HOW CLOSE ARE THE OPTION PRICING FORMULAS OF BACHELIER AND BLACK-MERTON-SCHOLES?

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Abstract. We compare the option pricing formulas of Louis Bachelier and Black-Merton-Scholes and observe – theoretically and by typical data – that the prices coincide very well. We illustrate Louis Bachelier's efforts to obtain applicable formulas for option pricing in pre-computer time. Furthermore we explain – by simple methods from chaos expansion – why Bachelier's model yields good short-time approximations of prices and volatilities.

1. Introduction

It is the pride of Mathematical Finance that L. Bachelier was the first to analyze Brownian motion mathematically, and that he did so in order to develop a theory of option pricing (see [2]). In the present note we shall review some of the results from his thesis as well as from his later textbook on probability theory (see [3]), and we shall explain the remarkable closeness of prices in the Bachelier and Black-Merton-Scholes model.

The "fundamental principle" underlying Bachelier's approach to option pricing is crystallized in his famous dictum (see [2], p.34):

"L'ésperance mathematique du spéculateur est nul",

i.e. "the mathematical expectation of a speculator is zero". His argument in favor of this principle is based on equilibrium considerations (see [2] and [10]), similar to what in today's terminology is called the "efficient market hypothesis" (see [9]), i.e. the use of martingales to describe stochastic time evolutions of price movements in ideal markets (see [2], p. 31):

"It seems that the market, the aggregate of speculators, can believe
in neither a market rise nor a market fall, since, for each
quoted price, there are as many buyers as sellers."

The reader familiar with today‘s approach to option pricing might wonder where the concept of "risk free interest rate" has disappeared to, which seems crucial in the modern approach of pricing by no arbitrage arguments (recall that the *discounted* price process should be a martingale under the risk neutral measure). The answer is that L. Bachelier applied his "fundamental principle" in terms of "true" prices (this is terminology from 1900 which corresponds to the concept of forward prices in modern terminology). Is is well-known that the passage to forward prices makes the riskless interest rate disappear: in the context of the Black-Merton-Scholes formula, this is what amounts to the so-called Black's formula (see [4]).

Summing up: Bachelier's "fundamental principle" yields *exactly* the same recipe for option pricing as we use today (for more details we refer to the first section of the St. Flour summer school lecture [10]): using discounted terms ("true prices") one obtains the prices of options (or of more general derivatives of European style) by taking expectations. The expectation pertains to a probability measure under which the price process of the underlying security (in discounted terms) satisfies the fundamental principle, i.e. is a martingale in modern terminology.

It is important to emphasize that, although the *recipes* for obtaining option prices are the same for Bachelier's as for the modern approach, the *arguments* in favour of them are very different: an equilibrium argument in Bachelier's case as opposed to the no arbitrage arguments in the Black-Merton-Scholes approach. With all admiration for Bachelier's work, the development of a theory of hedging and replication by dynamic strategies, which is the crucial ingredient of the Black-Merton-Scholes-approach, was far out of his reach (compare [10]).

In order to obtain option prices one has to specify the underlying model. We fix a time horizon $T > 0$. As is well-known, Bachelier proposed to use (properly scaled) Brownian motion as a model for stock prices. In modern terminology this amounts (using "true prices") to

$$S_t^B := S_0(1 + \sigma W_t), \tag{1.1}$$

for $0 \le t \le T$, where $(W_t)_{0\le t\le T}$ denotes standard Brownian motion and the superscript B stands for Bachelier. The parameter $\sigma > 0$ denotes the volatility in modern terminology. In fact, Bachelier used the normalization $H = S_0 \frac{\sigma}{\sqrt{2\pi}}$ and called this quantity the "coefficient of instability" or of "nervousness" of the security S.

The Black-Merton-Scholes model (under the risk-neutral measure) for the discounted price process is, of course, given by

$$S_t^{BS} = S_0 \exp(\sigma W_t - \frac{\sigma^2}{2} t), \tag{1.2}$$

for $0 \le t \le T$.

This model was proposed by P. Samuelson in 1965, after he had – led by an inquiry of J. Savage for the treatise [3] – personally rediscovered the virtually forgotten Bachelier thesis in the library of Harvard University. The difference between the two models is analogous to the difference between linear and compound interest, as becomes apparent when looking at the associated differential equations

$$dS_t^B = S_0^B \sigma dW_t,$$
$$dS_t^{BS} = S_t^{BS} \sigma dW_t.$$

This analogy makes us expect that, in the short run, both models should yield similar results while, in the long run, the difference should be spectacular. Fortunately, options usually have a relatively short time to maturity (the options considered by Bachelier had a time to expiration of less than 2 months), while in the long run we all are dead (to quote J.M. Keynes).

2. Bachelier versus Black-Merton-Scholes

We now have assembled all the ingredients to recall the derivation of the price of an option in Bachelier's framework. Fix a strike price K (of course, in "true", i.e. discounted terms), a horizon T and consider the European call C, whose pay-off at time T is modeled by the random variable

$$C_T^B = (S_T^B - K)_+.$$

Applying Bachelier‘s “fundamental principle” and using that S_T^B is normally distributed with mean S_0 and variance $S_0^2\sigma^2 T$, we obtain for the price of the option at time $t = 0$

$$C_0^B = E[(S_T^B - K)_+]$$

$$= \int_{K-S_0}^{\infty} (S_0 + x - K) \frac{1}{S_0\sigma\sqrt{2\pi T}} \exp(-\frac{x^2}{2\sigma^2 S_0^2 T}) dx \tag{2.1a}$$

$$= (S_0 - K)\Phi(\frac{S_0 - K}{S_0\sigma\sqrt{T}}) + S_0\sigma\sqrt{T}\phi(\frac{S_0 - K}{S_0\sigma\sqrt{T}}), \tag{2.1b}$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$ denotes the density of the standard normal distribution. We applied the relation $\phi'(x) = -x\phi(x)$ to pass from (2.1a) to (2.1b). For further use we shall need the Black-Merton-Scholes price, too,

$$C_0^{BS} = E[(S^{BS} - K)_+]$$

$$= \int_{-\infty}^{\infty} (S_0 \exp(-\frac{\sigma^2 T}{2} + \sigma\sqrt{T}x) - K)_+ \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx \tag{2.2a}$$

$$= \int_{\frac{\log \frac{K}{S_0} + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}}^{\infty} (S_0 \exp(-\frac{\sigma^2 T}{2} + \sigma\sqrt{T}x) - K) \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx \tag{2.2b}$$

$$= S_0\Phi(\frac{\log \frac{S_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}) - K\Phi(\frac{\log \frac{S_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}). \tag{2.2c}$$

Interestingly, Bachelier explicitly wrote down formula (2.1a), but did not bother to spell out formula (2.1b). The main reason seems to be that at his time option prices – at least in Paris – were quoted the other way around: while today the strike prices K is fixed and the option price fluctuates according to supply and demand, at Bachelier’s times the option prices were fixed (at 10, 20 and 50 Centimes for a “rente”, i.e., a perpetual bond with par value of 100 Francs) and therefore the strike price K fluctuates. Hence what Bachelier really needed was the inverse version of the above relation between $C = C_0^B$ and K.

Apparently there is no simple “formula” to express this inverse relationship. This is somewhat analogous to the situation in the Black-Merton-Scholes model, where there is also no “formula” for the inverse problem of calculating the implied volatility as a function of the given option price. We shall see below that L. Bachelier designed a clever series expansion for C_0^B as a function of the strike price K in order to derive (very) easy formulae which *approximate* this inverse relation and which were well suited to pre-computer technology.

2.1. **At the money options.** Bachelier first specializes to this case (called “simple options” in the terminology of 1900), when $S_0 = K$. In this case (2.1b) reduces to the simple and beautiful relation

$$C_0^B = S_0\sigma\sqrt{\frac{T}{2\pi}}.$$

As explicitly noticed by Bachelier, this formula can also be used, for a given price C of the at the money option with maturity T, to determine the “coefficient of nervousness of the security” $H = \frac{S_0\sigma}{\sqrt{2\pi}}$, i.e., to determine the implied volatility in modern technology. Indeed, it suffices to normalize the price C_0^B by $\sqrt{T}$ to obtain

$H = \frac{C_0^B}{\sqrt{T}}$. This leads us to our first result. For convenience we phrase it rather in terms of σ than of H.

Proposition 1. *The volatility σ in the Bachelier model is determined by the price C_0^B of an at the money option with maturity T by*

$$\sigma = \frac{C_0^B}{S_0}\sqrt{\frac{2\pi}{T}}. \tag{2.3}$$

In the subsequent Proposition, we compare, for fixed volatility $\sigma > 0$ and time to maturity T, the price of an at the money call option as obtained from the Black-Merton-Scholes and Bachelier's formula respectively. Furthermore we also compare the implied volatilities, for given price C_0 of an at the money call, in the Bachelier and Black-Merton-Scholes model.

Proposition 2. *Fix $\sigma > 0$, $T > 0$ and $S_0 = K$, and denote by C^B and C^{BS} the corresponding prices for a European call option in the Bachelier (1.1) and Black-Merton-Scholes model (1.2) respectively. Then*

$$0 \le C_0^B - C_0^{BS} \le \frac{S_0}{12\sqrt{2\pi}}\sigma^3 T^{\frac{3}{2}} = \mathcal{O}((\sigma\sqrt{T})^3). \tag{2.4}$$

Conversely, fix the price $0 < C_0 < S_0$ of an at the money option and denote by $\sigma^B := \sigma$ the implied Bachelier volatility and by σ^{BS} the implied Black-Merton-Scholes volatility, then

$$0 \le \sigma^{BS} - \sigma^B \le \frac{T}{12}(\sigma^{BS})^3. \tag{2.5}$$

Proof. (compare [10]). For $S_0 = K$, we obtain in the Bachelier and Black-Merton-Scholes model the following prices, respectively,

$$\begin{aligned} C_0^B &= \frac{S_0\sigma}{\sqrt{2\pi}}\sqrt{T} \\ C_0^{BS} &= S_0(\Phi(\frac{1}{2}\sigma\sqrt{T}) - \Phi(-\frac{1}{2}\sigma\sqrt{T})). \end{aligned}$$

Hence

$$\begin{aligned} 0 \le C_0^B - C_0^{BS} &= (\frac{S_0}{\sqrt{2\pi}}x - S_0(\Phi(\frac{x}{2}) - \Phi(-\frac{x}{2})))|_{x=\sigma\sqrt{T}} \le \\ &\le \frac{S_0}{\sqrt{2\pi}}\int_{-\frac{x}{2}}^{\frac{x}{2}} \frac{y^2}{2} dy|_{x=\sigma\sqrt{T}} = \\ &= \frac{S_0}{\sqrt{2\pi}}\frac{x^3}{12}|_{x=\sigma\sqrt{T}} = \frac{S_0}{12\sqrt{2\pi}}\sigma^3 T^{\frac{3}{2}} = \mathcal{O}((\sigma\sqrt{T})^3), \end{aligned}$$

since $e^y \ge 1 + y$ for all y, so that $\frac{y^2}{2} \ge 1 - e^{-\frac{y^2}{2}}$ for all y.

For the second assertion note that solving equation

$$C_0 = \frac{\sigma^B}{\sqrt{2\pi}}\sqrt{T} = \Phi(\frac{1}{2}\sigma^{BS}\sqrt{T}) - \Phi(-\frac{1}{2}\sigma^{BS}\sqrt{T})$$

given $\sigma^B > 0$ yields the Black-Merton-Scholes implied volatility σ^{BS}. We obtain similarly as above

$$\begin{aligned} 0 \leq \sigma^{BS} - \sigma^B &= \sigma^{BS} - \frac{\sqrt{2\pi}}{\sqrt{T}}(\Phi(\frac{1}{2}\sigma^{BS}\sqrt{T}) + \Phi(-\frac{1}{2}\sigma^{BS}\sqrt{T})) \\ &= \frac{\sqrt{2\pi}}{\sqrt{T}}(\frac{1}{\sqrt{2\pi}}x - (\Phi(\frac{x}{2}) - \Phi(-\frac{x}{2})))|_{x=\sigma^{BS}\sqrt{T}} \\ &\leq \frac{\sqrt{2\pi}}{\sqrt{T}}\frac{1}{12\sqrt{2\pi}}(\sigma^{BS})^3 T^{\frac{3}{2}} = \frac{(\sigma^{BS})^3 T}{12}. \end{aligned}$$

□

Proposition 1 and 2 yield in particular the well-known asymptotic behaviour of an at the money call price in the Black-Merton-Scholes model as described in [1].

Proposition 2 tells us that for the case when $(\sigma\sqrt{T}) \ll 1$ (which typically holds true in applications), formula (2.3) gives a satisfactory approximation of the implied Black-Merton-Scholes volatility, and is very easy to calculate. We note that for the data reported by Bachelier (see [2] and [10]), the yearly volatility was of the order of 2.4% p.a. and T in the order of one month, i.e $T = \frac{1}{12}$ years so that $\sqrt{T} \approx 0.3$. Consequently we get $(\sigma\sqrt{T})^3 \approx (0.008)^3 \approx 5 \times 10^{-7}$. The estimate in Proposition 2 yields a right hand side of $\frac{S_0}{12\sqrt{2\pi}} 5 \times 10^{-7} = 1.6 \times 10^{-8} S_0$, i.e. the difference of the Bachelier and Black-Merton-Scholes price (when using the same volatility $\sigma = 2.4\%$ p.a.) is of the order 10^{-8} of the price S_0 of the underlying security.

Remark 1. *Inequality (2.4) is an estimate of third order, whereas inequality (2.5) only yields an estimate of the* relative error *of second order*

$$\frac{\sigma^{BS} - \sigma^B}{\sigma^{BS}} \leq \frac{1}{12}(\sigma^{BS}\sqrt{T})^2.$$

On the other hand, for the time-standardized volatilities at maturity T we obtain an estimate of third order

$$\sigma^{BS}\sqrt{T} - \sigma^B\sqrt{T} \leq \frac{1}{12}(\sigma^{BS}\sqrt{T})^3.$$

3. Further results of L. Bachelier

We now proceed to a more detailed analysis of the option pricing formula (2.1b) for general strike prices K. We shall introduce some notation used by L. Bachelier for the following two reasons: firstly, it should make the task easier for the interested reader to look up the original texts by Bachelier; secondly, and more importantly, we shall see that his notation has its own merits and allows for intuitive and economically meaningful interpretations (as we have already seen for the normalization $H = \frac{S_0\sigma}{\sqrt{2\pi}}$ of the volatility, which equals the time-standardized price of an at the money option).

L. Bachelier found it convenient to use a parallel shift of the coordinate system moving S_0 to 0, so that the Gaussian distribution will be centered at 0. We write

$$a = \frac{S_0\sigma\sqrt{T}}{\sqrt{2\pi}}, \quad m := K - S_0, \quad P := m + C. \tag{3.1}$$

The parameter a equals, up to the normalizing factor $\frac{S_0}{\sqrt{2\pi}}$, the *time-standardized volatility $\sigma\sqrt{T}$ at maturity T*. Readers familiar, e.g. with the Hull-White model of stochastic volatility, will realize that this is a very natural parametrization for an option with maturity T.

In any case, the quantity a was a natural parametrization for L. Bachelier, as it is the *price of the at the money option* with the maturity T (see formula 2.3), so that it can be observed from market data.

The quantity m is the strike price K shifted by S_0 and needs no further explanation. P has a natural interpretation (in Bachelier's times it was called "écart", i.e. the "spread" of an option): it is the price P of a european put at maturity T with strike price K, as was explicitly noted by Bachelier (using, of course, different terminology): as nicely explained in [2], a speculator "á la hausse", i.e. hoping for a rise of S_T may buy a forward contract with maturity T. Using the "fundamental principle", which in this case boils down to elementary no arbitrage arguments, one concludes that the forward price must equal S_0, so that the total gain or loss of this operation is given by the random variable $S_T - S_0$ at time T.

On the other hand , a more prudent speculator might want to limit the maximal loss by a quantity $K > 0$. She thus would buy a call option with price C, which would correspond to a strike price K (here we see very nicely the above mentioned fact that in Bachelier's times the strike price was considered as a function of the option price C – la "prime" in french – and not vice versa as today). Her total gain or loss would then be given by the random variable

$$(S_T - K)_+ - C.$$

If at time T it indeed turns out that $S_T \geq K$, then the buyer of the forward contract is, of course, better off than the option buyer. The difference equals

$$(S_T - S_0) - [(S_T - K) - C] = K - S_0 + C = P,$$

which therefore may be interpreted as a "cost of insurance". If $S_T \leq K$, we obtain

$$(S_T - S_0) - [0 - C] = (S_T - K) + K - S_0 + C = (S_T - K) + P.$$

By the Bachelier's "fundamental principle" we obtain

$$P = E[(S_T - K)_-].$$

Hence Bachelier was led by no-arbitrage considerations to the put-call parity. For further considerations we denote the put price in the Bachelier model at time $t = 0$ by P_0^B or $P(m)$ respectively.

Clearly, the higher the potential loss C is, which the option buyer is ready to accept, the lower the costs of insurance P should be and vice versa, so that we expect a monotone dependence of these two quantities.

In fact, Bachelier observed that the following pretty result holds true in his model (see [3], p.295):

Proposition 3 (Theorem of reciprocity)**.** *For fixed $\sigma > 0$ and $T > 0$ the quantities C and P are reciprocal in Bachelier's model, i.e. there is monotone strictly decreasing and self-inverse, i.e. $I = I^{-1}$ function $I : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $P = I(C)$.*

Proof. Denote by ψ the density of $S_T - S_0$, then

$$C(m) = \int_m^\infty (x - m)\psi(x)dx,$$
$$P(m) = \int_{-\infty}^m (m - x)\psi(x)dx.$$

Hence we obtain that $C(-m) = P(m)$. We note in passing that this is only due to the symmetry of the density ψ with respect to reflection at 0. Since $C'(m) < 0$ (see the proof of Proposition 1) we obtain $P = P(m(C)) := I(C)$, where $C \mapsto m(C)$ inverts the function $m \mapsto C(m)$. C maps $\mathbb{R}$ in a strictly decreasing way to $\mathbb{R}_{>0}$ and P maps $\mathbb{R}$ in a strictly increasing way to $\mathbb{R}_{>0}$. The resulting map I is therefore strictly decreasing, and – due to symmetry – we obtain

$$I(P) = P(m(P)) = P(-m(C)) = C,$$

so I is self-inverse. □

Using the above notations (3.1), equation (2.1b) for the option price C_0^B (which we now write as $C(m)$ to stress the dependence on the strike price) obtained from the fundamental principles becomes

$$C(m) = \int_m^\infty (x - m)\mu(dx), \tag{3.2}$$

where μ denotes the distribution of $S_T - S_0$, which has the Gaussian density $\mu(dx) = \psi(x)dx$ with

$$\psi(x) = \frac{1}{S_0\sigma\sqrt{2\pi T}} \exp(-\frac{x^2}{2\sigma^2 S_0^2 T}) = \frac{1}{2\pi a} \exp(-\frac{x^2}{4\pi a^2}).$$

As mentioned above, Bachelier does not simply calculate the integral (3.2). He rather does something more interesting (see [3], Nr.445, p.294): "Si l'on développe l'integrale en série, on obtient

$$C(m) = a + \frac{m}{2} + \frac{m^2}{4\pi a} - \frac{m^4}{96\pi^2 a^3} + \frac{m^6}{1920\pi^3 a^5} + \ldots" . \tag{3.3}$$

In the subsequent theorem we justify this step. It is worth noting that the method for developing this series expansion is not restricted to Bachelier's model, but holds true in general (provided that μ, the probability distribution of $S_T - S_0$, admits a density function ψ, which is analytic in a neighborhood of 0).

Theorem 1. *Suppose that the law μ of the random variable S_T admits a density*

$$\mu(dx) = \psi(x)dx,$$

such that ψ is analytic in a ball of radius $r > 0$ around 0, and that

$$\int_{-\infty}^\infty x\psi(x)dx < \infty.$$

Then the function

$$C(m) = \int_m^\infty (x - m)\mu(dx)$$

is analytic for $|m| < r$ and admits a power series expansion

$$C(m) = \sum_{k=0}^\infty c_k m^k,$$

where $c_0 = \int_0^\infty x\psi(x)dx$, $c_1 = \int_0^\infty \psi(x)dx$ and $c_k = \frac{1}{k!}\psi^{(k-2)}(0)$ for $k \geq 2$.

Proof. Due to our assumptions C is seen to be analytic as sum of two analytic functions,

$$C(m) = \int_m^\infty x\psi(x)dx - m\int_m^\infty \psi(x)dx.$$

Indeed, if ψ is analytic around 0, then the functions $x \mapsto x\psi(x)$ and $m \mapsto \int_m^\infty x\psi(x)dx$ are analytic with the same radius of convergence r. The same holds true for the function $m \mapsto m\int_m^\infty \psi(x)dx$. The derivatives can be calculated by the Leibniz rule,

$$\begin{aligned} C'(m) &= -m\psi(m) - \int_m^\infty \psi(x)dx + m\psi(m) \\ &= -\int_m^\infty \psi(x)dx, \\ C''(m) &= \psi(m), \end{aligned}$$

whence we obtain for the k-th derivative,

$$C^{(k)}(m) = \psi^{(k-2)}(m),$$

for $k \geq 2$. □

Remark 2. *If we assume that $m \mapsto C(m)$ is locally analytic around $m = 0$ (without any assumption on the density ψ), then the density $x \mapsto \psi(x)$ is analytic around $x = 0$, too, by inversion of the above argument.*

Remark 3. *Arguing formally, the formulae for the coefficients c_k become rather obvious on an intuitive level: denote by $H_m(x) = (x - m)_+$ the "hockey stick" function with kink at m. Note the symmetry – up to the sign – in the variables m and x. In particular $\frac{\partial}{\partial m^k}H_m(x) = (-1)^k H_m^{(k)}(x)$, where the derivatives have to be interpreted in the distributional sense. We write $H_m^{(k)}$ for $\frac{\partial}{\partial x^k}H_m(x)$ and observe that $H_m'(x) = 1_{\{x \geq m\}}$, which is the Heaviside function centered at m, and $H_m''(x) = \delta_m(x)$, the "Dirac δ-function" centered at m. Applying this formal argument again, we obtain from $\frac{\partial^2}{\partial m^2}\langle \psi, H_m\rangle = \left\langle \psi, H_m^{(2)} \right\rangle = \psi(m)$, and consequently,*

$$\frac{\partial^k}{\partial m^k}| \langle \psi, H_m\rangle = \psi^{(k-2)}(m)$$

for any Schwartz test function ψ. Hence (under the assumption that the Taylor series makes sense)

$$\begin{aligned} C(m) &= \sum_{k=0}^\infty \frac{\partial^k}{\partial m^k}|_{m=0} \langle \psi, H_m\rangle \frac{m^k}{k!} \\ &= \sum_{k=0}^1 c_k \frac{m^k}{k!} + \sum_{k=2}^\infty \psi^{(k-2)}(0)\frac{m^k}{k!}, \end{aligned}$$

with $\left\langle \psi, H_m^{(0)} \right\rangle = c_0$ and $-\left\langle \psi, H_m^{(1)} \right\rangle = c_1$.

Remark 4. *In the case when ψ equals the Gaussian distribution, the calculation of the Taylor coefficients yields*

$$\frac{d}{dy}(\frac{1}{2\pi a}\exp(-\frac{y^2}{4\pi a^2})) = -\frac{1}{4}\frac{y}{\pi^2 a^3}e^{-\frac{1}{4}\frac{y^2}{\pi a^2}},$$
$$\frac{d^2}{dy^2}(\frac{1}{2\pi a}\exp(-\frac{y^2}{4\pi a^2})) = -\frac{1}{8}\frac{2\pi a^2 - y^2}{\pi^3 a^5}e^{-\frac{1}{4}\frac{y^2}{\pi a^2}},$$
$$\frac{d^3}{dy^3}(\frac{1}{2\pi a}\exp(-\frac{y^2}{4\pi a^2})) = \frac{1}{16}\frac{6\pi a^2 y - y^3}{\pi^4 a^7}e^{-\frac{1}{4}\frac{y^2}{\pi a^2}},$$
$$\frac{d^4}{dy^4}(\frac{1}{2\pi a}\exp(-\frac{y^2}{4\pi a^2})) = \frac{1}{32}\frac{12\pi^2 a^4 - 12y^2\pi a^2 + y^4}{\pi^5 a^9}e^{-\frac{1}{4}\frac{y^2}{\pi a^2}},$$

Consequently $\psi(0) = \frac{1}{2\pi a}$, $\psi'(0) = 0$, $\psi''(0) = -\frac{1}{4\pi^2 a^3}$, $\psi'''(0) = 0$ and $\psi''''(0) = \frac{3}{8}\frac{1}{\pi^3 a^5}$, hence with $C(0) = a$ and $C'(0) = -\frac{1}{2}$,

$$C(m) = a - \frac{m}{2} + \frac{m^2}{4\pi a} - \frac{m^4}{96\pi^2 a^3} + \frac{m^6}{1920\pi^3 a^5} + \mathcal{O}(m^8) \tag{3.4}$$

and the series converges for all m, as the Gaussian distribution is an entire function. This is the expansion indicated by Bachelier in [3]*. Since $P(-m) = C(m)$, we also obtain the expansion for the put*

$$P(m) = a + \frac{m}{2} + \frac{m^2}{4\pi a} - \frac{m^4}{96\pi^2 a^3} + \frac{m^6}{1920\pi^3 a^5} + \mathcal{O}(m^8). \tag{3.5}$$

Remark 5. *Looking once more at Bachelier's series one notes that it is rather a Taylor expansion in $\frac{m}{a}$ than in m. Note furthermore that $\frac{m}{a}$ is a dimensionless quantity. The series then becomes*

$$C(m) = a \quad F(\frac{m}{a})$$
$$F(x) = 1 - \frac{x}{2} + \frac{x^2}{4\pi} - \frac{x^4}{96\pi^2} + \frac{x^6}{1920\pi^3} + \mathcal{O}(x^8).$$

We note as a curiosity that already in the second order term the number π appears. Whence – if we believe in Bachelier's formula for option pricing – we are able to determine π at least approximately (see (3.6) below) – from financial market data (compare Georges Louis Leclerc Comte de Buffon's method to determine π by using statistical experiments).

Remark 6. *Denoting by $C(m, S_0)$ the price of the call in Bachelier's model as a function of $m = K - S_0$ and S_0, observe that there is another reciprocity relation namely*

$$C(m + b, S_0) = C(m, S_0 - b)$$

for all $b \in \mathbb{R}$. This is due to the linear dependence of S_T on S_0. Hence we obtain

$$\frac{\partial^k}{\partial m^k}C(m, S_0) = (-1)^k \frac{\partial^k}{\partial S_0^k}C(m, S_0).$$

Note that the derivatives on the right hand side are "Greeks" of the option in today's terminology. Hence in the Bachelier model – as in any other Markovian model with linear dependence on the initial value S_0 – the derivatives with respect to the strike price correspond to the calculation of certain "Greeks". Letting $S_0 = K$, we find that – in modern terminology – in Bachelier's model for at the money European call options the "delta" equals $\frac{1}{2}$ and the "gamma" equals $\frac{1}{2\pi}$. At this point, however, *we have to be careful not to overdo our interpretation of Bachelier's findings: the idea of "Greeks" and its relation to dynamic replication can – to the best of our knowledge – not be found in Bachelier's work.*

Let us turn back to Bachelier's original calculations. He first truncated the Taylor series (3.4) after the quadratic term, i.e.

$$C(m) \approx a - \frac{m}{2} + \frac{m^2}{4\pi a}. \tag{3.6}$$

This (approximate) formula can easily be inverted by solving a quadratic equation, thus yielding an explicit formula for m as a function of C. Bachelier observes that the approximation works well for small values of $\frac{m}{a}$ (the cases relevant for his practical applications) and gives some numerical estimates. We summarize the situation.

Proposition 4 (Rule of Thumb 1)**.** *For a given maturity T denote by $a = C(0)$ the price of the at the money option and by $C(m)$ the price of the call option with strike $K = S_0 + m$. Letting*

$$\widehat{C(m)} = a - \frac{m}{2} + \frac{m^2}{4\pi a} \tag{3.7}$$

$$= C(0) - \frac{m}{2} + \frac{m^2}{4\pi C(0)}, \tag{3.8}$$

we get an approximation of the Bachelier price $C(m)$ up to order m^3.

Remark 7. *Note that the value of $\widehat{C(m)}$ only depends on the price a of an at the money option (which is observable at the market) and the quantity $m = K - S_0$.*

Remark 8. *Given any stock price model under a risk neutral measure, the above approach of quadratic approximation can be applied if the density ψ of $S_T - S_0$ is locally analytic and admits first moments. The approximation then reads*

$$\widehat{C(m)} = C(0) - B(0)m + \frac{\psi(0)}{2}m^2$$

up to a term of order $\mathcal{O}(m^3)$. Here $C(0)$ denotes the price of the at the money european call option (pay-off $(S_T - S_0)_+$), $B(0)$ the price of the at the money binary option (pay-off $1_{\{S_T \geq S_0\}}$) and $\psi(0)$ the value of an at the money "Dirac" option (pay-off δ_{S_0}, with an appropriate interpretation as a limit).

Example 1. *Take for instance the Black-Merton-Scholes model, then the quadratic approximation can be calculated easily,*

$$\begin{aligned} C(0) &= S_0(\Phi(\frac{1}{2}\sigma\sqrt{T}) - \Phi(-\frac{1}{2}\sigma\sqrt{T})) \\ B(0) &= \Phi(-\frac{1}{2}\sigma\sqrt{T}), \\ \psi(0) &= \frac{1}{\sigma\sqrt{2\pi T}}\frac{1}{S_0}\exp(-\frac{1}{8}\sigma^2 T). \end{aligned}$$

Notice that the density ψ of $S_T - S_0$ in the Black-Merton-Scholes model is not an entire function, hence the expansion only holds with a finite radius of convergence.

Although Bachelier had achieved with formula (3.7) a practically satisfactory solution, which allowed to calculate (approximately) m as a function of C by only using pre-computer technology, he was not entirely satisfied. Following the reflexes of a true mathematician he tried to obtain better approximations (yielding still easily computable quantities) than simply truncating the Taylor series after the quadratic term. He observed that, using the series expansion for the put option (see formula 3.5)

$$P(m) = a + \frac{m}{2} + \frac{m^2}{4\pi a} - \frac{m^4}{96\pi^2 a^3} + ...$$

and computing the product function $C(m)P(m)$ or, somewhat more sophisticatedly, the triple product function $C(m)P(m)\frac{C(m)+P(m)}{2}$, one obtains interesting cancellations in the corresponding Taylor series,

$$\begin{aligned} C(m)P(m) &= a^2 - \frac{m^2}{4} + \frac{m^2}{2\pi} + \mathcal{O}(m^4), \\ C(m)P(m)\frac{C(m)+P(m)}{2} &= a^3 - \frac{m^2 a}{4} + \frac{3m^2 a}{4\pi} + \mathcal{O}(m^4). \end{aligned}$$

Observe that $(C(m)P(m))^{\frac{1}{2}}$ is the geometric mean of the corresponding call and put price, while $(C(m)P(m)\frac{C(m)+P(m)}{2})^{\frac{1}{3}}$ is the geometric mean of the call, the put and the arithmetic mean of the call and put price.

The latter equation yields the approximate identity

$$(C(m) + P(m))C(m)P(m) \approx 2a^3$$

which Bachelier rephrases as a cooking book recipe (see [3], p.201):

> On additionne l'importance de la prime et son écart.
> On multiplie l'importance de la prime par son écart.
> On fait le produit des deux résultats.
> Ce produit doit être le même pour toutes les primes
> qui ont même échéance.

This recipe allows to approximately calculate for $m \neq m'$ in a quadruple

$$(C(m), P(m), C(m'), P(m'))$$

any one of these four quantities as an easy (from the point of view of pre-computer technology) function of the other three. Note that, in the case $m = 0$, we have $C(0) = P(0) = a$, which makes the resulting calculation even easier.

We now interpret these equations in a more contemporary language (but, of course, only rephrasing Bachelier's insight in this way).

Proposition 5 (Rule of Thumb 2)**.** *For given $T > 0$, $\sigma^B > 0$ and $m = K - S_0$ denote by $C(m)$ and $P(m)$ the prices of the corresponding call and put options in the Bachelier model. Denote by*

$$\begin{aligned} a(m) &:= C(m)P(m) \\ b(m) &= C(m)P(m)(\frac{C(m)+P(m)}{2}) \end{aligned}$$

the products considered by Bachelier, then we have $a(0) = a^2$ *and* $b(0) = a^3$*, and*

$$\frac{a(m)}{a(0)} = 1 - \frac{(\pi - 2)(\frac{m}{a})^2}{4\pi} + \mathcal{O}(m^4),$$
$$\frac{b(m)}{b(0)} = 1 - \frac{(\pi - 3)(\frac{m}{a})^2}{4\pi} + \mathcal{O}(m^4).$$

Remark 9. *We rediscover an approximation of the reciprocity relation of Proposition 3.1 in the first of the two rules of the thumb. Notice also that this rule of thumb only holds up to order* $(\frac{m}{a})^2$*. Finally note that* $\pi - 3 \approx 0.1416$ *while* $\pi - 2 \approx 1.1416$*, so that the coefficient of the quadratic term in the above expressions is smaller for* $\frac{b(m)}{b(0)}$ *by a factor of* 8 *as compared to* $\frac{a(m)}{a(0)}$*. This is why Bachelier recommended this slightly more sophisticated product.*

4. Bachelier versus other models

We now aim to explain the remarkable asymptotic closeness encountered in Proposition 2 of the Black-Merton-Scholes and the Bachelier price for at the money options in a more systematic way. We therefore consider a general model $(S_t)_{0 \leq t \leq T}$ in true (i.e. discounted) prices (under a martingale measure) and provide conditions for asymptotic closeness to the Bachelier model $(S_t^B)_{0 \leq t \leq T}$, and systematic generalizations of these considerations. We shall show that – under mild technical conditions - we can always find a Bachelier model $(S_t^B)_{0 \leq t \leq T}$, i.e. a parameter $\sigma > 0$, such that the short-time asymptotics of the difference of the European style option prices in the general model and in the Bachelier model is of order $\mathcal{O}(S_0(\sigma\sqrt{t})^2)$ as $(\sigma\sqrt{t}) \to 0$. Since $(\sigma\sqrt{t})$ is typically small, this yields remarkably good approximations for prices.

We assume that $(S_t)_{0 \leq t \leq T}$ is modeled on a filtered probability space $(\Omega, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ such that $(\mathcal{F}_t)_{0 \leq t \leq T}$is generated by a standard Brownian motion $(W_t)_{0 \leq t \leq T}$. We assume that $(S_t)_{0 \leq t \leq T}$ is of the form

$$S_t := S_0 + \int_0^t \alpha(s) dW_s,$$

where $(\alpha_t)_{0 \leq t \leq T}$ is a predictable L^2-integrator, so that $(S_t)_{0 \leq t \leq T}$ is an L^2-martingale with continuous trajectories. First we analyse a condition, such that Bachelier's model yields a short-time asymptotics of order $\mathcal{O}(S_0(\sigma\sqrt{t})^2)$.

Proposition 6. *Assume that for some* $\sigma > 0$ *and fixed time horizon* $T > 0$ *we have* $E[\int_0^t (\alpha(s) - \sigma S_0)^2 ds]^{\frac{1}{2}} \leq C S_0 \sigma^2 t$ *for* $t \in [0, T]$ *and some constant* $C > 0$ *(which may depend on the horizon* T*). Then for any globally Lipschitz function* $g : \mathbb{R} \to \mathbb{R}$ *we have*

$$\left| E[g(S_t^B)] - E[g(S_t)] \right| \leq C ||g||_{Lip} S_0 \sigma^2 t,$$

for $0 \leq t \leq T$*, where* $(S_t^B)_{0 \leq t \leq T}$ *denotes the Bachelier model with constants* $\sigma > 0$ *and* $S_0^B = S_0$*.*

Proof. From the basic L^2-isometry of stochastic integration we obtain

$$\begin{aligned}\big|E[g(S_t^B)] - E[g(S_t)]\big| &\leq ||g||_{Lip} E[|S_t^B - S_t|] \\ &\leq ||g||_{Lip} E[(S_t^B - S_t)^2]^{\frac{1}{2}} \\ &= ||g||_{Lip} E[\int_0^t (\alpha(s) - \sigma S_0)^2 ds]^{\frac{1}{2}} \\ &\leq C||g||_{Lip} S_0 \sigma^2 t,\end{aligned}$$

for $0 \leq t \leq T$. □

Example 2. *If we compare the Bachelier and the Black-Merton-Scholes model, we observe that there is a constant $C = C(T) > 0$ such that*

$$E(\int_0^t (\sigma S_0 \exp(\sigma W_s - \frac{\sigma^2 s}{2}) - \sigma S_0)^2 ds) \leq C^2 S_0^2 \sigma^4 t^2,$$

hence the estimate holds for $0 \leq t \leq T$. As in Example 5 below we obtain an explicit sharp estimate, namely

$$C^2 = \sum_{m \geq 2} \frac{T^{m-2}}{m!} = \frac{\exp(T) - 1 - T}{T^2}.$$

Example 3. *Given the Hobson-Rogers model (see* [6]*), i.e.*

$$\begin{aligned} S_t^{HR} &= S_0 + \int_0^t S_0 \eta(Y_u) \exp(-\frac{1}{2}\int_0^u \beta(s)^2 ds + \int_0^u \beta(s) dW_s) dW_u, \\ \beta(s) &= \eta(Y_s), \\ dY_s &= (-\frac{\eta(Y_s)^2}{2} + \lambda Y_s) ds + \eta(Y_s) dW_s, \end{aligned}$$

with $\eta : \mathbb{R} \to \mathbb{R}$ a C^∞-bounded function and $\eta(Y_0) = \sigma$. Hence we obtain a similar estimate as in the Black-Merton-Scholes case (with a different constant C of course) in general, but we have to choose the Bachelier model with appropriate volatility $\sigma = \eta(Y_0)$.

Now we focus on possibly higher orders in the short-time asymptotics. We first replace Bachelier's original model by a general Gaussian martingale model and ask for the best Gaussian martingale model to approximate a given L^2-martingale. We then consider extensions of this Gaussian martingale model into higher chaos components to obtain higher order time asymptotics. We finally prove that, under a mild technical assumption on the general process $(S_t)_{0 \leq t \leq T}$, we are able to provide short-time asymptotics of any order.

If we analyse the crucial condition $E(\int_0^t (\alpha(s) - \sigma S_0)^2 ds)^{\frac{1}{2}} \leq C S_0 \sigma^2 t$ of Proposition 6 more carefully, we realize that we can improve this estimate if we allow for deterministic, time-dependent volatility functions $\sigma : [0, T] \to \mathbb{R}$ instead of a constant $\sigma > 0$, i.e. if we minimize

$$E(\int_0^t (\alpha(s) - \sigma(s) S_0)^2 ds)$$

along all possible choices of deterministic functions $s \mapsto \sigma(s)$. From the basic inertia identity $E[(X - E[X])^2] = \min_{a \in \mathbb{R}} E[(X - a)^2]$, which holds true for any random variable X with finite second moment, we infer that the best possible choice is given by

$$\sigma(s) := \frac{E(\alpha(s))}{S_0}. \tag{4.1}$$

If we are already given an estimate $E(\int_0^t (\alpha(s) - \sigma S_0)^2 ds)^{\frac{1}{2}} \leq C S_0 \sigma^2 t$ for some constant $\sigma > 0$, then $E(\int_0^t (\alpha(s) - \sigma(s) S_0)^2 ds)^{\frac{1}{2}} \leq C' S_0 \sigma^2 t$, with $C' \leq C$, for $\sigma(s)$ defined by (4.1).

Instead of solving the one-dimensional problem of finding the best Bachelier model with minimal L^2-distance to $(S_t)_{0 \leq t \leq T}$, we hence ask for the best Gaussian martingale model with respect to the L^2-distance, which will improve the constants in the short-time asymptotics. Notice, however, that for the Black-Merton-Scholes model with constant volatility σ, the choice $\sigma(s) = \sigma$ for $0 \leq s \leq T$ is the optimal.

Example 4. *In the Hobson-Rogers model (4.1) leads to*

$$\sigma(s) := E[\eta(Y_s) \exp(-\frac{1}{2} \int_0^s \beta(v)^2 dv + \int_0^s \beta(v) dW_v)]$$

for $0 \leq s \leq T$ and hence the best (in the sense of L^2-distance) Gaussian approximation of $(S_t^{HR})_{0 \leq t \leq T}$ is given through

$$(S_0(1 + \int_0^t \sigma(s) dW_s))_{0 \leq t \leq T}.$$

There are possible extensions of the Bachelier model in several directions, but extensions, which improve – in an optimal way – the L^2-distance and the short-time asymptotics, are favorable from the point of applications. This observation in mind we aim for best (in the sense of L^2-distance) approximations of a given general process $(S_t)_{0 \leq t \leq T}$ by iterated Wiener-Ito integrals up to a certain order. We demand that the approximating processes are martingales. The methodology results into the one of chaos expansion or Stroock-Taylor theorems (see [8]). Methods from chaos expansion for the (explicit) construction of price processes have proved to be very useful, see for instance [5]. In the sequel we shall call any Gaussian martingale a *(general) Bachelier model.*

Definition 1. *Fix $N \geq 1$. Denote by $(M_t^{(n)})_{0 \leq t \leq T}$ martingales with continuous trajectories for $0 \leq n \leq N$, such that $M_t^{(n)} \in \mathcal{H}_n$ for $0 \leq t \leq T$, where $\mathcal{H}_n$ denotes the n-th Wiener chaos in $L^2(\Omega)$. Then we call the process*

$$S_t^{(N)} := \sum_{n=0}^{N} M_t^{(n)}$$

an extension of degree N of the Bachelier model. *Note that $M_t^{(0)} = M_0$ is constant, and that $S_t^{(1)} = M_0 + M_t^{(1)}$ is a (general) Bachelier model.*

Given an L^2-martingale $(S_t)_{0 \leq t \leq T}$ and $N \geq 1$. There exists a unique extension of degree N of the Bachelier model (in the sense that the martingales $(M_t^{(n)})_{0 \leq t \leq T}$ are uniquely defined for $0 \leq n \leq N$) minimizing the L^2-norm $E[(S_t - S_t^{(N)})^2]$ for all $0 \leq t \leq T$. Furthermore $S_t^{(N)} \to S_t$ in the L^2-norm as $N \to \infty$, uniformly for $0 \leq t \leq T$.

Indeed, since the orthogonal projections $p_n : L^2(\Omega, \mathcal{F}_T, P) \to L^2(\Omega, \mathcal{F}_T, P)$ onto the n-th Wiener chaos commute with conditional expectations $E(.|\mathcal{F}_t)$ (see for instance [8]), we obtain that

$$M_t^{(n)} := p_n(S_t),$$

for $0 \leq t \leq T$, is a martingale with continuous trajectories, because

$$E(p_n(S_T)|\mathcal{F}_t) = p_n(E(S_T|\mathcal{F}_t) = p_n(S_t)$$

for $0 \leq t \leq T$. Consequently

$$S_t^{(N)} := \sum_{n=0}^{N} M_t^{(n)}$$

is a process minimizing the distance to S_t for any $t \in [0, T]$. Clearly we have that

$$S_t := \sum_{n=0}^{\infty} M_t^{(n)}$$

in the L^2-topology.

Example 5. *For the Black-Merton-Scholes model we obtain that*

$$M_t^{(n)} = S_0 \sigma^n t^{\frac{n}{2}} H_n(\frac{W_t}{\sqrt{t}}),$$

where H_n denotes the n-th Hermite polynomial, i.e. $(n+1)H_{n+1}(x) = xH_n(x) - H_{n-1}(x)$ and $H_0(x) = 1$, $H_1(x) = x$ for $n \geq 1$ (for details see [7]). Hence

$$\begin{aligned} M_t^{(0)} &= S_0, \\ M_t^{(1)} &= S_0 \sigma W_t, \\ M_t^{(2)} &= S_0 \frac{\sigma^2}{2}(W_t^2 - t). \end{aligned}$$

We recover Bachelier's model as extension of degree 1 minimizing the distance to the Black-Merton-Scholes model. Note that we have

$$||S_t - S_t^{(N)}||_2 \leq C S_0 \sigma^{\frac{N+1}{2}} t^{\frac{N+1}{2}},$$

for $0 \leq t \leq T$. Furthermore we can calculate a sharp constant C, namely

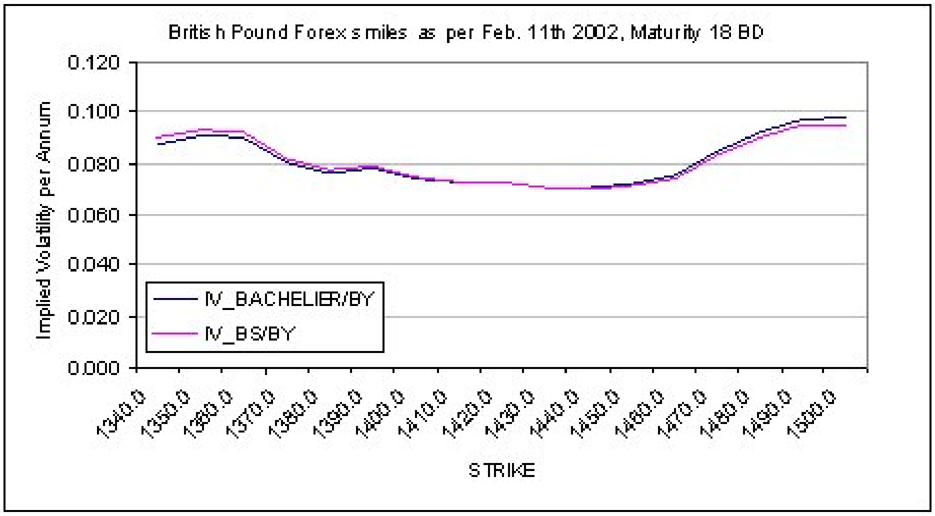
$$\begin{aligned} C^2 &= \sup_{0 \leq t \leq T} \sum_{m \geq N+1} t^{m-N-1} E[(H_m(\frac{W_t}{\sqrt{t}}))^2] \\ &= \sum_{m \geq N+1} T^{m-N-1} \frac{1}{m!}. \end{aligned}$$

since $E[(H_m(\frac{W_t}{\sqrt{t}}))^2] = \frac{1}{m!}$ for $0 < t \leq T$ and $m \geq 0$.

Due to the particular structure of the chaos decomposition we can prove the desired short-time asymptotics:

Theorem 2. *Given an L^2-martingale $(S_t)_{0 \leq t \leq T}$, assume that $S_T = \sum_{i=0}^{\infty} W_i(f_i)$ with symmetric L^2-functions $f_i : [0,T]^i \to \mathbb{R}$ and iterated Wiener-Ito integrals*

$$W_t^i(f_i) := \int_{0 \leq t_1 \leq \cdots \leq t_i \leq t} f_i(t_1, \ldots, t_i) dW_{t_1} \cdots dW_{t_i}$$



. If there is an index i_0 such that for $i \geq i_0$ the functions f_i are bounded and $K^2 := \sum_{i \geq i_0} \frac{T^{i-i_0}}{i!} ||f_i||_\infty^2 < \infty$, then we obtain $||S_t - S_t^{(N)}||_2 \leq K t^{\frac{n+1}{2}}$ for each $N \geq i_0$ and $0 \leq t \leq T$.

Proof. We apply that $E[(E(W_T^i(f_i)|\mathcal{F}_t))^2] = \frac{1}{i!}||1_{[0,t]}^{\otimes i} f_i||_{L^2([0,T]^i)}^2 \leq \frac{t^i}{i!}||f_i||_\infty^2$ for $i \geq i_0$. Hence the result by applying

$$||S_t - S_t^{(N)}||_2 \leq \sum_{i \geq i_0} E[(E(W_T^i(f_i)|\mathcal{F}_t))^2] \leq K t^{\frac{N+1}{2}}.$$

□

Remark 10. *Notice that the Stroock-Taylor Theorem (see [8], p.161) tells that for $S_T \in \mathcal{D}^{2,\infty}$ the series*

$$\sum_{i=0}^{\infty} W_T^i((t_1, \ldots, t_i) \mapsto E(D_{t_1,\ldots,t_i} S_T)) = S_T$$

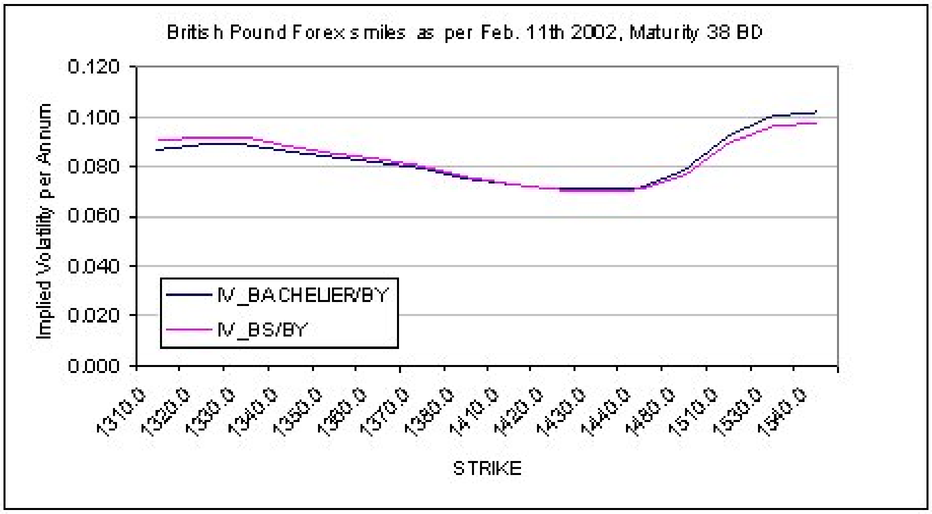
converges in $\mathcal{D}^{2,\infty}$. Hence the above condition is a statement about boundedness of higher Malliavin derivatives. The well-known case of the Black-Merton-Scholes model yields

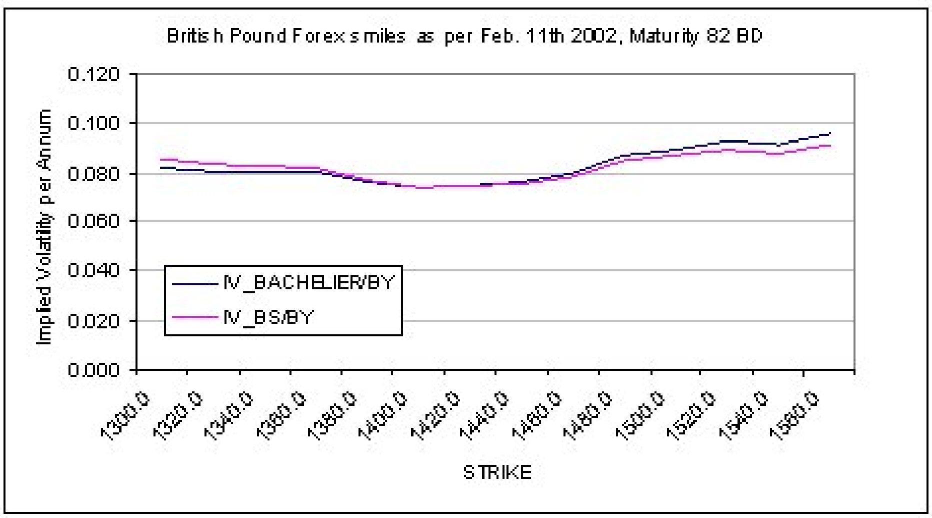
$$D_{t_1,\ldots,t_i} S_T^{BS} = 1_{[0,T]}^{\otimes i}$$

for $i \geq 0$, so the condition of the previous theorem is satisfied.

5. Appendix

We provide tables with implied Black-Merton-Scholes and Bachelier volatilities on the British Pound/ US $ foreign exchange market. The data refer to February 11, 2002. The spot price on this day was approximately 1.41 US$. In the graphical illustrations we indicate by K the strike price in "true prices". We measure time to maturity in Business days (BD). The graphs show the close match between implied volatilities for at the money options; this match becomes less precise when the options tends out of or into the money, and time to maturity T increases, as we have seen in Section 2.





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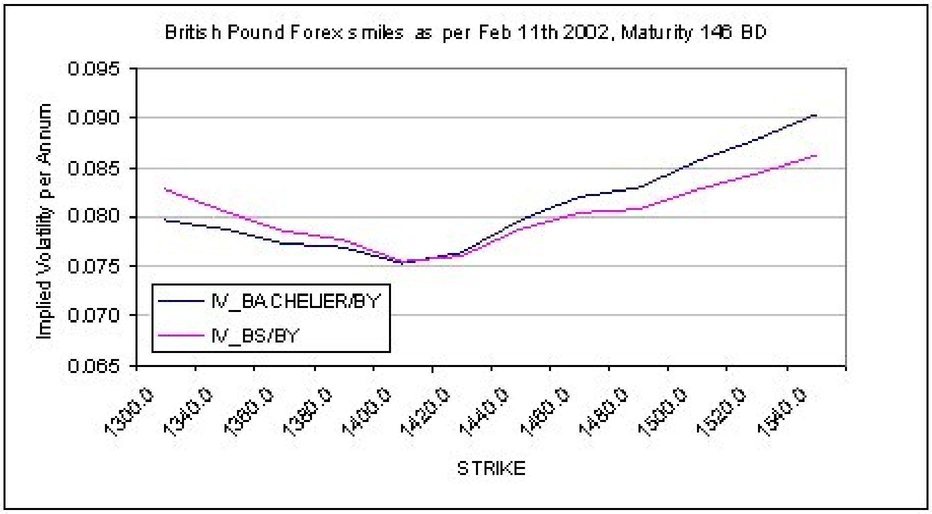

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