ASYMPTOTIC RUIN PROBABILITIES AND OPTIMAL INVESTMENT

J. GAIER, P. GRANDITS, AND W. SCHACHERMAYER

Abstract. We study the infinite time ruin probability for an insurance company in the classical Cramér-Lundberg model with finite exponential moments. The additional non-classical feature is that the company is also allowed to invest in some stock market, modeled by geometric Brownian motion. We obtain an exact analogue of the classical estimate for the ruin probability without investment, i.e. an exponential inequality. The exponent is larger than the one obtained without investment, the classical Lundberg adjustment coefficient, and thus one gets a sharper bound on the ruin probability.
A surprising result is that the trading strategy yielding the optimal asymptotic decay of the ruin probability simply consists in holding a fixed quantity (which can be explicitly calculated) in the risky asset, independent of the current reserve. This result is in apparent contradiction to the common believe that 'rich' companies should invest more in risky assets than 'poor' ones. The reason for this seemingly paradoxical result is that the minimization of the ruin probability is an extremely conservative optimization criterion, especially for 'rich' companies.

1. Introduction

Since 1903, when F. Lundberg [14] introduced a collective risk model based on a homogeneous Poisson claims process, the estimation of ruin probabilities has been a central topic in risk theory. It is known that, if the claim sizes have exponential moments, the ruin probability decreases exponentially with the initial surplus; see for instance the books by Gerber [8] and Asmussen [1]. If the claim sizes have heavier tails, there also exist numerous results in the literature (e.g., Embrechts and Veraverbeke [4]). In these models it is assumed that the insurance company may invest the reserve in a riskless bond yielding zero interest.

It has only been recently that a more general question has been asked: If an insurer additionally has the opportunity to invest in a risky asset (modeled, e.g., by geometric Brownian motion), what is the minimal ruin probability she can obtain? In particular, can she do better than keeping the funds in the bond? And if yes, how much can she do better?

S. Browne [2] investigated this problem, but under the assumption that the risk process follows a Brownian motion (the so called 'diffusion approximation'). In this simpler setting, the investment strategy which minimizes the ruin probability consists in holding a constant amount of wealth in the risky asset, and the corresponding minimal ruin probability is given by an exponential function.

Paulsen et al. ([16] and [15]) have investigated the question, but under the additional assumption that all of the surplus is invested in the risky asset; likewise did Kalashnikov and Norberg in [13]. Frovola, Kabanov and Pergamenshchikov [5] looked at the case where a constant fraction of wealth is invested in the stock described by geometric Brownian motion.

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In all of these cases it was shown that, even if the claim size has exponential moments, the ruin probability decreases only with some negative power of the initial reserve.
In [10] and [11], Hipp and Plum consider the general case and analyze the trading strategy which is optimal with respect to the criterion of minimizing the ruin probability. They derive the Hamilton-Jacobi-Bellman equation corresponding to the problem, prove the existence of a solution and a verification theorem. Then they give explicit solutions for cases with exponential claim size distribution and special parameter values (namely $c = \lambda + a^2/2b^2$, where c is the premium rate, λ the intensity of the Poisson process underlying the number of claims, a the drift and b the volatility of the geometric Brownian motion underlying the investment possibility). It turns out that for these explicit solutions with exponentially distributed claims the minimal ruin probability decreases exponentially.
In this paper we will consider the framework of a classical risk process, where the claims have exponential moments. We investigate whether there are constants $\hat{r}$ and C such that the probability of ruin $\Psi(x)$, obtained by starting from an initial reserve x and subsequently investing in an appropriate way, satisfies

$$\Psi(x) \leq Ce^{-\hat{r}x}. \tag{1}$$

Of course, there always is the possibility not to invest at all, resulting in an exponential bound for the ruin probability $\Psi(x)$ (with the so called Lundberg adjustment coefficient), under the assumption of a positive safety loading. We calculate the optimal (i.e. largest) coefficient $\hat{r}$ such that (1) holds true; it turns out that $\hat{r}$ is determined by a similar equation as the Lundberg adjustment coefficient (see (28) below). The trading strategy that corresponds to this optimal $\hat{r}$ consists in holding a – properly chosen – constant amount of wealth in the risky asset, independent of the current level of the reserve. We will show in Theorem 7 that this constant strategy is asymptotically optimal, resp. asymptotically unique, in the sense that every 'asymptotically different' Markovian strategy yields an exponentially worse decay of the ruin probability.
What is the message of our results from an actuarial point of view? After some discussions with H. Bühlmann, which are gratefully acknowledged, we attempt to make the following economic interpretation: minimizing the ruin probability is an extremely conservative approach to the insurance business. This is reflected by the - at least asymptotically - very conservative investment strategy of holding a constant amount of money in the risky asset. A more proper way to deal with the probability of ruin in the presence of control variables (such as the investment in a risky asset) apparently consists in imposing a certain threshold level on this probability while optimizing with respect to other criteria, e.g., the expected value of discounted dividends. This topic is left for future research.
Here is another remarkable fact, which follows from our analysis and bears practical relevance: by adding some additional risk (namely the investment in the risky stock) to the basis risk of the insurance business, it is possible to *decrease* the probability of ruin. In fact, this decrease is quite substantial and leads to a different order of the exponential decay in terms of the initial surplus. This stresses once more the importance of a proper asset-liability management of an insurance company.
A complementary result about the asymptotic ruin probabilities for large claims can be found in Gaier and Grandits [6].

2. The Model

We model the risk process of an insurance company in the classical way (see, e.g., Gerber [7], Asmussen [1]): the surplus process R is given by a Poisson process $N = (N(t))_{t\geq 0}$ with intensity $\lambda > 0$, and by a positive random variable X, independent of the process N, with distribution function F in the following way

$$R(t,x) = x + ct - \sum_{i=1}^{N(t)} X_i, \tag{2}$$

where $x \geq 0$ is the initial reserve of the insurance company, $c \in \mathbb{R}$ is the (constant) premium rate over time and X_i is an i.i.d. sequence of copies of X, modeling the size of ith claim incurred by the insurer.

The classical model does not account for interest on the reserve: in modern terms this may be expressed by saying that the insurance company may only invest in a bond with zero interest rate. Now we deviate from the classical setting and assume that the company may also invest in a stock or market index, described by geometric Brownian motion

$$dS(t) = S(t)(a\,dt + b\,dW(t)), \tag{3}$$

where $a, b \in \mathbb{R}$ are fixed constants and W is a standard Brownian motion independent of the process R.

We will denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by the processes R and S and use $\mathbb{E}_t[\,.\,]$ as a shorthand notation for the conditional expectation $\mathbb{E}[\,.\,|\mathcal{F}_t]$.

If at time t the insurer has wealth $Y(t)$, and invests an amount $K(t)$ of money in the stock and the remaining reserve $Y(t) - K(t)$ in the bond (which in the present model yields no interest), her wealth process Y can be written as

$$\begin{aligned} Y(t,x,K) &= x + ct - \sum_{i=1}^{N(t)} X_i + (\frac{K}{S} \cdot S)(t) \\ &= R(t,x) + (K \cdot W_{a,b})(t), \end{aligned} \tag{4}$$

where $W_{a,b}(t)$ denotes the generalized Wiener process $W_{a,b}(t) = at + bW(t)$ with drift a and standard deviation b and $(K \cdot W_{a,b})$ denotes the stochastic integral of the process K with respect to the process $W_{a,b}$ (see, e.g., Protter [17]).

We are interested in the infinite time ruin probability of the insurance company, defined by

$$\Psi(x,K) = \mathbb{P}[Y(t,x,K) < 0, \text{ for some } t \geq 0], \tag{5}$$

depending on the initial wealth x and the investment strategy K of the insurer. We further define the time of ruin

$$\tau(x,K) := \inf\{t : Y(t,x,K) < 0\}. \tag{6}$$

The set $\mathcal{K}$ of *admissible* strategies K is defined as

$$\begin{aligned} \mathcal{K} \;:=\; &\{K = (K(t))_{t\geq 0} :\; K \text{ is predictable and adapted to } \mathbb{F} \\ &\text{and } \mathbb{P}[\int_0^t K(s)^2 ds < \infty] = 1 \text{ for all } t \in [0,\infty[\}. \end{aligned} \tag{7}$$

Note that $K \in \mathcal{K}$ is necessary and sufficient for the stochastic integral $(K \cdot W_{a,b})$ w.r.t. the generalized Wiener process appearing in (4) to exist.

Furthermore we define

$$\Psi^*(x) = \inf_{K\in\mathcal{K}} \Psi(x,K). \tag{8}$$

If this infimum is attained for a certain strategy K^*, we will call this strategy an *optimal strategy* with respect to the initial reserve x.

Denoting by $h : \mathbb{R}_+ \to \mathbb{R}_+$ the moment generating function of the claim size X, shifted such that $h(0) = 0$,

$$h(r) = \mathbb{E}[e^{rX}] - 1, \tag{9}$$

we will make the classical assumption that there exists $r_\infty \in (0,\infty]$ such that $h(r) < \infty$, for $r < r_\infty$, and such that $h(r) \to \infty$, for $r \uparrow r_\infty$. The function h is increasing, convex, and continuous on $[0, r_\infty)$ (cf. Grandell [9]).

3. An Asymptotic Inequality

The classical Cramér-Lundberg model without investment possibility is, of course, a special case of the model described in Section 2, namely letting $a = b = 0$. There, one usually assumes that $c > \lambda\mathbb{E}[X]$, because otherwise the ruin probability is simply equal to one. Under this assumption, the ruin probability - defined by (5), which then is independent of the investment strategy K - can be bounded from above by $e^{-\nu x}$, where ν is the positive solution of the equation

$$\lambda h(r) = cr. \tag{10}$$

This is the famous *Lundberg inequality*, the exponent ν is called *Lundberg* or *adjustment coefficient* (Gerber [8], Asmussen [1] or Grandell [9]).

The main result of this paper is summarized in the following theorem. It will be a consequence of Theorem 3.

Theorem 1 (Main Theorem)**.** *For the model described in Section 2, assume that $b \neq 0$. Then the minimal ruin probability $\Psi^*(x)$ of an insurer, investing in a stock market, can be bounded from above by*

$$\Psi^*(x) \le e^{-\hat{r}x}, \tag{11}$$

where $0 < \hat{r} < r_\infty$ is the positive solution of the equation (compare Figure 1)

$$\lambda h(r) = cr + \frac{a^2}{2b^2}. \tag{12}$$

If $\mathbb{E}[X] < c/\lambda$, i.e., if the Lundberg coefficient $\nu > 0$ exists, we have that $\hat{r} > \nu$, if $a \neq 0$, so that one obtains a sharper bound for $\Psi^*(x)$. Dropping the assumption $\mathbb{E}[X] < c/\lambda$, for $a \neq 0$, we still obtain $\hat{r} > 0$, i.e. an exponential decay of the minimal ruin probability.

For later use, we introduce the following process, for fixed numbers $x, r \in \mathbb{R}_+$, and a fixed admissible strategy $K \in \mathcal{K}$,

$$M(t,x,K,r) := e^{-rY(t,x,K)}. \tag{13}$$

This process is already familiar from Gerber's approach to risk theory via martingale inequalities (Gerber [7]).

Lemma 2. *Let $x > 0$, and $a \neq 0, b \neq 0$. There exists a unique $0 < \hat{r} < r_\infty$ satisfying the equation*

$$\lambda h(\hat{r}) = \frac{a^2}{2b^2} + c\hat{r}. \tag{14}$$

For this $\hat{r}$ and the constant process $\hat{K}(t) \equiv a/\hat{r}b^2$, the process $M(t,x,\hat{K},\hat{r})$ is a martingale w.r.t. the filtration $\mathbb{F}$.

Proof. The existence and uniqueness of $\hat{r}$ are easy consequences of the properties of h (cf. Figure 1).
If we define $f : \mathbb{R} \times [0, r_\infty) \to \mathbb{R}$

$$f(K,r) := \lambda h(r) - (Ka + c)r + \frac{1}{2}K^2 b^2 r^2, \tag{15}$$

then it can be easily checked that $f(\hat{K},\hat{r}) = 0$. Now, in order to show that the process $M(t,x,\hat{K},\hat{r})$ is a martingale w.r.t. $\mathbb{F}$, we proceed as follows (see, e.g., the book by Asmussen [1]): for arbitrary $t \geq 0$,

$$\begin{aligned}
\mathbb{E}[M(t,0,\hat{K},\hat{r})] &= \mathbb{E}[e^{-\hat{r}(ct-\sum_{i=1}^{N(t)} X_i + \hat{K}W_{a,b}(t))}] \\
&= e^{-\hat{r}(c+\hat{K}a)t}\mathbb{E}[e^{\hat{r}\sum_{i=1}^{N(t)} X_i}]\mathbb{E}[e^{-\hat{r}\hat{K}bW(t)}] \\
&= e^{-\hat{r}(c+\hat{K}a)t}e^{h(\hat{r})\lambda t}e^{(\hat{r}^2\hat{K}^2b^2/2)t} \\
&= e^{f(\hat{K},\hat{r})t} \\
&= 1. \qquad (16)
\end{aligned}$$

Since $Y(t,x,\hat{K})$ has stationary independent increments, we obtain, for $0 \leq t \leq T$,

$$\begin{aligned}
\mathbb{E}_t[M(T,x,\hat{K},\hat{r})] &= \mathbb{E}_t[e^{-\hat{r}Y(T,x,\hat{K})}] \\
&= e^{-\hat{r}Y(t,x,\hat{K})}\mathbb{E}_t[e^{-\hat{r}(Y(T,x,\hat{K})-Y(t,x,\hat{K}))}] \\
&= e^{-\hat{r}Y(t,x,\hat{K})}\mathbb{E}[e^{-\hat{r}(Y(T-t,x,\hat{K})-Y(0,x,\hat{K}))}] \\
&= e^{-\hat{r}Y(t,x,\hat{K})}\mathbb{E}[e^{-\hat{r}Y(T-t,0,\hat{K})}] \\
&= e^{-\hat{r}Y(t,x,\hat{K})} \\
&= M(t,x,\hat{K},\hat{r}) \qquad (17)
\end{aligned}$$

and therefore $M(t,x,\hat{K},\hat{r})$ is a martingale w.r.t. the filtration $\mathbb{F}$. $\square$

Remark. The above argument also shows that for each $r \in [0,\hat{r})$, there exist two constant processes $K_{1,2}(r) \in \mathcal{K}$ such that the process $M(t,x,K_{1,2}(r),r)$ is a martingale. The values $K_{1,2}(r)$ are given in the following way

$$K_{1,2}(r) = \frac{a}{b^2 r} \pm \sqrt{\Delta(r)}, \tag{18}$$

where

$$\Delta(r) := \frac{2}{b^2 r^2}\left(\frac{a^2}{2b^2} + cr - \lambda h(r)\right) \geq 0 \text{ for } r \leq \hat{r}. \tag{19}$$

Note that for $r = \hat{r}$, we obtain $\Delta(\hat{r}) = 0$, and therefore $K_1(\hat{r}) = K_2(\hat{r}) = \hat{K}$.
From now on we shall always consider the processes M and Y, stopped at the time of ruin, so we define

$$\tilde{M}(t,x,K,r) := M(t \wedge \tau(x,K), x, K, r) \tag{20}$$

and

$$\tilde{Y}(t,x,K) := Y(t \wedge \tau(x,K), x, K), \tag{21}$$

where we use the standard notation $t \wedge \tau(x,K) := \min(t, \tau(x,K))$.

Theorem 3. *Let $a \neq 0, b \neq 0$. For the constant investment strategy $\hat{K}(t) \equiv a/\hat{r}b^2$, the ruin probability can be bounded from above by (for all $x \in \mathbb{R}_+$)*

$$\Psi(x, \hat{K}) \leq e^{-\hat{r}x}. \tag{22}$$

Proof. From Lemma 2 we know that $M(t, x, \hat{K}, \hat{r})$ is a martingale w.r.t. the filtration $\mathbb{F}$. Therefore, also the stopped process $\tilde{M}(t, x, \hat{K}, \hat{r})$ is a martingale w.r.t. $\mathbb{F}$ (Theorem (II.77.5) in Rogers and Williams Vol. 1 [18]; note that M is non–negative). Using this, we obtain similarly as in Gerber [7], for $t \geq 0$,

$$\begin{aligned} e^{-\hat{r}x} &= \tilde{M}(0, x, \hat{K}, \hat{r}) \\ &= \mathbb{E}[\tilde{M}(t, x, \hat{K}, \hat{r})] \\ &= \mathbb{E}[\tilde{M}(\tau(x, \hat{K}), x, \hat{K}, \hat{r})\chi_{\{\tau(x,\hat{K})<t\}}] \\ &\quad + \mathbb{E}[\tilde{M}(t, x, \hat{K}, \hat{r})\chi_{\{t \leq \tau(x,\hat{K})\}}] \\ &\geq \mathbb{E}[\tilde{M}(\tau(x, \hat{K}), x, \hat{K}, \hat{r})\chi_{\{\tau(x,\hat{K})<t\}}], \end{aligned} \tag{23}$$

where χ_A is the indicator function of the set A, and where we used the fact that the process $\tilde{M}$ is nonnegative.
Monotone Convergence yields that

$$\lim_{t\to\infty} \mathbb{E}[\tilde{M}(\tau(x, \hat{K}), x, \hat{K}, \hat{r})\chi_{\{\tau(x,\hat{K})<t\}}] = \mathbb{E}[\tilde{M}(\tau(x, \hat{K}), x, \hat{K}, \hat{r})\chi_{\{\tau(x,\hat{K})<\infty\}}]. \tag{24}$$

Hence

$$e^{-\hat{r}x} \geq \mathbb{E}[\tilde{M}(\tau(x, \hat{K}), x, \hat{K}, \hat{r})|\tau(x, \hat{K}) < \infty]\, \mathbb{P}[\tau(x, \hat{K}) < \infty]. \tag{25}$$

Thus we arrive at

$$\begin{aligned} \Psi(x, \hat{K}) &= \mathbb{P}[\tau(x, \hat{K}) < \infty] \\ &\leq \frac{e^{-\hat{r}x}}{\mathbb{E}[\tilde{M}(\tau(x, \hat{K}), x, \hat{K}, \hat{r})|\tau(x, \hat{K}) < \infty]}. \end{aligned} \tag{26}$$

Since the random variable $\tilde{M}(\tau(x, \hat{K}), x, \hat{K}, \hat{r})$ is always greater than or equal to 1 a.s. on the set $\{\tau(x, \hat{K}) < \infty\}$, the result follows. □

The Main Theorem now is an immediate consequence of Theorem 3, observing that $\hat{r} > \nu$ (assuming that $b \neq 0$ and $a \neq 0$).
As we have mentioned before, the classical Lundberg exponent ν is the positive solution to

$$h(r) = \frac{c}{\lambda} r. \tag{27}$$

If now, in addition, the insurance company has the opportunity to invest in the market, the corresponding exponent $\hat{r}$ is the positive solution of

$$h(r) = \frac{c}{\lambda} r + \frac{a^2}{2\lambda b^2}. \tag{28}$$

The right hand side of (28) is just the right hand side of (27), but shifted by the positive constant $a^2/2\lambda b^2$. From the properties of h it is obvious that $\hat{r} > \nu$ if $a \neq 0$ and that $\hat{r} = \nu$ for $a = 0$ (see also Figure 1).
What about the assumption $c > \lambda\mathbb{E}[X]$? In the classical setting without investment, this condition is equivalent to $h'(0) = \mathbb{E}[X] < c/\lambda$ and guarantees that h and the line with slope c/λ through 0 have a strictly positive intersection. In the present model with investment the picture

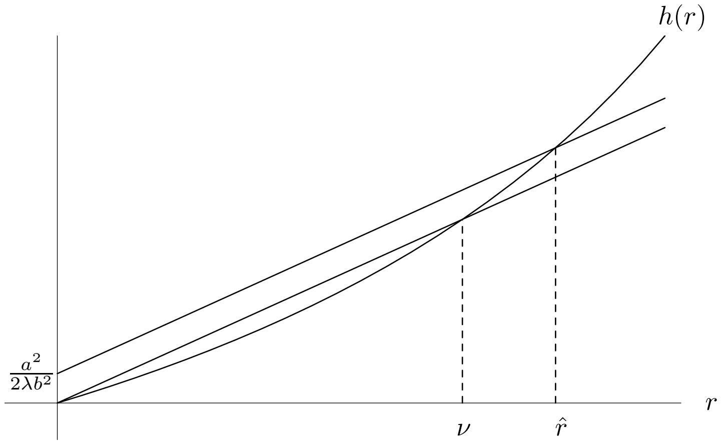

FIGURE 1. $h(r)$, $\frac{c}{\lambda}r$ and $\frac{c}{\lambda}r + \frac{a^2}{2\lambda b^2}$ for exponentially distributed claims and parameter values $\theta = 10$, $c = 15$, $\lambda = 1$, $a = 0.06$ and $b = 0.15$. In this case we obtain $\nu = 1/30 = 0.0\dot{3}$ and $\hat{r} = 0.041$.

changes (see Figure 2): it is easily seen that for $a \neq 0$, equation (28) always possesses a strictly positive solution $\hat{r}$.

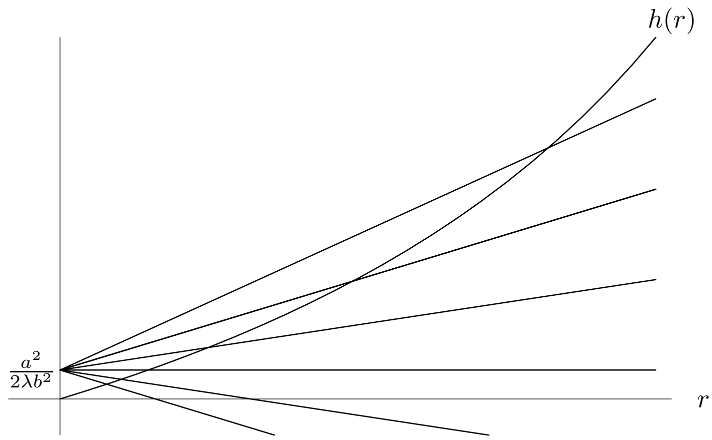

FIGURE 2. $h(r)$ and $\frac{c}{\lambda}r + \frac{a^2}{2\lambda b^2}$ for exponentially distributed claims, parameter values $\theta = 10$, $\lambda = 1$, $a = 0.06$, $b = 0.15$ and different values of c.

Thus we have completed the proof of the Main Theorem and now pass on to an illustrative example.

Example. Consider the situation for the classical Erlang model when claim sizes are exponentially distributed with parameter θ, i.e., $dF(x) = (e^{-x/\theta}/\theta)dx$. In this case $h(r) = \theta r/(1-\theta r)$, $r \in [0, 1/\theta)$. A plot of this function is shown in Figure 1 for $\theta = 10$. Equation (27) has two solutions, namely 0 and $\nu = \rho/(\rho+1)\theta$, where the relative safety loading ρ equals as $c/\lambda\theta - 1$. Note that ν is only positive if $c > \lambda\theta$. An elementary calculation reveals that on the other hand the coefficient $\hat{r}$ equals

$$\nu + \left(\sqrt{(\frac{\nu + a^2/2b^2c}{2})^2 + \frac{a^2}{2b^2c}(\frac{1}{\theta} - \nu)} - \frac{\nu + a^2/2b^2c}{2} \right). \tag{29}$$

Remarks.
1. At first sight it seems very amazing that one obtains an exponential bound on the ruin probability Ψ for *arbitrary* values of the parameters c, λ and $\mathbb{E}[X]$. The premium rate c might even be negative!
This stunning fact can be explained as follows: remember that the process $\hat{K}$ is given by $\hat{K}(t) \equiv a/\hat{r}b^2$. For 'unfavourable' parameters of the risk process, $\hat{r}$ is small and therefore $\hat{K}$ is large. This leads to an arbitrarily large drift of the wealth process from the investment. This way, the very large constant investment $\hat{K}$ leads eventually to an exponential decay of the ruin probability.
This result also gives some theoretical justification for the technique of '*cash flow underwriting*' which - at least from time to time - enjoys some popularity among re-insurers: according to this technique the re-insurer sometimes accepts contracts which will probably result in a technical loss, hoping that the financial gains obtained from a 'good' (i.e. a risky) investment of the premiums will outweigh this loss.
2. Note that $\hat{r}$ depends on the drift a of the risky investment via $|a|$! This can be explained as follows: If $a < 0$, then $\hat{K} = a/\hat{r}b^2$ is also less than zero, i.e. the investment strategy $\hat{K}$ prescribes to go short in the risky asset. This produces an arbitrarily large, positive drift $\hat{K} \cdot a$ of the wealth process Y (see item 1. above), which in turn leads to an exponential decay of the ruin probability at rate $\hat{r}$.
3. The investment strategy $\hat{K}$ consists in always holding a fixed amount of money in the risky asset: If $Y(t, x, \hat{K}) < \hat{K}$, i.e. if the wealth of the insurance company is less then the constant $\hat{K}$, it is still possible to hold the amount $\hat{K}$, since we have not imposed any short selling constraints on the set of admissible strategies.
4. If we drop the assumption that the bond yields zero interest rate, it turns out that the case of zero *real interest force* i, when the interest force on the bond is equal to the inflation force (cf. Delbaen and Haezendonck [3]), can be treated with essentially the same methods as the ones described above. The stochastic differential equation for the wealth process $Y^{(i)}$ with interest $i > 0$ is

$$(30)\ dY^{(i)}(t) = (ce^{it} + (i(Y^{(i)}(t-) - K(t)) + aK(t)))dt + bK(t)dW(t) - e^{it}X_{N(t)}dN(t).$$

If we introduce the *present value process* $\overline{Y}^{(i)}(t) := e^{-it}Y^{(i)}(t)$, we obtain

$$d\overline{Y}^{(i)}(t) = e^{-it}((ce^{it} + (a-i)K(t))dt + bK(t)dW(t) - e^{it}X_{N(t)}dN(t)). \tag{31}$$

Defining the process $\overline{M}^{(i)}(t) := e^{-r\overline{Y}^{(i)}(t)}$ for $r \in \mathbb{R}_+$, it follows the same way as with zero interest rate that $\overline{M}^{(i)}(t \wedge \tau, x, \hat{K}^{(i)}, \hat{r}^{(i)})$ is a martingale, where $\hat{r}^{(i)}$ is the solution to

$$\lambda h(r) = cr + \frac{(a-i)^2}{2b^2} \tag{32}$$

and the process $\hat{K}^{(i)} \in \mathcal{K}$ is given by

$$\hat{K}^{(i)}(t) = \frac{a-i}{\hat{r}^{(i)}b^2}e^{it}. \tag{33}$$

Then by the same line of argument as in the case of zero interest it can be shown that the ruin probability $\Psi(x, \hat{K}^{(i)})$ for the strategy $\hat{K}^{(i)}$ can be bounded from above by

$$\Psi(x, \hat{K}^{(i)}) \leq e^{-\hat{r}^{(i)}x}. \tag{34}$$

5. Actually, we need not assume that $h(r) \to \infty$ for $r \uparrow r_\infty$. If h were to jump to infinity at r_∞ (with $\lim_{r\uparrow r_\infty} h(r) = h(r_\infty) < \infty$), we still get an exponential bound on the ruin probability

$\Psi(x)$: If there exists $0 < \hat{r} < r_\infty$ such that $\lambda h(\hat{r}) = c\hat{r} + a^2/2b^2$, then the bound is $e^{-\hat{r}x}$, otherwise it is simply $e^{-r_\infty x}$.

4. Asymptotic Optimality and Asymptotic Uniqueness of the Constant Investment Strategy

In this section we want to show an asymptotic optimality resp. asymptotic uniqueness result for the constant investment strategy $\hat{K}$ and the exponent $\hat{r}$. We will need the following assumption on the exponential tail distribution of the claim sizes:

Definition. *Let $0 < r < r_\infty$ be given. We say that X has* a uniform exponential moment in the tail distribution for r, *if the following condition holds true*

$$\sup_{y \geq 0} \mathbb{E}[e^{-r(y-X)} | X > y] < \infty. \tag{35}$$

Remark. From now on we shall assume that the random variable X, that models the claim size, has a uniform exponential moment in the tail distribution for $\hat{r}$. Partly we do so for the ease of exposition, partly because we need the assumption: First to go from a local submartingale to a true submartingale in the proof of Theorem 4, and second in order to obtain a positive constant C in Theorem 6. In Appendix B, we present several of the results, that are proved in this section, without the assumption of a uniform exponential moment in the tail distribution. Under Assumption (35) (for $\hat{r}$), we can prove the following theorem.

Theorem 4. *Assume that X has a uniform exponential moment in the tail distribution for $\hat{r}$. Then for each $K \in \mathcal{K}$, the process $(\tilde{M}(t, x, K, \hat{r}))$ is a uniformly integrable submartingale.*

Proof. Application of Itô's Lemma to the process M yields, for arbitrary $K \in \mathcal{K}$ and $r \in \mathbb{R}_+$,

$$\begin{aligned} \frac{dM(t,x,K,r)}{M(t-,x,K,r)} &= \left(-(c + K(t)a)r + \frac{1}{2} r^2 b^2 K(t)^2 \right) dt \\ &\quad - rbK(t)dW(t) + \left(e^{rX_{N(t)}} - 1 \right) dN(t). \end{aligned} \tag{36}$$

This can be rewritten as

$$\begin{aligned} \frac{dM(t,x,K,r)}{M(t-,x,K,r)} &= \left(-(c + K(t)a)r + \frac{1}{2} r^2 b^2 K(t)^2 + \lambda h(r) \right) dt \\ &\quad -rbK(t)dW(t) \\ &\quad +(e^{rX_{N(t)}} - 1)dN(t) - \lambda \mathbb{E}[e^{rX_{N(t)}} - 1]dt \\ &= f(K(t), r)dt - rbK(t)dW(t) \\ &\quad +(e^{rX_{N(t)}} - 1)dN(t) - \lambda \mathbb{E}[e^{rX_{N(t)}} - 1]dt. \end{aligned} \tag{37}$$

Therefore the stopped process $\tilde{M}(t, x, K, \hat{r})$ can be expressed in terms of stochastic integrals as

$$\begin{aligned} &\tilde{M}(t, x, K, \hat{r}) - \tilde{M}(0, x, K, \hat{r}) \\ = &\int_0^{t \wedge \tau} M(s-, x, K, \hat{r}) f(K(s), \hat{r}) ds - \ rb \int_0^{t \wedge \tau} M(s-, x, K, \hat{r}) K(s) dW(s) \\ &+ \int_0^{t \wedge \tau} M(s-, x, K, \hat{r})(e^{\hat{r} X_{N(s)}} - 1) dN(s) - \mathbb{E}[e^{\hat{r}X} - 1] \int_0^{t \wedge \tau} M(s-, x, K, \hat{r}) \lambda ds. \end{aligned} \tag{38}$$

Since by assumption, the process $K \in \mathcal{K}$ is integrable with respect to the Brownian motion and since $0 \leq M(s-, x, \hat{K}, \hat{r}) \leq 1$ for $0 \leq s \leq \tau$, the stochastic integral w.r.t. the Brownian motion in (38) gives a local martingale. Furthermore, it is shown in Appendix A that the difference of the two processes

$$\int_0^{t\wedge\tau} \tilde{M}(s-,x,\hat{K},\hat{r})(e^{rX_{N(s)}}-1)dN(s) \tag{39}$$

and

$$\lambda\mathbb{E}[e^{\hat{r}X}-1]\int_0^{t\wedge\tau} \tilde{M}(s-,x,\hat{K},\hat{r})ds \tag{40}$$

is a martingale.
Finally, with the help of the defining equation (14) for $\hat{r}$, it is easy to show that for all $K \in \mathbb{R}$,

$$\begin{aligned} f(K,\hat{r}) &= \frac{1}{2}\hat{r}^2b^2(K-\hat{K})^2. \\ &\geq 0. \end{aligned} \tag{41}$$

Therefore, for all $0 \leq t \leq T$,

$$\int_{t\wedge\tau}^{T\wedge\tau} \tilde{M}(s-,x,K,\hat{r})f(K(s),\hat{r})ds \geq 0. \tag{42}$$

Putting the pieces together, it is an easy consequence that $\tilde{M}(t,x,K,\hat{r})$ is a local submartingale. To proceed from this to the conclusion that $\tilde{M}(t,x,K,\hat{r})$ indeed is a true submartingale, and even uniformly integrable, we use Assumption (35). Using the standard notation $\tilde{M}^* := \sup_{t\geq 0}|\tilde{M}(t)|$, it follows that

$$\begin{aligned} \mathbb{E}[\tilde{M}^*] &\leq \mathbb{E}[\tilde{M}(\tau,x,K,\hat{r})|\tau<\infty] \\ &\leq \mathbb{E}[\tilde{M}(\tau,x,K,\hat{r})|\tau<\infty, Y(\tau-)>0], \end{aligned} \tag{43}$$

since $M(\tau,x,K,\hat{r})$ is equal to 1 on $\{\tau<\infty, Y(\tau-)=0\}$, where ruin occurs a.s. through the Brownian motion, and $M(\tau,x,K,\hat{r}) \geq 1$ on $\{\tau<\infty, Y(\tau-)>0\}$, where ruin occurs through a jump.
Now we proceed similarly as in Asmussen [1], p. 77. Let $H(dt,dy)$ denote the joint probability distribution of τ and $Y(\tau-)$ conditional on the event that ruin occurs, and that it occurs through a jump. Then, given $\tau=t$ and $Y(\tau-)=y>0$, a claim has distribution function $dF(z)/\int_y^\infty dF(u)$ (for $z>y$). Therefore

$$\begin{aligned} \mathbb{E}[\tilde{M}^*] &\leq \mathbb{E}[\tilde{M}(\tau(x,K),x,K,\hat{r})|\tau<\infty, Y(\tau-)>0] \\ &= \int_0^\infty\int_0^\infty H(dt,dy)\int_y^\infty e^{-\hat{r}(y-z)}\frac{dF(z)}{\int_y^\infty dF(u)} \\ &\leq \left(\sup_{y\geq 0}\int_y^\infty e^{-\hat{r}(y-z)}\frac{dF(z)}{\int_y^\infty dF(u)}\right)\int_0^\infty\int_0^\infty H(dt,dy) \\ &= \sup_{y\geq 0}\int_y^\infty e^{-\hat{r}(y-z)}\frac{dF(z)}{\int_y^\infty dF(u)} \\ &< \infty \end{aligned} \tag{44}$$

by assumption (35).
A standard argument, using Dominated Convergence, gives that (44) implies that $\tilde{M}$ indeed is a uniformly integrable submartingale (see, e.g., Protter [17], Theorem I.47). □

The following lemma will be useful in the sequel (compare also the more general Proposition B.2 in Appendix B).

Lemma 5. *If X has a uniform exponential moment in the tail distribution for $\hat{r}$, then for arbitrary $K \in \mathcal{K}$ and $x \in \mathbb{R}_+$, the stopped wealth process $(\tilde{Y}(t,x,K))_{t\geq 0}$ converges almost surely on $\{\tau = \infty\}$ to ∞ for $t \to \infty$. In other words, either ruin occurs, or the insurer becomes infinitely rich.*

Proof. From Lemma 4 we know that $\tilde{M}(t,x,K,\hat{r})$ is a uniformly integrable submartingale. Applying Doob's Supermartingale Convergence Theorem (Rogers and Williams Vol. 1 [18], Theorem (II.69.1)) to $-\tilde{M}$, it follows that $\lim_{t\to\infty} \tilde{M}(t,x,K,\hat{r})$ exists a.s. Therefore, also the stopped wealth process $\tilde{Y}(t,x,K)$ converges a.s for $t \to \infty$.
There must exist $d > 0$ such that $\mathbb{P}[X > d] > 0$. If we define the events $E_n := \{X_n > d\}$, then $\mathbb{P}[E_n^c] < 1$, and the events $\{E_j\}_{j=1}^{\infty}$ are mutually independent. Therefore,

$$\mathbb{P}[\bigcup_{k=1}^{\infty} \bigcap_{n\geq k} E_n^c] = \lim_{k\to\infty} \mathbb{P}[\bigcap_{n\geq k} E_n^c] = \lim_{k\to\infty} \prod_{n\geq k} \mathbb{P}[E_n^c] = 0. \tag{45}$$

Hence, $\mathbb{P}[\bigcap_{k=1}^{\infty} \bigcup_{n\geq k} E_n] = 1$. In other words, with probability 1, a jump of size greater than d occurs infinitely often.
On the other hand, the stochastic integral $K \cdot W_{a,b}$ is a.s. continuous, and therefore the jumps of the compound Poisson process underlying the liabilities, greater than d, which will occur infinitely often a.s., cannot be compensated for by the a.s. continuous stochastic integral $K \cdot W_{a,b}$. As a result, the wealth process, stopped at time of ruin, cannot converge to a nonzero finite value with positive probability. $\square$

With the help of the two preceding lemmas we get the following result.

Theorem 6. *Assume that X has a uniform exponential moment in the tail distribution for $\hat{r}$. Then the ruin probability satisfies, for every admissible process $K \in \mathcal{K}$,*

$$\Psi(x,K) \geq C\, e^{-\hat{r}x}, \tag{46}$$

where

$$\begin{aligned} C &= \inf_{y\geq 0} \frac{\int_y^\infty dF(u)}{\int_y^\infty e^{-\hat{r}(y-z)} dF(z)} \\ &= \frac{1}{\sup_{y\geq 0} \mathbb{E}[e^{-\hat{r}(y-X)} | X > y]} > 0. \end{aligned} \tag{47}$$

Proof. As $\tilde{M}(t,x,K,\hat{r})$ is a uniformly integrable submartingale, it follows from Doob's Optional Sampling Theorem that (using τ as a shorthand notation for $\tau(x,K)$)

$$\begin{aligned} \tilde{M}(0,x,K,\hat{r}) &= e^{-\hat{r}x} \\ &\leq \mathbb{E}[\tilde{M}(\tau,x,K,\hat{r})]. \end{aligned} \tag{48}$$

Now we proceed similarly as in the proof of Theorem 3, but use Lemma 5.

$$\begin{aligned} &\mathbb{E}[\tilde{M}(\tau,x,K,\hat{r})] \\ = \; &\mathbb{E}[\tilde{M}(\tau,x,k,\hat{r}) | \tau < \infty] \mathbb{P}[\tau < \infty] \\ &+ \mathbb{E}[\lim_{t\to\infty} \tilde{M}(t,x,K,\hat{r}) | \tau = \infty] \mathbb{P}[\tau = \infty] \\ = \; &\mathbb{E}[\tilde{M}(\tau,x,K,\hat{r}) | \tau < \infty] \mathbb{P}[\tau < \infty]. \end{aligned} \tag{49}$$

Plugging this into (48), and using (43) and (44) we obtain

$$\Psi(x,K) \geq e^{-\hat{r}x} \frac{1}{\mathbb{E}[\tilde{M}(\tau,x,K,\hat{r}) | \tau < \infty]} \geq C e^{-\hat{r}x}. \tag{50}$$

This completes the proof. □

Remarks. 1. In the classical Erlang model, i.e., for claims with an exponential distribution (with parameter θ), one obtains the value $C = 1/(h(\hat{r})+1) = 1-\theta\hat{r}$.
2. For $K \equiv 0$, inequality (50) is the well–known lower bound for the ruin probability without investment, given e.g., in Asmussen [1], Theorem 6.3.
We now pass over to the asymptotic uniqueness of the constant investment strategy $\hat{K}$.
Hipp and Plum showed in [10] that, for the case of locally bounded density of the jump size, the problem of minimizing the ruin probability over all admissible trading strategies possesses a solution that is Markovian. That is to say that the trading strategy at time t depends on $\mathcal{F}_t$ only through the current level of wealth $Y(t-,x,K)$. Therefore from now on we shall restrict our attention to such strategies. We will write $k : \mathbb{R}_+ \to \mathbb{R}$ for the function that describes the dependency on wealth of a certain strategy $K \in \mathcal{K}$. Then the corresponding investment at time t equals $K(t) = k(Y(t-,x,K))$. We will show that, if the optimal strategy – as a function of wealth – converges to a constant as wealth tends to infinity, then the limiting constant must be $\hat{K} = a/b^2\hat{r}$ (Corollary 1). We will even show the stronger result that a Markovian strategy, which is asymptotically bounded away from this constant strategy, leads to an exponentially worse (i.e. larger) ruin probability than the one obtained by using the constant strategy $\hat{K}$.

Theorem 7. *Let X have a uniform exponential moment in the tail distribution for $\hat{r}$. Suppose further that $K \in \mathcal{K}$ is a Markovian strategy and let $k : \mathbb{R}_+ \to \mathbb{R}$ be its defining function. If there exist $\alpha > 0$ and $x_\alpha \geq 0$ such that*

$$|k(x) - \hat{K}| \geq \alpha \text{ for } x \geq x_\alpha, \tag{51}$$

then there are $r_\alpha < \hat{r}$ and $A_\alpha > 0$ such that

$$\Psi(x,K) \geq A_\alpha e^{-r_\alpha x}. \tag{52}$$

Proof. We split the proof into several steps.
Step 1. For α and x_α as in the Theorem, we define the stopping time

$$\tau_\alpha := \inf\{t : Y(t,x,K) \leq x_\alpha\}, \tag{53}$$

which is only nontrivial for $x > x_\alpha$.
Step 2. We show that, for $x > x_\alpha$, there exists $r_\alpha < \hat{r}$ such that $\tilde{M}(t\wedge\tau_\alpha, x, K, r_\alpha)$ is a uniformly integrable submartingale: we know that $f(\hat{K},\hat{r}) = 0$, that $f(k,\hat{r}) = \hat{r}^2b^2(k-\hat{K})^2/2 > 0$ for $k \neq \hat{K}$, and that $\lim_{k\to\infty} f(k,r) = \infty$ for all $r \in (0,\hat{r})$. Using these facts and the continuity of f it is straightforward to show that, for α as before, there exists some $0 < r_\alpha < \hat{r}$ such that, for $|k-\hat{K}| > \alpha$, we have $f(k,r_\alpha) \geq 0$. Now one proceeds the same way as in Section 4 to prove that $\tilde{M}(t\wedge\tau_\alpha, x, K, r_\alpha)$ is a uniformly integrable submartingale, using that $\tau_\alpha \leq \tau$ a.s., for $x > x_\alpha$, and Lemma A.1. Another consequence of $\tau_\alpha \leq \tau$ a.s. is that $\tilde{M}(t\wedge\tau_\alpha, x, K, r_\alpha) = M(t\wedge\tau_\alpha, x, K, r_\alpha)$ a.s.
Step 3. Using that the process $M(t\wedge\tau_\alpha, x, K, r_\alpha)$ is a uniformly integrable submartingale and Lemma 5, we obtain

$$\begin{aligned} e^{-r_\alpha x} &\leq \mathbb{E}[M(\tau_\alpha,x,K,r_\alpha)] \\ &= \mathbb{E}[\lim_{t\to\infty} M(t,x,K,r_\alpha)|\tau_\alpha=\infty]\mathbb{P}[\tau_\alpha=\infty] \\ &\quad +\mathbb{E}[M(\tau_\alpha,x,K,r_\alpha)|\tau_\alpha<\infty]\mathbb{P}[\tau_\alpha<\infty] \\ &\leq 0\cdot\mathbb{P}[\tau_\alpha=\infty] + \frac{1}{C_\alpha}e^{-r_\alpha x_\alpha}\mathbb{P}[\tau_\alpha<\infty], \end{aligned} \tag{54}$$

where the constant C_α is defined by

$$\frac{1}{C_\alpha} := \sup_{y\geq 0} \frac{\int_y^\infty e^{-r_\alpha(y-z)}dF(z)}{\int_y^\infty dF(u)} = \sup_{y\geq 0} \mathbb{E}[e^{-r_\alpha(y-X)}|X>y]. \tag{55}$$

Hence

$$\mathbb{P}[\tau_\alpha < \infty] \geq C_\alpha e^{-r_\alpha(x-x_\alpha)}. \tag{56}$$

Since $r_\alpha < \hat{r}$, the constant C_α satisfies $C_\alpha > C$ and therefore $C_\alpha > 0$ by assumption.
Step 4. The ruin probability then can be estimated as

$$\begin{aligned} \mathbb{P}[\tau(x,K)<\infty] &\geq \mathbb{P}[\tau(x,K)<\infty|\tau_\alpha<\infty]\mathbb{P}[\tau_\alpha<\infty] \\ &\geq \mathbb{P}[\tau(x_\alpha,K)<\infty]\mathbb{P}[\tau_\alpha<\infty] \\ &\geq \Psi^*(x_\alpha)C_\alpha e^{-r_\alpha(x-x_\alpha)}, \end{aligned} \tag{57}$$

where for the second inequality we have used that our setting is Markovian. Note that we only obtain the inequality $\mathbb{P}[\tau(x_\alpha,K)<\infty] \leq \mathbb{P}[\tau(x,K)<\infty|\tau_\alpha<\infty]$ since one can also fall below x_α after a jump and therefore arrive at a level strictly smaller than x_α.
Step 5. We use that $\Psi^*(x_\alpha) \geq Ce^{-\hat{r}x_\alpha}$ (Theorem 6) to show that $\Psi^*(x_\alpha) > 0$ and to finally obtain

$$\Psi(x,K) \geq D_\alpha e^{-r_\alpha x}, \tag{58}$$

for a constant $D_\alpha > 0$ and for all $x > x_\alpha$.
Step 6. It is obvious that for $x \leq x_\alpha$, we can bound $\Psi(x,K)$ from below by some constant $B_\alpha > 0$.
Step 7. Finally taking A_α as the minimum of B_α and D_α, we obtain the desired result. □

Corollary 8. *Assume that X has a uniform exponential moment in the tail distribution for $\hat{r}$. Let $k^* : \mathbb{R}_+ \to \mathbb{R}$ be the defining function of the optimal investment strategy K^*. If this function possesses a limit for $x \to \infty$, then this limit is given by*

$$\lim_{x\to\infty} k^*(x) = \hat{K}. \tag{59}$$

Proof. Assume that $\lim_{x\to\infty} k^*(x) \neq \hat{K}$. Then there exist $\alpha, x_\alpha > 0$ such that

$$|k^*(x) - \hat{K}| > \alpha \text{ for } x \geq x_\alpha. \tag{60}$$

Therefore, using Theorem 7 one obtains that

$$\Psi^*(x) \geq A_\alpha e^{-r_\alpha x}, \tag{61}$$

for some $r_\alpha < \hat{r}$, which together with the Main Theorem yields the apparent contradiction to the optimality of K^*

$$\lim_{x\to\infty} \frac{\Psi^*(x)}{e^{-\hat{r}x}} = \infty. \tag{62}$$

□

Remark. It has been shown recently (after the submission of this paper) by Hipp and Schmidli [12] that the function $k^*(x)$ possesses a limit, for $x \to \infty$.

Appendix A.

In this appendix we present the proof for the following lemma, which was used in the proof of Theorem 4, Section 4, in order to show that the process $\tilde{M}(t,x,K,\hat{r})$ is a local submartingale for all admissible trading strategies $K \in \mathcal{K}$.

Lemma A.1. *Let $0 \le r < r_\infty$ and $K \in \mathcal{K}$. The difference of the processes*

$$\lambda \mathbb{E}[e^{rX}-1]\int_0^{t\wedge\tau} M(s-,x,K,r)ds \tag{A.1}$$

and

$$\int_0^{t\wedge\tau} M(s-,x,K,r)(e^{rX_{N(s)}}-1)dN(s) \tag{A.2}$$

is a martingale w.r.t. the filtration $\mathbb{F}$.

Proof. Note that $N=(N(t))_{t\ge 0}$ is a finite variation process. Therefore the stochastic integral w.r.t. N in (A.2) makes sense (a.s.) as a pathwise Lebesgue-Stieltjes integral (see, e.g., Protter [17]). Let $\{T_n\}_{n=1}^\infty$ denote the arrival times of N. Then

$$\begin{aligned} &\int_0^{t\wedge\tau} M(s-,x,K,r)(e^{rX_{N(s)}}-1)dN(s)\\ = &\sum_{n=1}^\infty M(T_n-,x,K,r)(e^{rX_n}-1)\chi_{\{t\wedge\tau\ge T_n\}}. \end{aligned} \tag{A.3}$$

Taking expectations we obtain for $0 \le t \le T$

$$\begin{aligned} &\mathbb{E}_{t\wedge\tau}[\int_{t\wedge\tau}^{T\wedge\tau} M(s-,x,K,r)(e^{rX_{N(s)}}-1)dN(s)]\\ = &\mathbb{E}_{t\wedge\tau}[\sum_{n=1}^\infty M(T_n-,x,K,r)(e^{rX_n}-1)\chi_{\{T\wedge\tau\ge T_n>t\wedge\tau\}}]\\ = &\mathbb{E}_{t\wedge\tau}[\sum_{n=1}^\infty \mathbb{E}_{T_n-}[M(T_n-,x,K,r)(e^{rX_n}-1)\chi_{\{T\wedge\tau\ge T_n>t\wedge\tau\}}]]\\ = &\mathbb{E}_{t\wedge\tau}[\sum_{n=1}^\infty \mathbb{E}_{T_n-}[e^{\hat{r}X_n}-1]M(T_n-,x,K,r)\chi_{\{T\wedge\tau\ge T_n>t\wedge\tau\}}]\\ = &\mathbb{E}_{t\wedge\tau}[\sum_{n=1}^\infty \mathbb{E}[e^{\hat{r}X}-1]M(T_n-,x,K,r)\chi_{\{T\wedge\tau\ge T_n>t\wedge\tau\}}]\\ = &\mathbb{E}[e^{rX}-1]\mathbb{E}_{t\wedge\tau}[\int_{t\wedge\tau}^{T\wedge\tau} M(s-,x,K,r)dN(s)]\\ = &\mathbb{E}[e^{rX}-1]\mathbb{E}_{t\wedge\tau}[\int_{t\wedge\tau}^{T\wedge\tau} M(s-,x,K,r)\lambda\, ds], \end{aligned} \tag{A.4}$$

where from the fourth to the fifth line we have used that X_n and $\mathcal{F}_{T_n-}$ are independent and from the sixth to the seventh line we have used that $N(t)-\lambda t$ is a martingale (see, e.g., Protter [17], p. 39). Thus the difference of (A.1) and (A.2) is a martingale w.r.t. the stopped filtration $(\mathcal{F}_{t\wedge\tau})_{t\ge 0}$. A standard argument (Protter [17], p. 11) shows that then the difference of (A.1) and (A.2) also is a martingale w.r.t. the filtration $\mathbb{F}$. ☐

Appendix B.

In this appendix we shall examine, to which extent the results of Section 4 can be generalized, when the assumption of a uniform exponential moment in the tail distribution (see (35)) is dropped. In particular, we will show that the statement of Lemma 5 also holds true without this assumption, i.e., for every admissible trading strategy $K \in \mathcal{K}$, the insurer a.s. either gets infinitely rich or ruined (see Proposition B.2).

Proposition B.1.

(i) *Let $x > 0$, and let $\hat{r}$ be defined as in (14). For $z \in \mathbb{R}_+$, we define the stopping time*

$$\tau_z := \inf\{t \leq \tau(x, K) : \; \tilde{Y}(t, x, K) \geq z\},$$

which is only nontrivial, if $x < z$. For every $z \in \mathbb{R}_+$ and every admissible trading strategy $K \in \mathcal{K}$, the stopped process $\tilde{M}^{\tau_z}(t, x, K, \hat{r}) = \tilde{M}(t \wedge \tau_z, x, K, \hat{r})$ is a uniformly integrable submartingale.

Furthermore $\mathbb{P}[\{\tau_z \wedge \tau(x, K) < \infty\}] = 1$ for all $z \in \mathbb{R}_+$, i.e. with probability 1, either the insurer gets ruined or she reaches the level z.

(ii) *For all $K \in \mathcal{K}$, the process $\tilde{M}(t, x, K, \hat{r})$ satisfies the submartingale inequality (for $0 \leq s \leq t$)*

$$\tilde{M}(s, x, K, \hat{r}) \leq \mathbb{E}_s[\tilde{M}(t, x, K, \hat{r})],$$

however, we also allow for the possibility that the above expressions may equal ∞.

Proof.
(i) We have already shown in the proof of Theorem 4 that, for all $K \in \mathcal{K}$, the process $\tilde{M}(t, x, K, \hat{r})$ is a local submartingale. Therefore, the stopped process $\tilde{M}^{\tau_z}(t, x, K, \hat{r})$ is also a local submartingale, for all $K \in \mathcal{K}$. Observe that, for the stopped process $\tilde{M}^{\tau_z}(t, x, K, \hat{r})$, we have a uniform estimate for the exponential tail moments, namely

$$\sup_{0 \leq y \leq z} \mathbb{E}[e^{-r(y-X)} | X > y] < \infty, \; r \in [0, r_\infty). \tag{B.1}$$

Hence (cf. (44))

$$\mathbb{E}[\sup_{0 \leq t < \infty} |\tilde{M}^{\tau_z}(t, x, K, \hat{r})|] < \infty, \tag{B.2}$$

and therefore $\tilde{M}^{\tau_z}(t, x, K, \hat{r})$ is a uniformly integrable submartingale (Protter [17], p. 35). Exactly the same way as in the proof of Lemma 5, we apply Doob's Supermartingale Convergence Theorem to show that $\lim_{t \to \infty} M^{\tau_z}$ exists a.s. Then, we deduce that, for $t \to \infty$, the insurer a.s. either gets ruined or reaches the level z from the fact that, with probability 1, infinitely many jumps of size greater than d occur, which cannot be compensated for by the a.s. continuous stochastic integral w.r.t. the Brownian motion or the a.s. continuous drift term.
(ii) We know from (i) that, for $n \in \mathbb{N}$, $\tau_n := \inf\{t : \tilde{Y}(t, x, K) \geq n\}$, and $0 \leq s \leq t$,

$$\tilde{M}(s \wedge \tau_n, x, K, \hat{r}) \leq \mathbb{E}_s[\tilde{M}(t \wedge \tau_n, x, K, \hat{r})]. \tag{B.3}$$

The l.h.s. of (B.3) converges a.s. to $\tilde{M}(s, x, K, \hat{r})$. The r.h.s. of (B.3) can be rewritten as

$$\begin{aligned} & \mathbb{E}_s[\tilde{M}(t \wedge \tau_n, x, K, \hat{r})] \\ = \;& \mathbb{E}_s[\tilde{M}(t \wedge \tau_n, x, K, \hat{r})\chi_{\{t \wedge \tau_n < \tau(x,K)\}}] + \mathbb{E}_s[\tilde{M}(\tau(x, K), x, K, \hat{r})\chi_{\{\tau(x,K) \leq t \wedge \tau_n\}}]. \end{aligned} \tag{B.4}$$

Letting $n \to \infty$, we can apply the conditional version of the reverse Fatou Lemma to the first term in (B.4) and (conditional) Monotone Convergence to the second term to obtain

$$\lim_{n \to \infty} \mathbb{E}_s[\tilde{M}(t \wedge \tau_n, x, K, \hat{r})\chi_{\{t \wedge \tau_n < \tau(x,K)\}}] \leq \mathbb{E}_s[\tilde{M}(t, x, K, \hat{r})\chi_{\{t < \tau(x,K)\}}], \tag{B.5}$$

and

$$\lim_{n\to\infty} \mathbb{E}_s[\tilde{M}(\tau(x,K),x,K,\hat{r})\chi_{\{\tau(x,K)\leq t\wedge\tau_n\}}] = \mathbb{E}_s[\tilde{M}(\tau(x,K),x,K,\hat{r})\chi_{\{\tau(x,K)\leq t\}}]. \tag{B.6}$$

To sum it up, we obtain

$$\tilde{M}(s,x,K,\hat{r}) \leq \mathbb{E}_s[\tilde{M}(t,x,K,\hat{r})] \text{ a.s..} \tag{B.7}$$

□

Proposition B.2. *Let $x > 0$ and $K \in \mathcal{K}$ be given. On the set $\{\tau(x,K) = \infty\}$, the process $Y(t,x,K)$ converges a.s. to ∞ for $t \to \infty$: either the insurer gets ruined or infinitely rich.*

Proof. Assume that $\lim_{t\to\infty} \tilde{Y}(t,x,K)$ is not a.s. equal to ∞ on the set $\{\tau(x,K) = \infty\}$ for some process $K \in \mathcal{K}$ and some initial reserve $x \in \mathbb{R}_+$. Let us work towards a contradiction.
We know from Proposition B.1 (i) that, for all admissible trading strategies $K \in \mathcal{K}$ and all $n \in \mathbb{N}$,

$$\lim_{t\to\infty} \tilde{Y}^{\tau_n}(t,x,K) = n \text{ a.s. on } \{\tau(x,K) = \infty\}, \tag{B.8}$$

where $\tilde{Y}^{\tau_n}$ denotes the process Y, stopped at time $\tau_n := \inf\{t : \tilde{Y}(t,x,K) \geq n\}$. Therefore for x and K as above, there have to exist numbers $d > 0$, $\delta > 0$, and a subsequence $(n_k)_{k=1}^{\infty}$ of the natural numbers such that

$$\mathbb{P}[\bigcap_{k=1}^{\infty}\{\exists t : \tau_{n_k} \leq t < \tau_{n_{k+1}}, Y(t,x,K) \leq d\} \cap \{\tau(x,K) = \infty\}] > \delta. \tag{B.9}$$

This means that on the set $\{\tau(x,K) = \infty\}$, where ruin a.s. never occurs, the insurer has to reach each level $n \in \mathbb{N}$ - a consequence of Proposition B.1 (i) - but on the other hand she has to fall below the level d in each of the stochastic intervals $]\tau_{n_k}, \tau_{n_{k+1}}[$ with positive probability.
The idea of the subsequent argument is the following: if the insurer falls below the level d too often, she will get ruined with too high probability. For this purpose we define the following stopping times

$$\sigma_k := \inf\{t : \tau_{n_k} \leq t < \tau_{n_{k+1}}, \tilde{Y}(t,x,K) \leq d\} \wedge \tau(x,K) \wedge \tau_{n_{k+1}},\ k \in \mathbb{N}. \tag{B.10}$$

Note that, for all $k \in \mathbb{N}$, the stopping times σ_k are finite a.s.
Next, we define another sequence of stopping times

$$\rho_k := \inf\{t : t > \sigma_k, \tilde{Y}(t,x,K) \geq 2d\},\ k \in \mathbb{N}. \tag{B.11}$$

As a consequence of Proposition B.1 (i), for all $k \in \mathbb{N}$, the stopping times $\rho_k \wedge \tau(x,K)$ are finite a.s. Furthermore, there exists $k_1 \in \mathbb{N}$ such that, for $k \geq k_1$, $\rho_k \wedge \tau(x,K) \leq \tau_{n_{k+1}}$ a.s.
We know from Proposition B.1 (i) that, for each $k \in \mathbb{N}$, the stopped process $\tilde{M}^{\tau_{n_{k+1}}}(t,x,K,\hat{r})$ is a uniformly integrable submartingale, so we can apply Doob's Optional Sampling Theorem (Protter [17]) to the process $\tilde{M}^{\tau_{n_{k+1}}}(t,x,K,\hat{r})$ and the two stopping times σ_k and $\rho_k \wedge \tau(x,K)$, $\sigma_k \leq \rho_k \wedge \tau(x,K) \leq \tau_{n_{k+1}}$, to obtain

$$M(\sigma_k) \leq \mathbb{E}_{\sigma_k}[M(\rho_k \wedge \tau(x,K),x,K,r)],\ k \geq k_1. \tag{B.12}$$

Now, we define the following events

$$A_j := \{\sigma_j < \tau; \sigma_j < \tau_{n_{j+1}}\},\ j \in \mathbb{N}, \tag{B.13}$$

and

$$A^k := \bigcap_{j=1}^{k} A_j,\ k \in \mathbb{N}. \tag{B.14}$$

For all $k \in \mathbb{N}$, the event A^k lies in $\mathcal{F}_{\sigma_k}$.
We multiply inequality (B.12), for each $k \geq k_1$, with the indicator function χ_{A^k} and take expectations to obtain

$$\mathbb{E}[M(\sigma_k, x, K, r)\chi_{A^k}] \leq \mathbb{E}[\mathbb{E}_{\sigma_k}[M(\rho_k \wedge \tau(x, K), x, K, r)]\chi_{A^k}]. \tag{B.15}$$

The left hand side of (B.15) can be bounded by

$$e^{-rd}\mathbb{P}[A^k] \leq \mathbb{E}[M(\sigma_k, x, K, r)\chi_{A^k}], \;\; k \geq k_1, \tag{B.16}$$

since $M(\sigma_k, x, K, r) \geq e^{-rd}$ on the set A^k.
Our aim is to show that the probability, conditional on the event A^k, to get ruined before reaching $2d$ is strictly greater than zero, independent of k. In order to get this estimate we proceed as follows with the right hand side of (B.15). By definition of the conditional expectation

$$\mathbb{E}[\mathbb{E}_{\sigma_k}[M(\rho_k \wedge \tau(x, K), x, K, r)]\chi_{A^k}] = \mathbb{E}[M(\rho_k \wedge \tau(x, K), x, K, r)\chi_{A^k}]. \tag{B.17}$$

Now we argue in a similar fashion as in the proof of Theorem 6

$$\begin{aligned} & \mathbb{E}[M(\rho_k \wedge \tau(x, K), x, K, r)\chi_{A^k}] \\ = \;& \mathbb{E}[M(\tau(x, K), x, K, r)\chi_{A^k}\chi_{\{\tau(x,K)<\rho_k\}}] \\ & + \mathbb{E}[M(\rho_k, x, K, r)\chi_{A^k}\chi_{\{\tau(x,K)\geq\rho_k\}}] \\ \leq \;& \mathbb{E}[M(\tau(x, K), x, K, r)\chi_{A^k}\chi_{\{\tau(x,K)<\rho_k\}}] + e^{-2rd}, \;\; k \geq k_1, \end{aligned} \tag{B.18}$$

using that, for $k \geq k_1$, on the set A^k, the random variable $M(\rho_k, x, K, r)$ equals $\exp(-2rd)$. Then,

$$\begin{aligned} & \mathbb{E}[M(\tau(x, K), x, K, r)\chi_{A^k}\chi_{\{\tau(x,K)<\rho_k\}}] \\ = \;& \mathbb{E}[M(\tau(x, K), x, K, r)|A^k \cap \{\tau(x, K) < \rho_k\}]\,\mathbb{P}[A^k \cap \{\tau(x, K) < \rho_k\}]. \end{aligned} \tag{B.19}$$

Finally, we need the following inequality

$$\mathbb{E}[M(\tau(x, K), x, K, r)|A^k \cap \{\tau(x, K) < \rho_k\}] \leq \sup_{0\leq y\leq 2d} \mathbb{E}[e^{-r(y-X)}|y > X], \tag{B.20}$$

which holds true, because the insurer's wealth is below the level $2d$ on the set $A^k \cap \{\tau(x, K) < \rho_k\}$.
Putting (B.12), (B.16), (B.18) and (B.20) together, we obtain

$$\mathbb{P}[\tau(x, K) < \rho_k|A^k] \geq \frac{e^{-rd} - e^{-2rd}}{\sup_{0\leq y\leq 2d} \mathbb{E}[e^{-r(y-X)}|y > X]} \geq \beta, \;\; k \geq k_1, \tag{B.21}$$

for some constant $\beta > 0$, that just depends on d and not on k.
Now, the proof of Proposition B.2 is almost finished. In order to see that (B.9) cannot hold true for $\delta > 0$, just use

$$\begin{aligned} & \mathbb{P}[\bigcap_{k_1\leq k\leq n} \{\exists t : \tau_{n_k} \leq t < \tau_{n_{k+1}}, Y(t, x, K) \leq d\} \cap \{\tau(x, K) = \infty\}] \\ \leq \;& \mathbb{P}[\bigcap_{k_1\leq k\leq n} \{\exists t : \tau_{n_k} \leq t < \tau_{n_{k+1}}, Y(t, x, K) \leq d\}] \\ = \;& \mathbb{P}[A^n] \\ = \;& \mathbb{P}[A_n|A^{n-1}]\,\mathbb{P}[A^{n-1}]. \end{aligned} \tag{B.22}$$

Since the event $\{\tau(x, K) < \rho_{n-1}\}$ excludes the event A_n, the following holds

$$\begin{aligned} \mathbb{P}[A_n|A^{n-1}]\,\mathbb{P}[A^{n-1}] & \leq (1 - \mathbb{P}[\tau(x, K) < \rho_{n-1}|A^{n-1}])\,\mathbb{P}[A^{n-1}] \\ & \leq (1 - \beta)\,\mathbb{P}[A^{n-1}]. \end{aligned} \tag{B.23}$$

The bottom line is that $\lim_{n\to\infty} \mathbb{P}[A^n] = 0$ and therefore

$$\begin{aligned} & \lim_{n\to\infty} \mathbb{P}[\bigcap_{k_1\le k\le n} \{\exists t : \tau_{n_k} \le t < \tau_{n_{k+1}}, Y(t,x,K) \le d\} \cap \{\tau(x,K) = \infty\}] \\ = \;& \mathbb{P}[\bigcap_{k_1\le k} \{\exists t : \tau_{n_k} \le t < \tau_{n_{k+1}}, Y(t,x,K) \le d\} \cap \{\tau(x,K) = \infty\}] \\ = \;& 0, \end{aligned} \tag{B.24}$$

which is an apparent contradiction to (B.9). Thus we have completed the proof of Proposition B.2. □

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Department for Financial and Actuarial Mathematics, Vienna University of Technology, Wiedner Hauptstrasse 8-10, A-1040 Vienna, Austria

E-mail address: gaier@fam.tuwien.ac.at