On Utility Based Pricing of Contingent Claims in Incomplete Markets

J. Hugonnier[1] D. Kramkov[2] W. Schachermayer[3]

March 5, 2004

[1]HEC Montréal and CIRANO, 3000 Chemin de la Côte S[te] Catherine, Montréal, QC H3T2A7. Financial support from the United States National Science Foundation under grant DMS·9819950 and from IFM[2] is gratefully acknowledged.

[2]Department of Mathematical Sciences, Carnegie Mellon University, 5000 Forbes Avenue,Pittsburgh, PA 15213. Financial support from the United States National Science Foundation under grant DMS·0139911 is gratefully acknowledged.

[3]Vienna University of Technology, Wiedner Hauptstrasse 8−10/105, A−1040 Wien, Austria. Financial support from the FWF Austrian Science Fund under the Wittgenstein-Prize program Z36−MAT and grant P15889 and from the Austrian National Bank under grant Jubiläumsfondprojekt N. 8699 is gratefully acknowledged.

Abstract

We study the uniqueness of the marginal utility based price of contingent claims in a semimartingale model of incomplete financial market. In particular, we obtain that a necessary and sufficient condition for all bounded contingent claims to admit a unique marginal utility based price is that the solution to the dual problem defines an equivalent local martingale measure.

Introduction

As is well known, in a complete financial market every contingent claim can be perfectly replicated by a controlled portfolio of the traded securities and therefore admits a well-defined arbitrage free price. In an incomplete market, to every contingent claim is associated an interval of arbitrage free prices and, unless the contingent claim is replicable in which case this set consists of a single point, arbitrage arguments alone are not sufficient to determine a unique price. Since the endpoints of the arbitrage free interval coincide with the sub and super replication costs of the contingent claim, the buying or selling of the claim at any price in the interior of this set leads to a possible loss at the maturity. Hence, in this case the choice of a price can only be made with respect to some risk functional representing the preferences and endowments of the agent under consideration.

Relying on this observation different utility based valuation approaches have been developed and studied by Hodges and Neuberger [9], Davis [2], Karatzas and Kou [11], Frittelli [7], Foldes [6] and Kallsen [10] among others. We study in this paper the concept of *marginal utility based price* which is defined as such an amount p that, given the possibility of buy & hold trading at p, the agent's optimal demand for the contingent claim is equal to zero. Note that the basic idea underlying this valuation principle is well-known in economics and finance, see, for example, the classic work [8] by Sir Hicks.

While marginal utility based prices exist under very minimal assumptions on the financial market model, the agent's preferences and the contingent claim, their uniqueness remains a more delicate question. In our main theorem we show that the uniqueness of the marginal utility based price is closely related to the property that contingent claims be dominated by the terminal wealth of a portfolio such that the product of its capital process and the solution to a dual problem is a uniformly integrable martingale. As a corollary to this result, we obtain that a necessary and sufficient condition for *all* bounded contingent claims to admit a unique marginal utility based price is that the solution to the dual problem defines an equivalent local martingale measure.

The rest of the paper is organized as follows. In Section 1 we present the model of financial market and recall the definition of various classes of trading strategies. In Section 2 we state and discuss our main results. Section 3 contains all the proofs.

We are indebted to two anonymous referees and an associate editor for pertinent comments and suggestions.

1 The Model

We consider a finite horizon model of a financial market which consists of $d+1$ securities: one savings account and d stocks. As usual in mathematical finance, we shall assume that the price process of the savings account is normalized to one. The price process $S := (S^i)_{i=1}^n$ of the stocks is assumed to be a locally bounded semimartingale on a given filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ where the filtration $\mathbf{F} := (\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions. Hereafter, we let $\mathcal{F}_T = \mathcal{F}$ and denote by $\mathbf{L}^0$ the set of $\mathcal{F}-$measurable random variables.

A probability measure $\mathbb{Q}$ is called an equivalent local martingale measure if it is equivalent to $\mathbb{P}$ and if S is a local martingale under $\mathbb{Q}$. We denote by $\mathcal{M}$ the family of all such measures and assume that

$$\mathcal{M} \neq \emptyset. \tag{1}$$

This rather mild condition is essentially equivalent to the absence of arbitrage opportunities in the financial market, see [3] for precise statements and further references.

A self financing portfolio is defined by a pair (x, H) where $x \in \mathbf{R}$ represents the initial capital and $H = (H^i)_{i=1}^d$ is a predictable and $S-$integrable process specifying the number of shares of each of the stocks held in the portfolio. The value process of a self financing portfolio evolves in time as the stochastic integral of the process H with respect to the stock price:

$$X_t := x + (H \cdot S)_t = x + \int_0^t H_\tau dS_\tau, \qquad t \in [0,T]. \tag{2}$$

A process X is called *admissible* if it is the value of a self financing portfolio and is almost surely non negative. For every $x > 0$ we denote by $\mathcal{X}(x)$ the class of admissible processes whose initial value is equal to x, that is

$$\mathcal{X}(x) := \Big\{X \geq 0 : \ X \text{ satisfies (2) for some } H \text{ and } X_0 = x\Big\}. \tag{3}$$

We shall use a shorter notation $\mathcal{X}$ for the set $\mathcal{X}(1)$. A process $X \in \mathcal{X}(x)$ is said to be *maximal* if its terminal value cannot be dominated by that of any other process in $\mathcal{X}(x)$, that is if $X' \in \mathcal{X}(x)$ and $X_T \leq X'_T$ imply $X' = X$. Finally, a process X is said to be *acceptable* if it admits a decomposition of the form $X = X' - X''$ where X' is admissible and X'' is maximal. For details on maximal and acceptable processes we refer to [4].

We consider an economic agent whose preferences over terminal wealth are represented by a utility function $U : (0, \infty) \to \mathbf{R}$ which is assumed to be strictly increasing, strictly concave, continuously differentiable and to satisfy the Inada conditions:

$$U'(0) := \lim_{x \to 0} U'(x) = \infty, \qquad U'(\infty) := \lim_{x \to \infty} U'(x) = 0. \tag{4}$$

In what follows we set $U(0) := \lim_{x \to 0} U(x)$ and $U(x) = -\infty$ for all $x < 0$.

The *convex conjugate* function of the agent's utility function is defined to be the Legendre transform of the convex function $-U(-\cdot)$, that is

$$V(y) := \sup_{x > 0} \Big\{ -xy + U(x) \Big\}, \qquad y > 0. \tag{5}$$

It is well known that under the Inada conditions (4), the conjugate function is a continuously differentiable, strictly decreasing and strictly convex function satisfying $-V'(0) = \infty$, $V'(\infty) = 0$ and $V(0) = U(\infty)$, $V(\infty) = U(0)$ as well as the following bidual relation

$$U(x) = \inf_{y > 0} \Big\{ xy + V(y) \Big\}, \qquad x > 0. \tag{6}$$

We also note that under the Inada conditions (4), the agent's marginal utility is the inverse of minus the derivative of the convex conjugate: $(U')^{-1} = -V'$.

2 Main Results

Assume that the agent has some initial capital $x > 0$. In accordance with the model set forth in the previous section, the maximal expected utility that this agent can achieve by trading in the financial market is given by

$$u(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E}\big[U\big(X_T\big)\big], \qquad x > 0. \tag{7}$$

Let $\mathcal{Y}$ denote the set of non negative semimartingales Y with initial value one and such that for any $X \in \mathcal{X}$ the product XY is a supermartingale:

$$\mathcal{Y} := \Big\{ Y \geq 0 : Y_0 = 1 \text{ and } XY \text{ is a supermartingale } \forall X \in \mathcal{X} \Big\}.$$

Note that since $1 \in \mathcal{X}$ the semimartingales in $\mathcal{Y}$ are supermartingales. Define a non increasing convex function by setting

$$v(y) = \inf_{Y \in \mathcal{Y}} \mathbb{E}\big[V\big(yY_T\big)\big], \qquad y > 0. \tag{8}$$

Hereafter we shall assume that v is finitely valued on $(0, \infty)$, that is

$$v(y) < \infty, \qquad y > 0. \tag{9}$$

Under this assumption the value function u is strictly increasing, strictly concave and continuously differentiable and v is its convex conjugate, that is

$$v(y) = \sup_{x>0} \Big\{ -xy + u(x) \Big\}. \tag{10}$$

The solutions $\widehat{X}(x)$ and $\widehat{Y}(y)$ to, respectively, (7) and (8) exist and are unique for any $x > 0$ and $y > 0$. Furthermore, if $y = u'(x)$ then

$$U'(\widehat{X}_T(x)) = y\widehat{Y}_T(y), \qquad \mathbb{E}\left[\widehat{Y}_T(y)\widehat{X}_T(x)\right] = x. \tag{11}$$

For details on these results we refer to [13].

Let now $B \in \mathbf{L}^0$ denote a European contingent claim which matures at the terminal time T of the model. For a pair $(x, q) \in \mathbf{R}^2$ we denote by $\mathcal{X}(x, q|B)$ the set of acceptable processes with initial value x and whose terminal value dominates the random variable $-qB$, that is

$$\mathcal{X}(x, q|B) := \Big\{ X \text{ is acceptable with } X_0 = x \text{ and } X_T + qB \geq 0 \Big\}. \tag{12}$$

Note that with $x > 0$ and $q = 0$ this set coincides with the set of admissible processes with initial capital equal to x.

Definition 1 Let $B \in \mathbf{L}^0$ and $x > 0$. A real number p is called a *marginal utility based price for B given the initial capital x* if

$$\mathbb{E}\Big[U\Big(X_T + qB\Big)\Big] \leq u(x), \qquad q \in \mathbf{R},\ X \in \mathcal{X}\big(x - qp, q|B\big). \tag{13}$$

The interpretation of this definition is that p is a marginal utility based price for a contingent claim if, given the possibility of buy & hold trading in the claim at that price, the agent's optimal demand is equal to zero.

The following theorem provides sufficient conditions for the uniqueness of the marginal utility based price and constitutes our main result.

Theorem 1 *Assume that the conditions* (1), (4), (9) *hold true. Fix* $x > 0$, *define* $y = u'(x)$ *and let* $\widehat{Y}(y)$ *denote the corresponding solution to* (8). *Then for any maximal admissible process* $\widetilde{X} \in \mathcal{X}$ *we have:*

(i) *If the product process* $\widehat{Y}(y)\widetilde{X}$ *is a uniformly integrable martingale then every contingent claim with the property that* $|B| \leq c\widetilde{X}_T$ *for some* $c > 0$ *admits a unique marginal utility based price with respect to the initial capital* x *which is given by*

$$p(B|x) := \mathbb{E}\left[\widehat{Y}_T(y)B\right]. \tag{14}$$

(ii) *If the product process* $\widehat{Y}(y)\widetilde{X}$ *fails to be a uniformly integrable martingale, that is if there exists a constant* $\delta > 0$ *such that*

$$\mathbb{E}\left[\widehat{Y}_T(y)\widetilde{X}_T\right] = 1 - \delta,$$

then there exists a contingent claim $0 \leq B \leq \widetilde{X}_T$ *and a constant* $\alpha \geq 0$ *such that every* $\alpha \leq \pi \leq \alpha + \delta$ *is a marginal utility based price for* B.

The proof of the theorem will be given in the next section. We conclude this section with an important corollary and some remarks.

Corollary 1 *Assume that the conditions of Theorem 1 hold. Fix an arbitrary* $x > 0$, *define* $y = u'(x)$ *and let* $\widehat{Y}(y)$ *denote the corresponding solution to* (8). *Then the following assertions are equivalent:*

(i) *The dual optimizer* $\widehat{Y}(y)$ *is the density process of an equivalent local martingale measure* $\widehat{\mathbb{Q}}(y)$ *with respect to* $\mathbb{P}$.

(ii) *Every bounded contingent claim admits a unique marginal utility based price given the initial capital* x.

Moreover, if any of the above assertions hold true, then the marginal utility based price of any bounded contingent claim B has the representation:

$$p(B|x) = \mathbb{E}_{\widehat{\mathbb{Q}}(y)}[B]. \tag{15}$$

Proof. The result follows from Theorem 1 by taking $\widetilde{X} \equiv 1$. ■

Remark 1 Let $\widetilde{X} \in \mathcal{X}$ be a maximal admissible process, fix a contingent claim $|B| \leq \widetilde{X}_T$ and denote by $\mathcal{B}$ the interior of the set of points (x, q) such that the family $\mathcal{X}(x, q|B)$ is non empty. With this notation, the function defined by

$$u(x,q) := \sup_{X \in \mathcal{X}(x,q|B)} \mathbb{E}\Big[U\Big(X_T + qB\Big)\Big], \qquad (x,q) \in \mathcal{B} \tag{16}$$

represents the maximal expected utility that an agent whose endowment is given by x units of cash and q units of the contingent claim can achieve by trading in the financial market.

Using standard arguments from the theory of convex functions we deduce that the set of marginal utility based prices for the contingent claim is given by

$$\mathcal{P}(B|x) := \left\{p = \frac{r}{y} : (y,r) \in \partial u(x,0)\right\} \tag{17}$$

where ∂u denotes the subdifferential of the function u. Uniqueness of the marginal utility based price for the contingent claim B given an initial capital $x > 0$ is thus equivalent to the differentiability of u at the point $(x, 0)$ and our main result can be seen as providing sufficient conditions for this property to hold. If the function u is indeed differentiable at the point $(x, 0)$, then it follows from (17) that the contingent claim's marginal utility based price is uniquely given by

$$p(B|x) = \frac{u_q(x,0)}{u_x(x,0)}. \tag{18}$$

Remark 2 If B is replicable in the sense that there exists an acceptable process X such that $-X$ is also acceptable and $X_T = B$, then B admits a unique marginal utility based price which is independent of both the initial capital and the utility function and is given by $b := X_0$.

Relying on the above observation, one may be tempted to think that for a replicable claim $B \geq 0$ the marginal utility based price is always given by equation (14). Unfortunately, this guess is wrong as can be seen by taking

$B = X_T$ for some maximal admissible process such that $\mathbb{E}[\widehat{Y}_T(y)X_T] < X_0$. For example, in the setting of [12, Example 5.1] one may take $B = 1$. Note that in this case the representation (18) for the marginal utility based price still holds true.

Remark 3 In Theorem 1 above we considered the case of a utility function $U : (0, \infty) \to \mathbf{R}$, satisfying $U'(0) = \infty$ so that we extended U to the whole real line by letting $U(x) = -\infty$, for $x < 0$ (see (4)).

Another important class of utility functions are those $U : \mathbf{R} \to \mathbf{R}$ which are finitely valued on $\mathbf{R}$, and satisfy — apart from increasingness, strict concavity and differentiability — the Inada conditions $U'(-\infty) = \infty$ and $U'(\infty) = 0$. The prime examples of such functions are the constant absolute risk aversion utility functions given by $U(x) = -\exp(-\alpha x)$ for some positive α.

For this class of utility functions it was shown in [1] (see also [14]) that — under mild regularity conditions — the dual optimizer $\widehat{Y}_T(y)$ to (8) equals the density of a probability measure $\widehat{Q}(y) \in \mathcal{M}$. In other words, condition (i) of Corollary 1 is automatically satisfied in this context. Using similar arguments as in the proof of Theorem 1 below one may therefore show that, in the setting of [1] or [14], every bounded contingent claim B admits a unique marginal utility based price with respect to a given initial capital x. In other words the "kink phenomenon" described in Remark 1 only appears in the present case of utility functions satisfying (4) and not for utility functions which are finitely valued on all of $\mathbf{R}$.

Remark 4 In the present paper we restrict to the assumption of local boundedness for the process S, mainly in order to avoid technicalities. It is, however, possible to generalize our results to the case of arbitrary semi-martingales S if the following provisos are taken: in the definition of the set $\mathcal{M}$ (as well as in the statement of Corollary 1) the notion of "local martingale measure" has to be replaced by the term "separating measure". A probability measure $\mathbb{Q}$ is called a "separating measure" if any admissible X is a supermartingale under $\mathbb{Q}$. It is not hard to check that the results of [4], on which the proof of the theorem below relies, carry over to this setting, so that one does not need the local boundedness assumption there either.

3 Proof of the main theorem

For the convenience of the reader we start by recalling some results on maximal processes.

Lemma 1 *Assume* (1) *and let* $(X^n)_{n\geq 1}$ *be a sequence of maximal admissible processes. Then the set of probability measures* $\mathbb{Q} \in \mathcal{M}$ *under which all these processes are uniformly integrable martingales is non empty and dense in* $\mathcal{M}$ *with respect to the variation norm.*

Proof. The result follows from Corollary 2.16, Theorem 5.2 and Corollary 5.3 of [4]. ■

Lemma 2 *Assume* (1). *Let* $\widetilde{X} \in \mathcal{X}$, $Y \in \mathcal{Y}$, $B \in \mathbf{L}^0$ *be such that* $|B| \leq \widetilde{X}_T$ *and* $x > 0$, $q \in \mathbf{R}$ *be such that the set* $\mathcal{X}(x, q|B)$ *is not empty. If the product* $Y\widetilde{X}$ *is a uniformly integrable martingale, then*

$$\mathbb{E}[Y_T X_T] \leq x, \quad X \in \mathcal{X}(x, q|B).$$

Proof. Fix $X \in \mathcal{X}(x, q|B)$ and consider the process $X + |q|\widetilde{X}$. As is easily seen from the assumption made on the contingent claim, this process has a positive terminal value. On the other hand, the process X being acceptable it can be written as $X' - X''$ where X' is admissible and X'' is maximal. By Lemma 1 we have that there exists at least one probability measure $\mathbb{Q} \in \mathcal{M}$ under which X'' is a uniformly integrable martingale. Using this in conjunction with the fact that admissible processes are supermartingales under $\mathbb{Q}$ we obtain

$$0 \leq \mathbb{E}_{\mathbb{Q}}\Big[\Big(X_T + |q|\widetilde{X}_T\Big)\Big|\mathcal{F}_t\Big] \leq X_t + |q|\widetilde{X}_t$$

and conclude that $X + |q|\widetilde{X}$ is an admissible process. The definition of $\mathcal{Y}$ and the assumption of the lemma then imply that

$$\begin{aligned} \mathbb{E}\Big[Y_T X_T\Big] &= \mathbb{E}\Big[Y_T\Big(X_T + |q|\widetilde{X}_T\Big)\Big] - |q| \cdot \mathbb{E}\Big[Y_T \widetilde{X}_T\Big] \\ &= \mathbb{E}\Big[Y_T\Big(X_T + |q|\widetilde{X}_T\Big)\Big] - |q| \leq x. \end{aligned}$$

■

Proof of Theorem 1.(i). Let $B \in \mathbf{L}^0$ be a European contingent claim satisfying the assumption of the statement and assume without loss of generality that $c = 1$. Fix also $x > 0$ and set $y = u'(x)$. We start by showing that the quantity $p = p(B|x)$ defined by (14) is indeed a marginal utility based price for the contingent claim. Using (6) in conjunction with (14) and (10) we obtain that

$$\begin{aligned}
\mathbb{E}\Big[U\Big(X_T + qB\Big)\Big] &\leq \mathbb{E}\Big[V\left(y\widehat{Y}_T(y)\right) + y\widehat{Y}_T(y)\Big(X_T + qB\Big)\Big] \\
&= v(y) + y\left(\mathbb{E}\Big[\widehat{Y}_T(y)X_T\Big] + qp\right) \\
&= u(x) + y\left(\mathbb{E}\Big[\widehat{Y}_T(y)X_T\Big] - (x - qp)\right)
\end{aligned}$$

holds for all $q \in \mathbf{R}$, $X \in \mathcal{X}(x - qp, q|B)$ and the desired inequality (13) now follows from Lemma 2.

We start the proof of uniqueness by showing that p defines the minimal marginal utility based price in the sense that for any $\pi < p$ one can find a positive number q such that the inequality

$$u(x - q\pi, q) = \sup_{X \in \mathcal{X}(x - q\pi, q|B)} \mathbb{E}\Big[U\Big(X_T + qB\Big)\Big] > u(x) \tag{19}$$

holds true. As is easily seen, the existence of such a positive number will follow once we have shown that

$$D(x, \pi) := \liminf_{q \searrow 0} \left\{ \frac{u(x - q\pi, q) - u(x)}{q} \right\} > 0. \tag{20}$$

Let $0 \leq (q_n)_{n=1}^{\infty} \leq \frac{x}{1+|\pi|}$ be an arbitrary sequence of positive numbers decreasing to zero and define a sequence of acceptable processes by setting

$$X^n := \widehat{X}(x - q_n(1 + \pi)) + q_n\widetilde{X}$$

where $\widehat{X}(x - q_n(1+\pi))$ is the optimal wealth process for the no–contingent claim problem (7) with initial capital $x - q_n(1 + \pi)$. As is easily seen, this process belongs to the set $\mathcal{X}(x - q_n\pi, q_n|B)$ for each $n \geq 1$ and it thus follows from the concavity of the agent's utility function that we have

$$\begin{aligned}
u(x - q_n\pi, q_n) &\geq \mathbb{E}\left[U\Big(X_T^n + q_nB\Big)\right] \\
&\geq \mathbb{E}\left[U\left(\widehat{X}_T(x - q_n(1 + \pi))\right) + q_n\Big(\widetilde{X}_T + B\Big)U'\Big(X_T^n + q_nB\Big)\right] \\
&= u(x - q_n(1 + \pi)) + \mathbb{E}\left[q_n\Big(\widetilde{X}_T + B\Big)U'\Big(X_T^n + q_nB\Big)\right].
\end{aligned}$$

Using the above inequality in conjunction with (14), the definition of $D(x,\pi)$ and the differentiability of the value function $u = u(x)$ we obtain

$$
\begin{aligned}
D(x,\pi) \geq\ & -u'(x)(1+\pi) + \liminf_{n\to\infty} \mathbb{E}\left[\left(\widetilde{X}_T + B\right) U'\left(X_T^n + q_n B\right)\right] \\
\geq\ & -u'(x)(1+\pi) + \mathbb{E}\left[\left(\widetilde{X}_T + B\right) U'\left(\widehat{X}_T(x)\right)\right] \\
=\ & u'(x)(p-\pi) > 0.
\end{aligned}
$$

where the second inequality follows from the assumption $|B| \leq \widetilde{X}_T$, the non negativity of U', the fact that X_T^n converges to $\widehat{X}_T(x)$ almost surely and Fatou's Lemma.

We thus have shown that for any contingent claim B' dominated by $\widetilde{X}_T$ the quantity $p(B'|x)$ defined by (14) determines the lower bound of marginal utility based prices. In particular, we have that $-p = p(-B|x)$ is the minimal marginal utility based price for $-B$. However, as one can easily see from Definition 1, this implies that p is the maximal marginal utility based price for B. This finishes the proof of uniqueness. ■

Proof of Theorem 1.(ii). Assume that the product $\widehat{Y}(y)\widetilde{X}$ is not a uniformly integrable martingale. Since this process is a supermartingale by definition of the set $\mathcal{Y}$ it follows that there exists $0 < \delta < 1$ such that

$$
\mathbb{E}\left[\widehat{Y}_T(y)\widetilde{X}_T\right] = 1 - \delta < 1. \tag{21}
$$

Let $\mathcal{M}'$ be the set of equivalent probability measures $\mathbb{Q} \in \mathcal{M}$ under which the maximal admissible process $\widetilde{X}$ is a uniformly integrable martingale. As is easily seen, this set is closed under countable convex combinations and by Lemma 1 it is dense in $\mathcal{M}$ with respect to the variation norm. It thus follows from [12, Propositions 3.1 and 3.2] that

$$
\sup_{\mathbb{Q}\in\mathcal{M}'} \mathbb{E}_{\mathbb{Q}}[X_T] = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[X_T] = \sup_{Y\in\mathcal{Y}} \mathbb{E}[Y_T X_T]
$$

holds for all admissible processes $X \in \mathcal{X}$ and

$$
\begin{aligned}
v(y) &= \min_{Y\in\mathcal{Y}} \mathbb{E}\big[V\big(yY_T\big)\big] \\
&= \mathbb{E}\left[V\left(y\widehat{Y}_T(y)\right)\right] = \inf_{\mathbb{Q}\in\mathcal{M}'} \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right].
\end{aligned}
$$

Let now $(\mathbb{Q}^n)_{n=1}^{\infty}$ be an arbitrary minimizing sequence for the above problem, that is a sequence in $\mathcal{M}'$ with the property that

$$\lim_{n\to\infty} \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}^n}{d\mathbb{P}}\right)\right] = v(y) = \mathbb{E}\Big[V\left(y\widehat{Y}_T(y)\right)\Big]. \tag{22}$$

Passing if necessary to a subsequence and relying on Komlòs lemma, we may assume without loss of generality that $\left(\frac{d\mathbb{Q}_n}{d\mathbb{P}}\right)_{n=1}^{\infty}$ converges in probability to a non negative function Y_*. Applying [12, Lemma 3.4] and taking into account the uniqueness of the solution to (8) we deduce that $Y_* = \widehat{Y}_T(y)$. Using this fact in conjunction with (21) and the definition of the set $\mathcal{M}'$, we conclude that even though it is convergent in probability and bounded in expectations, the sequence

$$(Z^n)_{n=1}^{\infty} := \left(\widetilde{X}_T\frac{d\mathbb{Q}^n}{d\mathbb{P}}\right)_{n=1}^{\infty}$$

is not uniformly integrable. Therefore, by Rosenthal's subsequence splitting lemma (see for example [5]) we can find a subsequence $(Z^k)_{k=1}^{\infty}$ and a sequence of pairwise disjoint measurable sets $(A_k)_{k=1}^{\infty}$ with the properties that

$$\lim_{k\to\infty} \mathbb{E}\Big[1_{A_k}Z^k\Big] = \delta, \tag{23}$$

and for any bounded random variable R

$$\lim_{k\to\infty} \mathbb{E}\left[\left(1_{A_k^c}Z^k - \widehat{Y}_T(y)\widetilde{X}_T\right)R\right] = 0. \tag{24}$$

After these lengthy preparations we now turn to the construction of a contingent claim satisfying the assertions of the statement. Let $2\mathbb{N}$ denote the set of even natural numbers and define

$$A := \bigcup_{k\in 2\mathbb{N}} A_k \subset \Omega.$$

For our contingent claim we take the positive random variable $B := 1_A\widetilde{X}_T$ and finally, for the non negative constant α we take

$$\alpha := \mathbb{E}\Big[\widehat{Y}_T(y)B\Big] = \mathbb{E}\Big[1_A\widehat{Y}_T(y)\widetilde{X}_T\Big].$$

In order to complete our proof, it is now sufficient to show that the inequality $u(x-qp,q) \le u(x)$ holds true for all strictly positive q when $p=\alpha$ and for all strictly negative q when $p = \alpha + \delta$. To this end, we start by observing that it follows from (5), the definition of $\mathcal{M}'$ and Lemma 2 that

$$\begin{aligned}\mathbb{E}\Big[U\Big(X_T + qB\Big)\Big] &\leq \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}^k}{d\mathbb{P}}\right)\right] + y\,\mathbb{E}\left[\frac{d\mathbb{Q}^k}{d\mathbb{P}}\Big(X_T + qB\Big)\right] \\ &\leq \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}^k}{d\mathbb{P}}\right)\right] + y(x - qp) + yq\,\mathbb{E}\left[\frac{d\mathbb{Q}^k}{d\mathbb{P}}B\right] \qquad (25)\end{aligned}$$

holds for all $(q,p) \in \mathbf{R}^2$, $X \in \mathcal{X}(x - qp, q|B)$ and $k \geq 1$.

Now let $p = \alpha$ and fix an arbitrary $q > 0$. Taking limits on both sides of the above expression as $k \notin 2\mathbb{N} \to \infty$ and using (22) in conjunction with the definition of the set A and (24) with $R = 1_A$ we obtain that

$$\begin{aligned}\mathbb{E}\Big[U\Big(X_T + qB\Big)\Big] &\leq v(y) + y(x - q\alpha) + \lim_{k\notin 2\mathbb{N}\to\infty}\mathbb{E}\Big[yq\Big(1_A Z^k\Big)\Big] \\ &= u(x) - yq\alpha + \lim_{k\notin 2\mathbb{N}\to\infty}\mathbb{E}\Big[yq\Big(1_{A\cap A_k}Z^k + 1_{A\cap A_k^c}Z^k\Big)\Big] \\ &= u(x) - yq\alpha + \lim_{k\notin 2\mathbb{N}\to\infty}\mathbb{E}\Big[yq\Big(1_{A\cap A_k^c}Z^k\Big)\Big] = u(x)\end{aligned}$$

holds for all $X \in \mathcal{X}(x - q\alpha, q|B)$ and the desired inequality follows. Similarly, fix an arbitrary $q < 0$ and let $p = \alpha + \delta =: \beta$. Taking limits on both sides of (25) as $k \in 2\mathbb{N}$ increases to infinity and using (22) in conjunction with the definition of A, (23) and (24) with $R = 1_A$ we obtain that

$$\begin{aligned}\mathbb{E}\Big[U\Big(X_T + qB\Big)\Big] - u(x) &\leq yq\left(-\beta + \lim_{k\in 2\mathbb{N}\to\infty}\mathbb{E}\Big[1_A Z^k\Big]\right) \\ &= yq\left(-\beta + \lim_{k\in 2\mathbb{N}\to\infty}\mathbb{E}\Big[1_{A\cap A_k}Z^k + 1_{A\cap A_k^c}Z^k\Big]\right) \\ &= yq\left(-\beta + \lim_{k\in 2\mathbb{N}\to\infty}\mathbb{E}\Big[1_{A_k}Z^k + 1_{A\cap A_k^c}Z^k\Big]\right) = 0\end{aligned}$$

holds for all $X \in \mathcal{X}(x - q\beta, q|B)$ and our proof is complete. ■

References

[1] F. Bellini and M. Frittelli. On the existence of minimax martingale measures. *Mathematical Finance*, 12(1):1–21, 2002.

[2] M.H.A. Davis. Option pricing in incomplete markets. In M.A.H. Dempster and S.R. Pliska, editors, *Mathematics of Derivative Securities*, pages 216–226. Cambridge University Press, 1997.

[3] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 300:463–520, 1994.

[4] F. Delbaen and W. Schachermayer. The banach space of workable contingent claims in arbitrage theory. *Ann. Inst. H. Poincaré Probab. Statist.*, 33(1):113–144, 1997.

[5] J. Diestel and J.J. Uhl. *Vector Measures*. Mathematical Surveys. American Mathematical Society, 1997.

[6] L. Foldes. Valuation and martingale properties of shadow prices: An exposition. *Journal of Economic Dynamics and Control*, 24:1641–1701, 2000.

[7] M. Frittelli. Introduction to a theory of value coherent with the no-arbitrage principle. *Finance and Stochastics*, 4(3):275–297, 2000.

[8] Sir J. R. Hicks. *A Revision of Demand Theory*. Oxford University Press, 1956.

[9] S.D. Hodges and A. Neuberger. Optimal replication of contingent claims under transaction costs. *The Review of Futures Markets*, 8:222–239, 1989.

[10] J. Kallsen. Derivative pricing based on local utility maximization. *Finance and Stochastics*, 6:115–140, 2002.

[11] I. Karatzas and S. G. Kou. On the pricing of contingent claims under constraints. *Annals of Applied Probability*, 6(2):321–69, 1996.

[12] D.O. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, 9(3):904–950, 1999.

[13] D.O. Kramkov and W. Schachermayer. Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *Annals of Applied Probability*, 13(4):1504–1516, 2003.

[14] W. Schachermayer. Optimal investment in incomplete markets when wealth may become negative. *Annals of Applied Probability*, 11(3):694–734, 2001.