompkins [20] compared static and dynamic hedging strategies for a wider range of exotic options. In the Tompkins [20] paper, compound options were specifically considered.

Thomas [18] first proposed the static hedge of compound options (which can be thought of as a two-payment installment options). Thomas suggests that a standard European option will provide an upper bound on the value of a compound call on a call (put) and Tompkins [20] confirmed this. Tompkins [20] extends this to consider compound puts. This research extends this static hedging approach to installment options and considers the implications for this hedge providing an approximation for the pricing of these products.

In section 2 the pricing formula for installment options was derived under the perfect market assumptions made by Black and Scholes. The unique solution to pricing is intimately associated with the construction of a riskless dynamic hedging portfolio. In Section 3 a static hedging strategy was suggested using standard European options. The purpose of this section is to

examine and compare these hedging strategies in detail. By way of introduction, let us consider the standard example of section 2 with two equal installments. The fair value is $p_0 = p_1 = 5.855$. Our proposed static hedge is as follows: at time 0 we receive p_0 and buy a 1-year call option with strike $K' = K + p_1 = 105.855$. This will cost 7.627, so we have to borrow $p' = 1.772$. Figure 4 shows the P&L of the hedged position at time $t_1 = 0.5$ as a function of the price $S(t_1)$. The P&L is equal to $\min\{-p' + C', -p' + C' + p_1 - C\}$, where $C' = C(t_1, T, S(t_1), K'), C = C(t_1, T, S(t_1), K)$. The maximum loss is p', which is 17.72% of the Black-Scholes premium for the underlying option, and the maximum gain is 20.3% of this premium. Figure 5 shows the distribution of P&L under the risk-neutral measure; this turns out to be close to the uniform distribution. This quick analysis shows that the static hedge has some attractive features.

In the rest of this section we will examine hedging of multi-installment options under more realistic scenarios than those of section 2. This will be done by simulation of the dynamic hedge of section 2 and the static hedge of 3. In these simulations many of the assumptions of perfect market conditions are relaxed. The simulations will consider discrete (daily) hedge rebalancing, transaction costs when dealing in the underlying asset or the option, and stochastic volatility. The objective is to assess the sensitivity of the pricing solutions and bounds on valuation under more realistic market conditions.

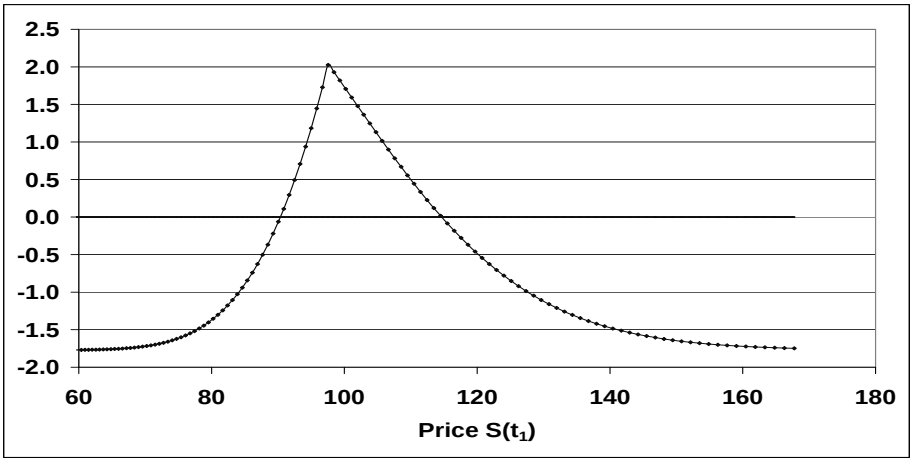

Figure 4: *The Profit/Loss Profile of a Compound Option Static Hedge as a function of the underlying Asset Price* $S(t_1)$

4.1 Dynamic Hedging of Compound Options

A practical study on the effectiveness of dynamic hedging for compound options was done in Tompkins [20]. As with that study, this paper will consider the cost of hedging compound and installment options from the standpoint of the option writer. In this paper, we will relax the assumptions of continuous and frictionless financial markets and will assess how the cost of hedging deviates from the theoretical values derived under perfect (continuous) financial markets.

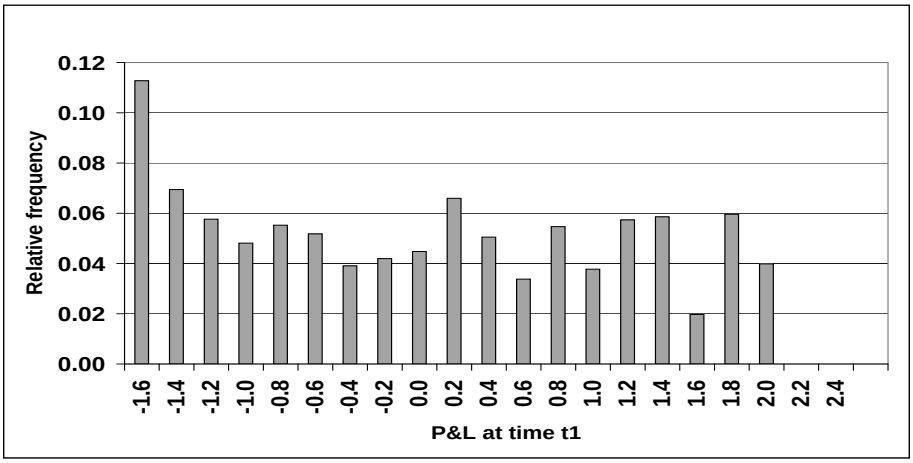

Figure 5: *The Distribution of Static Hedge Profits/Losses under the risk neutral measure*

To evaluate the hedging performance of compound and installment options, hedging was done by simulation. Underlying stock price series were simulated for 1000 paths of 180 days. These simulations used the anti-thetic approach suggested by Boyle [2] and the control variate method suggested by Hull & White [11] to increase efficiency. In a similar spirit to Hull [12], we assumed that the stock did not pay dividends and compared the cost to the writer of hedging the option to the theoretical value. The price process for the asset was assumed to conform to equation (1).

Apart from the discrete increments also assumed by Hull [12], we included for the same set of simulated underlying prices, transactions costs when dealing in the underlying asset (or options in the static hedge). For this simulation, we assumed that the spread between the bid and offer price (of both the underlying and options) was fixed at $1/16^{th}$. In addition, a proportional cost reflecting a commission of 0.01% was charged (of both the underlying values of the stock purchase or of the option) whenever a transaction took place[3].

The hedging in discrete time with transactions costs was further distinguished by examination of two cases. The first case assumed that the volatility $[(\sigma) > 0]$ is a fixed constant (the Black-Scholes model) and the second case allowed the volatility to be stochastic. The stochastic process modeling the volatility $(\sigma_t)_{0\leq t\leq T}$ assumed was a mean reverting process described by equation (18) [10] [Hull & White (1987) model]. When this case was considered 1000 paths of stochastic volatility were simulated [using the Euler discrete approximation in equation (19)] and 1000 new price paths were determined with $\widetilde{\sigma}$ replacing the

[3]Thus, when price paths were simulated, the determination of the delta was based upon the simulated price. However, when the quantity of stock was purchased (sold), the dealing price was fixed at $1/32^{nd}$ more (less) than the simulated price levels (the units in which the price was quoted). As an example, if the simulated price were 100, the buying price was assumed to be $100 - 1/32^{nd}$ and the selling price was $99 - 31/32^{nd}$. This was assumed for the entire amount of the asset purchased or sold. The percentage commission of 0.01% was charged on the total amount purchased or sold of the stock and in this simulation we assumed the number of shares in the contracts was 100,000 shares. The same spread and commission rate applied whenever options were purchased or sold.

constant volatility σ in equation (1). For both cases, the same random draws from the Wiener Process (w_t) were used to determine the price paths. The stochastic volatility model chosen can be expressed as:

$$d\sigma = \kappa\sigma(\theta - \sigma)dt + \xi\sigma d\widetilde{w}_t \tag{18}$$

and the path of volatilities was determined using an Euler approximation of the form:

$$\widetilde{\sigma}_t = \widetilde{\sigma}_{t-1} + \Delta\sigma \tag{19}$$

In equation (18), κ represents the rate of mean reversion, which was set to 16 for this simulation. The term ξ reflects the volatility of volatility input and this was set to 1.0. The term θ is the long-term level of the instaneous volatility and this was set to 20% per annum. These parameters were similar to ones reported by Hull & White [11] for stocks. The term σ is the instaneous volatility realised by the stochastic process and given that we are examining this model in discrete time, is replaced by $\widetilde{\sigma}$. In equation (18), the volatility inputs in the right-hand side of the equation are replaced by $\widetilde{\sigma}_{t-1}$. The term $\widetilde{w}_t$ reflects draws from a Wiener Process independent of the draws $(\widetilde{w}_t)$ used to determine the price paths.

The pricing model determined the compound or the installment theoretical price and the respective delta (relative to the underling stock price) using a binomial approximation to the Geske [9] model. The prices of the instruments and their derivatives required for dynamic hedging were estimated from the Monis software system (associated with the London Business School). A 90-day compound call option (with the right to purchase a 3 month 100 European call) was considered. The starting value of the underlying stock was set to 98, with a fixed interest rate of 5% , no dividends and a starting volatility of 20%. Using these inputs, the initial theoretical value of the compound option $(\widehat{p})$ was equal to 3.293 with the striking price of the compound set equal to this amount $(p_1 = 3.293)$. As a comparison, the underlying (180-day) European call $C(s, t, K)$ was equal to 5.695 at the inception of the compound call.

Following the customary dynamic hedging approach (with daily rebalancing of the portfolio), an arbitrary notional amount of 100000 shares was assumed and the delta amount of the stock was purchased. This was partially paid for by the receipt of the premium and the remaining funds required to purchase the delta amount of the stock was borrowed. Each subsequent day, the new estimated delta was determined and the hedge portfolio was revised. As with Hull [12], the costs of rebalancing were accumulated until either the compound expired (unexercised and dynamic hedging ceased) or the compound was exercised and the hedger dynamically hedged the underlying European call option to expiration.

In the absence of transactions costs, the expected cost of hedging the option will exactly be zero. In the stochastic volatility case, Merton [16] and Hull & White [10] demonstrate that the theoretical value of the option will be the Black Scholes [1] value evaluated using the average realised volatility of the holding period. The stochastic volatility process in equation (17) will yield an expected realised volatility of θ (which in this instance is 20%) and the expectation of a zero hedging cost is retained. Across the 1000 simulations, we determined the average cost of hedging and divided this by the theoretical value of the underlying

call option at the inception. This can be expressed as:

$$AD = HC/C(s,t,K) \tag{20}$$

Where AD is the average percentage difference, HC is the average hedging cost and $C(t,S,K)$, is the theoretical value of the European call option underlying the compound call. Given that the expectation of the hedging cost in the absence of transactions costs is zero, to reduce the errors introduced by the selection of the 1000 random price paths, the average hedging cost was standardised to zero. This control variate approach (suggested by Hull & White [11]) assured that for the first simulation scenario (solely discrete time rebalancing with no transactions costs and constant volatility) the average hedging cost would be exactly equal to the theoretical value[4]. This can be found in Table 1 (under the column labelled *Discrete Dynamic Hedging, No Transaction Costs*) with an average hedging cost equal to 0.00%. In addition, we were also interested in assessing the variability of this hedge result. We measure this variability (or hedge performance) by dividing the standard deviation of the cost of hedging the option by the theoretical value of the underlying call option. This can be expressed as:

$$HP = \frac{\sigma_{HC}}{C(s,t,K)} \tag{21}$$

This is the same measure of hedging performance proposed by Hull [12]. In this table, the reader can see that this standard deviation was equal to 7.11% of the theoretical value of the option in the case of constant volatility and no transactions costs[5].

Using the same paths of 180 days (1000 trials), transactions costs were included in the dynamic hedging strategy. The delta used to construct the hedge portfolio was determined using the simulated price but when purchases of stock were required, the hedger was required to pay $1/32^{nd}$ (units) more and when selling would receive $1/32^{nd}$ less. In all instances, a 0.01% commission was charged relative to the value of the transaction. In the table, under the column *Discrete Dynamic Hedging Transaction Costs*, one can see that the average cost of hedging the call option for 1000 simulations has risen to 3.07% of the theoretical premium value. The standard deviation (hedge performance) is essentially unchanged at 7.35% (from 7.11% for discrete hedging without transactions costs). This is to be expected as the inclusion of transactions costs to the same price paths only serves to increase the average cost of hedging and will not affect the deviation of the performance.

Of further interest was the impact of stochastic volatility on the dispersion of hedge costs. In Table 1, the reader will find the results of this hedging simulation

[4]Without the control variate method, a slight difference in the average hedging cost was found but this was not significantly different from 0.0.

[5]Using this standard deviation it is possible to use a T-test to assess if the average cost of hedging is significantly different from zero. In this case, the use of the control variate approach assures that the average hedging cost is zero. When a T-test is applied to the other cases, the inclusion of transactions costs significantly increases the costs of hedging and for the case with stochastic volatility with no transactions costs, the difference is by design equal to zero. Even without the use of the control variate adjustment this difference was not significantly different from zero.

	Constant Volatility		Stochastic Volatility	
	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs
Average Difference (%) Hedging Cost / Theoretical Value	0.00 %	-3.07 %	0.00 %	-3.14 %
Standard Deviation (%) of Hedging Cost / Theoretical Value	7.11 %	7.35 %	13.10 %	13.18 %
Average Loss (%) Hedging Cost / Theoretical Value	-4.77 %	-5.95 %	-11.71 %	-11.93 %
Standard Deviation (%) of Hedging Loss /	4.62 %	5.46 %	9.70 %	10.34 %
Average Gain (%) Hedging Cost / Theoretical Value	4.77 %	2.88 %	11.71 %	8.79 %
Standard Deviation (%) of Hedging Gain	5.35 %	5.03 %	6.70 %	6.00 %

Table 1: *Results of 1000 Simulation Runs for Delta Hedging a Compound Call*

under the column *Stochastic Volatility*. As was stated previously, the averaged expected hedging cost will be zero (Merton [16] and Hull & White [10, 11]). This allows the use of the control variate method to increase the efficiency of the simulation. As would be expected, the standard deviation of the hedging costs has risen to 13.10%. Clearly, the variability of the hedging costs indicates that when volatility changes the price path that could occur for the underlying asset, the dispersion of hedging results increases two-fold.

Finally, to assess both the impacts of transactions costs and stochastic volatility on the hedge costs the compound option; the 1000 simulation runs included both conditions. The results of this simulation can be seen under the column *Stochastic Volatility Discrete Dynamic Hedging Transaction Costs*. As one would expect, both the hedging costs and the variability of the difference between the costs and the theoretical value rise. The average hedging cost was 3.14% higher than the theoretical value of the option (similar to the constant volatility discrete hedging with transactions costs) and the standard deviation of hedging costs was 13.18% (similar to the stochastic volatility discrete hedging without transactions costs).

An interesting issue is whether the hedging errors will be distributed in a symmetrical manner. To examine this, the hedging results were split into losses and gains. Hence, we divided the 1000 sample paths into two types: those where the final hedging result (without and with transactions costs) were negative and

positive. The average for both the losses and gains along with their standard deviations appear in the lower rows of Table 1. From this hedging simulation, it appears that the standard deviation of losses and gains when dynamically hedging are similar when constant volatility is assumed. However, this is not the case when stochastic volatility is considered.

The standard deviation of losses is 30–40% higher compared to gains when stochastic volatility is considered. From an economic point of view, it seems reasonable to assume that option writers would be more concerned about the variability of losses (as opposed to gains). These results suggest that the inclusion of stochastic volatility in the state space complicates expected hedge variability, as the variability of losses is higher than gains.

4.2 Static Hedging of Compound Options

In the previous section of this paper, where bounds on compound and installment options are considered, an alternative static hedging strategy is suggested. Thomas [18] has previously proposed that writers of compound call options could hedge the option with a static hedge consisting of the purchase of a European call option with a term to expiration of $T - t_0$. and a strike price of $K + e^{r(T-t_1)}p_1$. The proof of this upper bound on the compound price is given in Proposition 1.

For the simulation of this hedge, exactly the same price paths examined for the dynamic hedge were used. In addition, the same assumptions regarding the transactions costs and stochastic volatility process were made. From the previous section on the upper boundary of the compound price, a 180 day European call with a striking price of 103.293 was purchased upon the sale of the 90 day compound call ($p_1 = 3.293$ and the underlying European call $K = 100$). To fund the purchase of the static hedging call, borrowing at a constant rate of 5% was required in addition to the receipt of the compound premium.

If at the expiration of the compound option, the value of the underlying call (strike price $K = 100$) was less than the compound payment (of 3.293), the purchased static call was sold at the current market price. The value of the European call was determined using the simulated underlying price (and volatility) at that point and used the Black Scholes [1] formula. Otherwise, the compound payment was made to the seller who invests it into her cash account and the terminal payoffs of the two European options were examined. Table 2 displays the hedging results for the four alternative scenarios of the 1000 simulated price paths.

In this table, the cost of the hedge was slightly (but not substantially) higher than the theoretical value of the underlying 100 call. The almost negligible impact of the transactions is due to the fact that these costs only apply when the call is purchased and at expiration when it is sold. Regarding the average performance due to changes in volatility levels, the reader may recall that the expected level of volatility was 20%. Therefore, even though a stochastic volatility process was introduced, that would not change the expected (or average result).

As opposed to the previous dynamic hedging simulations, the standard deviation of the hedging costs has increased but is not substantively impacted when volatility levels are allowed to vary. This is to be expected as the purchase of the

	Constant Volatility		Stochastic Volatility	
	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs
Average Difference (%) Hedging Cost / Theoretical Value	-0.07 %	-0.63 %	-0.25 %	-1.09 %
Standard Deviation (%) of Hedging Cost / Theoretical Value	21.34 %	21.34 %	21.13 %	21.18 %
Average Loss (%) Hedging Cost / Theoretical Value	-14.02 %	-14.43 %	-14.05 %	-14.60 %
Standard Deviation (%) of Hedging Loss /	4.79 %	4.99 %	4.83 %	5.06 %
Average Gain (%) Hedging Cost / Theoretical Value	13.95 %	13.80 %	13.79 %	13.51 %
Standard Deviation (%) of Hedging Gain	15.87 %	15.64 %	16.08 %	15.95 %

Table 2: *Results of 1000 Simulation Runs for Static Hedging a Compound Call*

European call allows hedging against stochastic volatility. As with the previous simulation of the dynamic hedging of compound options (see Table 1), an asymmetry exists in the hedging performance for losses and gains. In this instance, the standard deviation of hedging losses is $1/3^{rd}$ of the standard deviation of hedging gains, for all four hedging scenarios. In the instance of the stochastic volatility scenarios, the standard deviations in hedging losses is one half the standard deviation of hedging losses compared to the dynamic hedging of these products (see Table 1). When volatility is assumed to be constant the levels of the standard deviation of hedging losses are similar for the dynamic and static hedging strategies.

Essentially, the static hedge losses are bounded and are equal to the initial borrowing required to establish the hedge. The maximum loss is related to the no-arbitrage bounds considered in the previous section. The potential gains are unbounded as profits exist when the compound fails to be exercised and the static hedger sells the static European call in the market. Even in the instance of possible losses, for the static hedge the average hedging loss was 14.60% (with stochastic volatility and transactions costs) and for dynamic hedging the average hedging loss was 11.93%. One could argue that the dynamic hedging strategy might still be preferred as the average expected loss is less. However, the assumptions of transactions costs levels made in this simulation are somewhat low. It is conceivable for higher levels of transactions costs; the static hedge would

be preferred. In addition (and maybe most importantly), the maximum loss for the static hedge was only 18.97% of the underlying European call value, while the maximum loss for the dynamic hedging strategy was 64.35%. A moment's reflection on the logic for the establishment of the static hedge will make this result obvious. The static hedge is defined to bound potential losses and this is not sample dependent, but the maximum loss amount is known a priori and is known from Proposition 1 (and the subsequent remarks).[6] Such asymmetrical potential realisations may induce option writers to prefer static hedging strategies to dynamic strategies when stochastic volatility and transactions costs are included in the state space.

4.3 Dynamic Hedging of Installment Options

As was the case for compound options, installment options can be hedged either dynamically or statically. In a Black Scholes world, the dynamic hedging can be seen as simply hedge as you go. When the initial installment is paid, the seller dynamically hedges the option in the standard way and at the time of the next installment the premium should be exhausted (the costs of hedging will exactly equal the premium receipt). If the option is not worth the next installment, the option is cancelled and no further hedging is required. On the other hand, if the next installment is paid, the seller continues dynamically hedging with the replenished funds introduced by the installment payment. This continues until the option expires or the holder fails to pay the next installment.

There are some differences in the dynamic hedging ratios of installment options compared to standard European calls. The delta of an installment option at launch and throughout its life is generally lower than the delta for a standard European call. However, once the last payment is made, the installment option actually becomes a classical Black-Scholes vanilla option, and the deltas of both types of options are at that point equal.

Using the same approach as was used for the dynamic hedging of the compound option, two 180-day installment options with 6 payments was considered. As with the previous example, the underlying European call option had a strike price of 100, the starting price of the stock was 98, interest rates were constant at 5% and the starting volatility was assumed to be 20%.

The first installment option considered had a relatively higher initial payment of 2.532 with five further payments at one-month intervals of 0.80. If all six installments were paid, in month 5, the holder will be in possession of a standard European call ($K = 100$) expiring in one month's time. The second installment option considered had all six payments set to the same level. Such an installment option (given the initial market conditions) would have equal payments of 1.219.

As with the compound hedge, the delta of the installment was also estimated

[6] In this example, the initial amount to be borrowed was $99,217$ without transactions costs and $102,342$ with transactions costs. Given the accumulation of interest during the period of 180 days at 5% (continuously compounded), this results in a terminal cost of $101,694$ for the no transactions costs case and $104,897$ for the transactions costs case. These represent 17.86% and 18.42% of the Black Scholes price respectively. The fact that the maximum loss was found to be 18.97% reflects the additional transactions cost associated with unwinding the static hedge.

and the delta amount of the underlying stock was purchased (with a notional amount of 100000 shares). The purchase of the shares was partially met by the receipt of the installment premium with the remainder borrowed.

For both examples of installment options, if at the installment payment dates, the value of the installment option was greater than the fixed payments, the installment premium was paid to the seller and dynamic hedging continued until the next installment date. If the installment was not paid, the hedging account was closed but the costs where financed to the terminal date to allow comparisons with the underlying European call. Table 3 displays the results of the hedging simulations for the four scenarios for the first variant of the installment option (with a higher upfront payment).

	Constant Volatility		Stochastic Volatility	
	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs
Average Difference (%) Hedging Cost / Theoretical Value	0.00 %	-2.36 %	0.00 %	-2.40 %
Standard Deviation (%) of Hedging Cost / Theoretical Value	8.09 %	8.36 %	13.57 %	13.90 %
Average Loss (%) Hedging Cost / Theoretical Value	-4.23 %	-5.34 %	-10.42 %	-11.11 %
Standard Deviation (%) of Hedging Loss /	5.10 %	5.67 %	9.73 %	10.24 %
Average Gain (%) Hedging Cost / Theoretical Value	4.23 %	2.98 %	10.42 %	8.71 %
Standard Deviation (%) of Hedging Gain	6.52 %	6.33 %	8.23 %	7.85 %

Table 3: *Results of 1000 Simulation Runs for Delta Hedging the first Variant of an Installment Call*

In the case of a 180-day installment option with unequal payments, results are roughly similar to those found for the compound option. The inclusion of transactions costs when hedging significantly increases the cost of hedging relative to the theoretical value of the underlying call option. However, this impact is substantially reduced relative to the compound. The standard deviation of hedging costs is slightly higher for the constant volatility scenarios but essentially the same for the stochastic volatility scenarios.

The reason the impacts of transactions costs on dynamic hedging of this installment option is reduced is due to the large number of installments (6 in this

instance). In many instances, the second installment payment is never made. Thus, dynamic hedging was only required for the first 30 days. This reduction in the time period of dynamic hedging would clearly reduce the accumulated transactions costs. In Table 4, the number of payments made for the 1000 price paths are reported.

# Payments	1	2	3	4	5	6	Total
Constant σ	437	123	81	51	33	275	1000
Stochastic σ	459	118	86	47	38	252	1000

Table 4: *Number of Installments paid for 1000 sample price paths of Constant and Stochastic Volatility for the First Variant of the Installment Call*

In approximately one half (45%) of the sample price paths, the second payment in the installment option was not paid and the dynamic hedging ceased. For the remaining half (55%) of the simulations, dynamic hedging continued. However, less than 28% of the installment options remained active to the final expiration. Thus, the impacts of transactions costs relative to the compound option must be less, as the compound must be dynamically hedged for a minimum of 90 days, while installment option rarely required hedging beyond 30 days. As with the compound option dynamic hedging simulation, the standard deviation of hedging errors is higher for losses compared to gains. This difference is not as extreme as for the compound option, as the average period of dynamic hedging is less for the installment option. Table 5 displays the results of the hedging simulations for the four scenarios for the second variant of the installment option (with six equal payments).

In the case of a 180-day installment option with six equal payments, results are roughly similar to those previously found for the compound option and the first installment option examined. The inclusion of transactions costs when hedging significantly increases the cost of hedging relative to the theoretical value of the underlying call option. However, this impact is substantially reduced relative to that of the compound dynamic hedges. When equal payment installment options are considered, it is even more likely that the second installment payment will never be made. In this instance, for approximately 75% of the 1000 price paths, dynamic hedging was only required for the first 30 days. This further reduction in the average life of these products reduces the accumulated transactions costs associated with dynamic hedging. In Table 6, the number of payments made for the 1000 price paths are reported.

In approximately 80% of the sample price paths, the second payment in the installment option was not paid and the dynamic hedging ceased. For the remaining 20% of the simulations, dynamic hedging continued. However, less than 10% of the installment options remained active to the final expiration. Thus, the impacts of transactions costs relative to the compound option must be less, as the compound must be dynamically hedged for a minimum of 90 days, while installment option rarely required hedging beyond 30 days.

As with the compound option dynamic hedging simulation, the standard deviation of hedging errors is higher for losses compared to gains. This difference

	Constant Volatility		Stochastic Volatility	
	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs
Average Difference (%) Hedging Cost / Theoretical Value	0.00 %	-1.53 %	0.00 %	-1.52 %
Standard Deviation (%) of Hedging Cost / Theoretical Value	7.94 %	7.89 %	11.90 %	11.99 %
Average Loss (%) Hedging Cost / Theoretical Value	-2.95 %	-3.61 %	-7.90 %	-8.41 %
Standard Deviation (%) of Hedging Loss /	3.29 %	3.82 %	7.37 %	7.94 %
Average Gain (%) Hedging Cost / Theoretical Value	2.95 %	2.09 %	7.90 %	6.89 %
Standard Deviation (%) of Hedging Gain	7.51 %	7.30 %	8.73 %	8.38 %

Table 5: *Results of 1000 Simulation Runs for Delta Hedging the second Variant of an Installment Call*

# Payments	1	2	3	4	5	6	Total
Constant σ	769	65	29	21	18	98	1000
Stochastic σ	784	71	31	16	17	81	1000

Table 6: *Number of Installments paid for 1000 sample price paths of Constant and Stochastic Volatility for the Second Variant of the Installment Call*

is not as extreme as for the compound option, as the average period of dynamic hedging is less for the installment option. As with the compound option, the inclusion of stochastic volatility and transactions costs in the state space complicates the expected hedge variability when dynamic hedging and agents willingness to employ such hedging strategies for installment options.

4.4 Static Hedging of Installment Options

The static hedging of an installment option can be seen as an extension of the static hedging approach for simple compound calls. Consider extending the logic for the static hedge of the two-payment installment option appearing in Table 2. From Proposition 2 and accompanying equation (15), the appropriate (purchased) European option to hedge the installment option is simply the strike of the underling option (K) plus the summation of the future value of the pos-

sible installment payments $\left(\widehat{p}_1 = p_1 \sum_{i=1}^{n-1} e^{r(T-t_i)}\right)$[7]. Consider the two 180-day installment calls with six payments, we are examining. After the initial payment, subsequent payments occur would be due in each of the next five months. After five months, if the final installment is paid, the option is a simple European option thereafter.

As with the compound static hedge, if the purchase of the European option is funded solely by the initial installment payment an arbitrage exists. Therefore, in the absence arbitrage, the European option purchase must be funded partially through the initial installment premium and borrowing.

For the simulation of the static hedge for the unequal payments installment option, exactly the same price paths examined for the dynamic hedge were used. In addition, the same assumptions regarding the transactions costs and stochastic volatility process were made. ¿From the previous section on the upper boundary of the installment price, a 180-day European call with a striking price of 104.05 was purchased upon the sale of the installment call (with an initial price of 2.532 and five subsequent payments of 0.80). This option was priced using the Black-Scholes [1] formula. To fund the purchase of the static hedging call, borrowing was required in addition to the receipt of the initial installment premium.

If at each installment payment date, the value of the installment option at that point was less than required payment (of 0.80), the purchased static call was sold at the current market price. The price of the option was determined by using the simulated underlying price and volatility (at that point) that were entered in the Black-Scholes [1] model. Otherwise, the payment was made to the seller and evaluation continued at all subsequent payment dates. If all installments were made, the terminal payoffs of the two European options were examined. Table 7 displays the hedging results for the four scenarios associated with the 1000 simulated price paths.

The results of this hedging approach are similar to those found for the compound option in Table 3. The cost of the hedge tends to be slightly higher than the theoretical value of the underlying 100 call. As with the previous comparison of the dynamic and static hedging strategies for compound options, the standard deviation of the hedging costs has increased but is not substantially impacted when volatility levels are allowed to vary.

As with comparison of the dynamic and static hedging strategies for the compound call, the standard deviations of hedging losses and gains is asymmetric. While not as dramatic as in the case of the compound option, the standard deviation of the losses is approximately 40% less than the standard deviation of the hedging gains. Comparisons of Tables 3 and 7 indicate that the average loss (for the stochastic volatility and transactions costs scenario) is 16.78% for the static hedge and 11.11% for the dynamic hedge. However, the standard deviation of hedging losses is essentially the same for the static hedge (10.20%) compared to the dynamic hedge (10.24%). From these simulations, the maximum realised loss for the static hedge was 27.01% of the underlying European call value and for the dynamic hedge the maximum loss was 70.34%.

Finally, the same static hedging strategy was applied to the second type of

[7]Plus for an installment call and minus this amount for an installment put.

	Constant Volatility		Stochastic Volatility	
	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs
Average Difference (%) Hedging Cost / Theoretical Value	0.21 %	-0.41 %	0.08 %	-0.98 %
Standard Deviation (%) of Hedging Cost / Theoretical Value	20.19 %	20.11 %	20.82 %	20.82 %
Average Loss (%) Hedging Cost / Theoretical Value	-15.98 %	-16.30 %	-17.25 %	-17.76 %
Standard Deviation (%) of Hedging Loss /	10.09 %	10.19 %	10.10 %	10.20 %
Average Gain (%) Hedging Cost / Theoretical Value	16.19 %	15.89 %	17.33 %	16.78 %
Standard Deviation (%) of Hedging Gain	12.02 %	12.84 %	12.84 %	12.81 %

Table 7: *Results of 1000 Simulation Runs for Static Hedging for the First Variant of the Installment Call*

installment call option, with equal payments. For this hedge, a 180-day European call with a striking price of 106.53 was purchased upon the sale of the installment call (with an initial price of 1.291 and five subsequent payments of 1.291). To fund the purchase of the static hedging call, borrowing was required in addition to the receipt of the initial installment premium. Table 8 displays the hedging results for the four hedging scenarios.

The results of this hedging approach are similar to those found for the previous static hedges in Tables 3 and 7. When compared to the previous examples, the average hedging errors indicate a small but insubstantial loss. As with the previous comparisons of the dynamic and static hedging strategies, the standard deviation of the hedging costs has increased but is invariant to stochastic volatility.

As with comparison of the dynamic and static hedging strategies for the compound call, the standard deviations of hedging losses and gains is asymmetric. While not as dramatic as in the case of the compound option, the standard deviation of the losses is approximately 40% less than the standard deviation of the hedging gains. Comparisons of Tables 5 and 8 indicate that the average loss (for the stochastic volatility and transactions costs scenario) is twice as high (18.09%) for the static hedge compared to the dynamic hedge (9.25%). However (and as was the case for the previous comparisons), the standard deviation of hedging

	Constant Volatility		Stochastic Volatility	
	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs
Average Difference (%) Hedging Cost / Theoretical Value	-0.42 %	-1.05 %	0.311 %	-0.756 %
Standard Deviation (%) of Hedging Cost / Theoretical Value	23.84 %	23.74 %	23.37 %	23.34 %
Average Loss (%) Hedging Cost / Theoretical Value	-17.00 %	-17.49 %	-17.54 %	-18.09 %
Standard Deviation (%) of Hedging Loss /	10.16 %	10.19 %	10.26 %	10.43 %
Average Gain (%) Hedging Cost / Theoretical Value	16.58 %	16.44 %	17.85 %	17.33 %
Standard Deviation (%) of Hedging Gain	16.34 %	16.21 %	16.60 %	16.53 %

Table 8: *Results of 1000 Simulation Runs for Static Hedging for the Second Variant of the Installment Call*

losses is only 10.43% for the static hedge and is 14.01% for the dynamic hedge. Even though the average loss for the static hedge is twice as high, comparable losses could exist if the level of transactions costs were higher. As is the nature of the static hedging strategy, the losses are also bounded, while the maximum loss for the dynamic hedge was three times as great. These simulations confirm that the bounds for the maximal loss suggested in section 3 indeed hold true.

5 Concluding Remarks

We believe that the analysis given in this paper justifies the claims made in the Introduction about the attractiveness of the installment option as a product. From the buyer's point of view the attractions are obvious: the rights implied by the option contract can be acquired at very low initial cost. For the seller, the simple static hedging strategy suggested by the arbitrage arguments of Section 3 is remarkably effective, as the evidence presented in Section 4 shows. The main point is that the downside risk is bounded, and bounded by a model-independent number that is generally no more than a modest fraction of the Black-Scholes premium for the underlying option. Another possibility, which we have not investigated in detail, would be to apply the static hedge and then to dynamically hedge the residual risk. This would certainly be much more efficient than dy-

namically hedging *all* the risk.

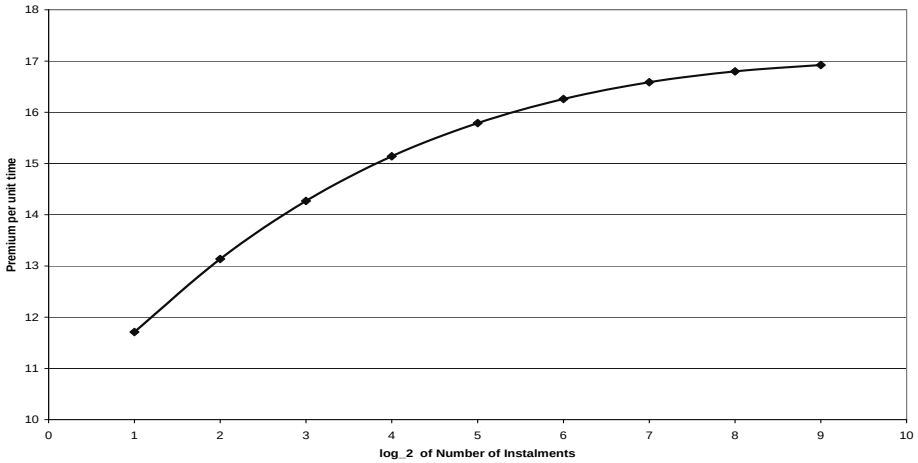

Figure 6: *Dependence of the present value of the total premium on number of installments*

A further question of some independent interest is to investigate what happens in the limit as the number of installments increases. Because of increasing optionality, the total premium (i.e. the present value of all installments) increases with the number of installments. Figure 6 shows this for the standard example of Section 2. The horizontal scale is logarithmic, the data points being for $2, 4, 8, \ldots$ installments. As can be seen, the total premium – or, equivalently, the premium per unit time – appears to converge to an upper bound $\hat{p}$. In fact, $\hat{p}$ is the fair premium for a *continuously-paid installment option* in which the premium is paid at *rate* $\hat{p}$ per unit time, with the right to terminate payments at any time. This is equivalent to holding the underlying European option, purchased at price $q(0)$, plus the right to sell this option at strike price $q(t)$ at any time $t < T$. Here $q(t) = \hat{p}(1 - e^{-r(T-t)})/r$ is the NPV at time t of the future premium payment stream. The value of this American option can be characterized as the solution of a variational equality, but a rigorous theory is complicated: because it is a compound option with time-varying strike, standard theory for American options cannot be applied in any straightforward way. We plan to return to this topic in a later paper.

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