In which Financial Markets do Mutual Fund Theorems hold true?

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Abstract

The Mutual Fund Theorem (MFT) is considered in a general semimartingale financial market S with a finite time horizon T, where agents maximize expected utility of terminal wealth. The main results are:

(i) Let N be the wealth process of the numéraire portfolio (i.e. the optimal portfolio for the log utility). If any path-independent option with maturity T written on the numéraire portfolio can be replicated by trading *only* in N and the risk-free asset, then the (MFT) holds true for general utility functions, and the numéraire portfolio may serve as mutual fund. This generalizes Merton's classical result on Black-Merton-Scholes markets as well as the work of Chamberlain in the framework of Brownian filtrations [4].

Conversely, under a supplementary weak completeness assumption, we show that the validity of the (MFT) for general utility functions implies the replicability property for options on the numéraire portfolio described above.

(ii) If for a given class of utility functions (i.e. investors) the (MFT) holds true in all complete Brownian financial markets S, then all investors use the same utility function U, which must be of HARA type. This is a result in the spirit of the classical work by Cass and Stiglitz.

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1 Introduction

The *Mutual Fund Theorem* (MFT), also sometimes called the "two fund theorem" or the "separation theorem", is one of the pearls of Mathematical Finance. Under suitable assumptions (for which we attempt to give a better understanding in the present paper), the optimal investment strategy of a utility maximizing agent has the following simple form:

The agent will only invest in two funds: the risk-free asset as well as a second mutual fund which is a linear combination of the risky assets available on the financial market. The crucial

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feature is that the same (second) mutual fund, i.e., the same linear combination (portfolio) of risky assets applies to all utility maximizing agents, independently of the special form of their utility function as well as their initial endowment.

This theme goes back to the work of Tobin [32], who considered a cash and console bond market described by a mean-variance single-period model as analyzed in the work of Markowitz ([23], [24]). Their original work focused on the analysis of the mean and variance of the asset returns and was restricted to the rather limited framework of single-period markets and quadratic utility functions. Starting from these seminal papers, there were essentially two directions into which the subsequent research can be divided.

On the one hand, an important positive result was obtained by Merton (see [25] and [26]) following a direct approach based on dynamic programming arguments: in the framework of a continuous-time multi-dimensional Black-Merton-Scholes model, with deterministic drift and volatility coefficients, the mutual fund theorem does hold true (see Theorem 2 of [25] and Theorem 1 of [26]). The results of Merton were further developed: we refer here the reader to [20] and Chapter 3 Remark 8.9 of [16] for generalizations within the class of deterministic drift and volatility models, and to [9] (see Theorem 3.10) for mutual fund theorems in the framework of bond markets. Another important positive result was obtained by Chamberlain [4], using an approach based on the *predictable representation property of Brownian motion*: assuming that there exists a martingale measure whose density is adapted to the filtration generated by a (small) number of independent Brownian motions and that all random variables measurable with respect to the density of this martingale measure are marketed claims, the author concludes that all optimal strategies are actually stochastic integrals with respect to the above Brownian motions plus the drifts determined by the specified martingale measure (see [4], Lemma 3). A further discussion along these lines can also be found in Back ([1], Section 14.3).

On the other hand, Cass and Stiglitz [3] (see also [12] and [28]) obtained negative results: for general discrete-time financial markets, the mutual fund theorem only holds true for very restrictive classes of utility functions (see Theorem 3.15 below for a result in this spirit).

In the present paper we want to develop a better understanding of this discrepancy and to obtain necessary and sufficient conditions (on a financial model) for the mutual fund theorem to hold true, thus providing a *unifying view* of the classical work of Merton [25], [26], Chamberlain [4] and Cass and Stiglitz [3]. Our paper is closest in spirit to the work of Chamberlain, being based on *martingale representations*. We identify the martingale measure in [4] for which all random variables measurable with respect to its density can be replicated by trading (if such a measure exists) as being the *dual minimizer* corresponding to logarithmic utility, so its density is the inverse of the terminal value of the numéraire portfolio process. We recall that the numéraire portfolio is the optimal portfolio of a logarithmic utility maximizer, and we refer the reader to [2] and [18] (and the references given there) for more details. In this framework, our main positive results states that if every (bounded) path-independent option on the numéraire portfolio which expires at the final time horizon T can be replicated *by trading only in the numéraire portfolio and the risk-free asset*, then the mutual fund theorem holds true with respect to arbitrary utility functions (see Theorem 3.1 below). In the sequel, we will call "European option" any option which is both path-independent and has fixed maturity. Also, whenever we say "trading in a mutual fund" (e.g. the numéraire portfolio) we actually mean "trading in the mutual fund and the risk-free asset", even if the risk-free asset is not mentioned in an explicit way.

We present an example which shows that a direct converse to the implication described above does not hold true in a general incomplete market (see Example 4.5 below). However, if we assume that all European options on the numéraire portfolio which expire at maturity T can be replicated by trading in the whole market (condition which is similar to the assumption (R_N) in Chamberlain [4], p. 1288, and that we call *weak completeness*), we can prove a reverse result (Theorem 3.6 below). In addition, we obtain as a by-product of our analysis a dual characterization of the weak-completeness condition (see Proposition 3.10 below). This dual characterization relates directly the weak-completeness condition to the properties of *utility-based prices* and their approximations, as described in [22].

As explained in Remark 3.7, our positive results can be extended to accommodate the case when a larger number of mutual funds is needed to describe the optimal behavior of all investors, as in Chamberlain [4]. Compared to the literature, our results generalize the work of Chamberlain and Merton in several directions.

Firstly, our findings not only apply to continuous price processes, but to general semimartingales, allowing for the possibility of jumps. This fact also allows us to get a better insight why in the setting of Cass and Stiglitz [3] things go wrong: there are only few one-dimensional processes with jumps (which play the role of the numéraire portfolio in our case) which satisfy the replicability property needed for the (MFT) to hold true, if we assume weak completeness. Typical examples are the binomial model in discrete time and the (compensated) Poisson process in continuous time, which actually generate one-dimensional complete markets. We also provide a *unifying view* for the work of Merton and Chamberlain, emphasizing that Merton's results are a consequence of the martingale representation approach, thus extending beyond Markov models. In our definition of a mutual fund we insist on the fact that it should be *a trading strategy* (i.e a stochastic integral with respect to the traded assets), which is slightly different from the setting of Chamberlain, where the processes Y appearing in Lemma 3 (which are Brownian motions with drifts) are not necessarily trading strategies. Although there is no simple relation between the assumptions of Theorem 3.1 and Chamberlain's Lemma 3, conceptually, Theorem 3.1 can be regarded as a generalization of Lemma 3 in [4]. On a technical level, we remove a number of assumptions (like boundedness of the martingale measure), needed in the work of Chamberlain even in the Brownian filtration framework. We also provide a result on *necessary conditions*, namely Theorem 3.6. Finally, we raise the important question of *information* needed to invest in a mutual fund. We show that, in general, *all information* available may be needed to take decisions on how to invest in the mutual fund (Proposition 3.14). The basic message is that the validity of the mutual fund theorem is an *informational* issue, pertaining to the fact which kind of derivatives (in terms of their *measurability* properties) can be replicated by only trading in a mutual fund, which usually is the *numéraire portfolio*, and what kind of *information* is needed in order to achieve the replication.

Turning to the negative side, we also provide new insight. Theorem 3.15 shows that the same negative assertions as in Cass and Stiglitz [3], in the framework of discrete-time models, can be proved in the framework of continuous processes in continuous time.

2 The Mathematical Model

We consider a financial market, on a finite time interval $[0, T]$, with one risk-free asset S^0 called the bond (or better, money market account) and d risky assets called stocks. We choose S^0 as numéraire (which means we normalize $S^0 = 1$) and denote by $S^1, \dots, S^d$ the prices of the risky assets measured in units of S^0. The price process of the stocks $S = (S^i)_{1 \le i \le d}$ is assumed to be a *locally bounded* semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_{0 \le t \le T}$ satisfies the *usual conditions* (right continuous and saturated) and $\mathcal{F} = \mathcal{F}_T$.

A portfolio is defined as a pair (x, H), where the constant x represents the initial capital and $H = (H^i)_{1 \le i \le d}$ is a $(\mathcal{F}_t)_{0 \le t \le T}$-predictable and S-integrable process in the vector integration sense. The wealth process $X = (X_t)_{0\leq t\leq T}$ of the portfolio evolves in time as the stochastic integral of H with respect to S:

$$X_t = x + \int_0^t (H_u, dS_u), \quad 0 \leq t \leq T. \tag{2.1}$$

For each $x > 0$ we denote by $\mathcal{X}(x)$ the family of wealth processes $X = (X_t)_{0\leq t\leq T}$ with nonnegative capital at any instant and with initial value equal to x:

$$\mathcal{X}(x) \triangleq \{X \geq 0 : \ X \text{ is defined by (2.1)}\}. \tag{2.2}$$

A probability measure $\mathbf{Q} \sim \mathbf{P}$ is called an *equivalent local martingale measure* if S is a local martingale under $\mathbf{Q}$. We denote by $\mathcal{M}^e$ the set of equivalent local martingale measures and assume, as usually, that there exists an equivalent local martingale measure for S:

Assumption I

$$\mathcal{M}^e \neq \emptyset. \tag{2.3}$$

We would like to point out that all the results in the present paper can be proven without the local boundedness assumption on S, by changing slightly the definition of equivalent martingale measures (the technical details related to the case of unbounded semimartingales are described and analyzed in [8]). We decided to avoid here unnecessary technicalities by assuming that the stock price process S is *locally bounded.*

In this financial model we consider economic agents whose preferences are modeled by *expected (deterministic) utility from terminal wealth.* The cases of *expected utility from consumption* as well as *random utility functions* are left to forthcoming work. A generic (deterministic) utility function $U : (0, \infty) \to \mathbb{R}$ is assumed to be strictly concave, continuously differentiable and strictly increasing. In addition, it satisfies the Inada conditions

$$\lim_{x\searrow 0} U'(x) \to \infty \quad \text{and} \quad \lim_{x\to\infty} U'(x) = 0, \tag{2.4}$$

as well as the *reasonable asymptotic elasticity condition* (see [21]),

$$\limsup_{x\to\infty} xU'(x)/U(x) < 1. \tag{2.5}$$

We resume this in the subsequent statement.

Assumption II *The utility function $U : (0, \infty) \to \mathbb{R}$ is strictly increasing, strictly concave, differentiable on $(0, \infty)$ and formulas (2.4) and (2.5) hold true.*

We now introduce the problem of *optimal investment from terminal wealth* for an economic agent and define the *indirect utility function* u by:

$$u(x) = \sup_{X\in\mathcal{X}(x)} \mathbf{E}[U(X_T)], \ x > 0. \tag{2.6}$$

Assumption III *We assume that $u(x) < \infty$, for some $x > 0$ (or, equivalently, for all $x > 0$).*

Remark 2.1 The problem of optimal investment from terminal wealth (2.6) can be studied for different classes of utility functions. For example, one can consider utility functions $U : (a, \infty) \to \mathbb{R}$, for some finite a. In this case, the Inada conditions have the form

$$\lim_{x\searrow a} U'(x) \to \infty \quad \text{and} \quad \lim_{x\to\infty} U'(x) = 0, \tag{2.7}$$

and the optimization problem can be regarded as a translation with respect to x of an optimization problem for $a = 0$. All the results related to the optimization problem (2.6) and our results below on Mutual Fund Theorems hold true modulo some obvious modifications: for example, in Proposition 3.15 the utilities would be translations in x of affine transformations of power utilities, etc.

On the other hand, the case of utilities which are finite-valued on the whole real line, $U : \mathbb{R} \to \mathbb{R}$, such as the exponential utility $U(x) = -e^{-x}$, requires a slightly different duality approach, described in detail in [30]. For the sake of presentation, we decided to not include this case here: however, we want to point out that our results carry over to this setting too.

We denote by $\hat{X}(x, U)$ the optimal wealth process in (2.6), which exists and is unique under Assumptions I, II and III (see [21]).

A particularly important utility function U is the *logarithmic utility*

$$U(x) = \ln(x), \quad x > 0. \tag{2.8}$$

In this case, Assumption III takes the form

Assumption IV *Assume that*

$$\sup_{X \in \mathcal{X}(1)} \mathbf{E}[\ln(X_T)] < \infty. \tag{2.9}$$

Assumption IV, which is an assumption on the model, is equivalent to the dual condition

$$\inf_{\mathbf{Q} \in \mathcal{M}^e} \mathbf{E}[-\ln(\frac{d\mathbf{Q}}{d\mathbf{P}})] < \infty. \tag{2.10}$$

The equivalence between Assumption IV and (2.10) follows from the duality arguments in [21]. Some of these duality arguments are briefly described in Section 4 below, since they are needed to proof our own results.

Assuming that Assumption IV is satisfied, we denote by N the optimal wealth process for the logarithmic maximizer with initial endowment 1, i.e. $N = \hat{X}(1, \ln)$. The wealth process N is often called in the literature the *numéraire portfolio process* (we refer the reader to [2] and [18] for a thorough analysis of the properties of N). The numéraire portfolio (which can even be defined in the absence of either the local boundedness assumption on S or Assumption IV) can be characterized as the unique numéraire that makes any positive wealth process (measured in terms of this numéraire) a supermartingale under the original probability measure $\mathbf{P}$. In particular, the constant wealth process $X_t = 1$ for $0 \leq t \leq T$, measured in terms of the numéraire N, is a supermartingale, which means that the process Z defined by

$$Z_t = \frac{1}{N_t}, \quad 0 \leq t \leq T,$$

is a supermartingale under $\mathbf{P}$.

Assumption V *The process Z is a martingale, i.e*

$$\mathbf{E}[Z_T] = 1.$$

If Assumption V is satisfied, we can define the probability measure $\mathbf{Q}^{(m)} \sim \mathbf{P}$ by

$$\frac{d\mathbf{Q}^{(m)}}{d\mathbf{P}} = Z_T.$$

Under our standing assumption that S is locally bounded, the measure $\mathbf{Q}^{(m)}$ is an *equivalent martingale measure*, i.e $\mathbf{Q}^{(m)} \in \mathcal{M}^e$. In particular, each *bounded* wealth process is a *martingale* under $\mathbf{Q}^{(m)}$. For the case of continuous stock price process S, the measure $\mathbf{Q}^{(m)}$ was introduced in [11], Definition 3.2 under the name of *minimal martingale measure*.

Remark 2.2 Our semimartingale market model includes the particular case of models driven by Brownian Motions. We describe this briefly: suppose that the price process $(S_t)_{0\le t\le T} = (S_t^1, \dots, S_t^d)_{0\le t\le T}$ satisfies the stochastic differential equation (SDE)

$$\frac{dS_t^i}{S_t^i} = \sum_{1\le j\le N} \sigma_t^{i,j}(\gamma_t^j\, dt + dW_t^j), \quad 1 \le i \le d \tag{2.11}$$

where $W = (W^1, \dots, W^N)$ is a standard N-dimensional Brownian motion with respect to the filtered probability space $(\Omega, \mathbf{P}, \mathcal{F}_T, (\mathcal{F}_t)_{0\le t\le T})$, where $(\mathcal{F}_t)_{0\le t\le T}$ is a right continuous and saturated filtration. The $d \times N$-matrix-valued volatility process σ with matrix elements σ_t^{ij} at time $0 \le t \le T$ and the N-dimensional market price of risk process $\gamma = (\gamma_t)_{0\le t\le T}$, where $\gamma_t = (\gamma_t^1, \dots, \gamma_t^N)$, are assumed to be optional with respect to $(\mathcal{F}_t)_{0\le t\le T}$ and locally square integrable. In general, the market price of risk γ_t is not uniquely determined by the process S via (2.11). Indeed, two processes $(\gamma_t)_{0\le t\le T}$, and $(\tilde{\gamma}_t)_{0\le t\le T}$ such that $\gamma_t - \tilde{\gamma}_t \in \ker(\sigma_t)$ a.s, for all $0 \le t \le T$ define the same process S via (2.11). There is a unique γ such that γ_t is a.s. orthogonal to the kernel of σ_t, for all $0 \le t \le T$. We denote it by $\gamma^{(m)}$ and call it *Minimal Market Price of Risk* (MMPR). The numéraire portfolio process N and its inverse process Z can be computed in this case in terms of the minimal market price of risk:

$$dN_t = N_t \left((\gamma_t^{(m)}, dW_t) + \tfrac{1}{2}\|\gamma_t^{(m)}\|^2 dt \right), \ 0 \le t \le T, \ N_0 = 1, \tag{2.12}$$

and

$$Z_t = \frac{1}{N_t} = \exp\left(-\int_0^t (\gamma_s^{(m)}, dW_s) - \frac{1}{2}\int_0^t \|\gamma_s^{(m)}\|^2 ds \right), \quad 0 \le t \le T. \tag{2.13}$$

Going back to general semimartingale models, we now can formalize the idea of the Mutual Fund Theorem. In order to do this we need two definitions:

Definition 2.3 *We call a mutual fund for the market S any positive wealth process M with initial capital equal to one, i.e any $M \in \mathcal{X}(1)$.*

Definition 2.4 *Let the financial market S satisfy Assumption I and let $\mathcal{U}$ be a family of utility functions satisfying Assumptions II, III. We say that the financial market S satisfies the Mutual Fund Theorem (MFT) with respect to $\mathcal{U}$ if there exists a mutual fund M such that for each $U \in \mathcal{U}$ and $x > 0$ there exists a a real-valued $(\mathcal{F}_t)_{0\le t\le T}$ predictable and M-integrable process $k = k(x, U)$ such that*

$$\hat{X}_t(x, U) = x + \int_0^t k_u dM_u, \ 0 \le t \le T. \tag{2.14}$$

The process M is then called a mutual fund for the market S and the class of utility functions $\mathcal{U}$.

The interpretation of (2.14) is that the optimal investment strategy is to invest in the mutual fund M and the money market only. In the case when the logarithmic utility function $U(x) = \ln x$ is in $\mathcal{U}$ we will show that one may choose the *numéraire portfolio* as mutual fund, i.e. $M = N$, under a mild technical assumption (according to Remark 3.12 below).

3 Results

In what follows, for any stochastic process $X = (X_t)_{0\le t\le T}$, we will denote by $(\mathcal{F}_t^X)_{0\le t\le T}$ the right-continuous and saturated filtration generated by the process X, and by $\mathcal{G}_t^X = \sigma(X_t)$

the sigma-algebra generated by the random variable X_t for a fixed $0 \le t \le T$. Therefore, the space $L^\infty(\Omega, \mathcal{G}_T^N, \mathbf{P})$ or, briefly, $L^\infty(\mathcal{G}_T^N)$ can be interpreted as the collection of all bounded European options on the numéraire portfolio N expiring at time T:

$$L^\infty(\mathcal{G}_T^N) = \{h(N_T), \text{ where } h : \mathbb{R} \to \mathbb{R} \text{ is a bounded Borel measurable function}\}.$$

We now define $\mathcal{R}(S)$ to be the set of all bounded random variables that are replicable by trading in the market S, i.e. the set of bounded random variables f for which there exists a number $p(f)$ and d-dimensional $(\mathcal{F}_t)_{0\le t\le T}$ -predictable and S-integrable process H^f such that

$$f = p(f) + \int_0^T (H_u^f, dS_u),$$

and the stochastic integral is a bounded process. The process $P(f) = (P_t(f))_{0\le t\le T}$ defined by

$$P_t(f) = p(f) + \int_0^t (H_u^f, dS_u), \quad 0 \le t \le T$$

is called the price process of f, and $p(f)$ the initial price of f. For a fixed mutual fund M (see Definition 2.3) we also define $\mathcal{R}(M)$ to be the set of bounded random variables which can be replicated by trading in this mutual fund (and the risk-free asset) only, i.e.

$$f = p(f) + \int_0^T k_u^f dM_u, \tag{3.1}$$

where k^f is a one-dimensional $(\mathcal{F}_t)_{0\le t\le T}$ -predictable process, which is M-integrable, and such that the stochastic integral is bounded. The set of random variables $f \in \mathcal{R}(M)$ where the integrand k^f in (3.1) can be chosen to be $(\mathcal{F}_t^M)_{0\le t\le T}$-predictable is denoted by $\mathcal{R}(M, \mathcal{F}^M)$. It is obvious that

$$\mathcal{R}(M, \mathcal{F}^M) \subset \mathcal{R}(M) \subset \mathcal{R}(S).$$

Finally, for a fixed model S satisfying Assumption I, we denote by $\mathcal{A}(S)$ the set of all utility functions satisfying Assumptions II and III. We emphasize that the class $\mathcal{A}(S)$ depends on the model S, due to Assumption III. We can now state our first main result, which provides *sufficient conditions* for the validity of the mutual fund theorem:

Theorem 3.1 *Let the semimartingale financial market S satisfy Assumptions I , IV and V. If there is a mutual fund M such that each bounded European option f with maturity T and written on the numéraire portfolio N can be replicated by trading only in the mutual fund M, then the financial model S satisfies the Mutual Fund Theorem with respect to the class $\mathcal{A}(S)$ and M is the mutual fund for the market S and the class of utility functions $\mathcal{A}(S)$.*

Speaking more formally, if the one-dimensional replicability condition

(R) *there exists a mutual fund M such that $L^\infty(\mathcal{G}_T^N) \subset \mathcal{R}(M)$,*

is satisfied then then S satisfies the (MFT) with respect to the class $\mathcal{A}(S)$ of utility functions, i.e.,

$$(\mathbf{R}) \Longrightarrow (\mathbf{MFT}).$$

As already mentioned in the introduction,Theorem 3.1 is similar in spirit to Lemma 3 in Chamberlain [4]. It has the following obvious consequence which implies assertion (i) of the abstract.

Corollary 3.2 *Let the semimartingale financial market S satisfy Assumptions I , IV and V. Suppose that each bounded European option f with maturity T and written on the numéraire portfolio N can be replicated by trading only in N, i.e.*

$$(\mathbf{R^N}) \qquad\qquad L^\infty(\mathcal{G}_T^N) \subset \mathcal{R}(N).$$

Then, the financial model S satisfies the (MFT) with respect to the set of all utility functions $\mathcal{A}(S)$, and the numéraire portfolio N can be chosen as Mutual Fund, i.e.,

$$(\mathbf{R^N}) \Longrightarrow (\mathbf{MFT}).$$

Specializing one step further we can identify a well known concept in finance, namely the completeness of N with respect to its own filtration, as a sufficient condition for (MFT):

Corollary 3.3 *Let the financial market S satisfy Assumptions I, IV and V. If the numéraire portfolio process N defines a complete market with respect to its own filtration $(\mathcal{F}_t^N)_{0\le t\le T}$, then the financial model S satisfies the (MFT) with respect to the set of all utility functions $\mathcal{A}(S)$, and the numéraire portfolio N can be chosen as Mutual Fund.*

Remark 3.4 Let us analyze the economic message of Corollary 3.3 in the context of a d-dimensional model driven by an N-dimensional Brownian Motion, as described in Remark 2.2. If S is of the form (2.11), where the volatility matrix σ_t and the drift vector $\mu_t = \sigma_t\gamma_t$ are constant (or just deterministic), then the minimal market price of risk $\gamma^{(m)}$ is going to be a constant (or deterministic) vector-valued process. According to (2.12), the numéraire portfolio process N is a geometric Brownian Motion with volatility $\|\gamma^{(m)}\|$, so it defines a complete market with respect to its natural filtration. Corollary 3.3 applies, so (MFT) holds true with respect to the set of all utility functions $\mathcal{A}(S)$. This is the classical result of Merton (see [25], [26]), which, in its original proof, exploits the Markov structure of the market via the Bellman equation.

A closer look at the formula (2.12), reveals that the numéraire portfolio process N depends *only* on the *minimal market price of risk* process $\gamma^{(m)}$ and *not* on the volatility matrix σ. Therefore, we can apply Corollary 3.3 to the case when the minimal market price of risk is a constant (or deterministic) vector-valued process, but the volatility matrix is *stochastic*. In other words, through the use of duality, we exploit the knowledge of the *law* of the numéraire portfolio process N despite the fact the market does *not necessarily* have a Markov structure (so the original arguments based on the Bellman equation cannot be applied). In fact, our generalization goes even further: if the minimal market price of risk process $(\gamma_t^{(m)})_{0\le t\le T}$ does not vanish $\lambda \otimes \mathbf{P}$-a.e., we can define the one-dimensional Brownian Motion $W^{(m)}$ by

$$dW^{(m)} = \frac{1}{\|\gamma_t^{(m)}\|}(\gamma_t^{(m)}, dW_t), \;\; 0 \le t \le T,$$

and rewrite (2.12) as

$$\frac{dN_t}{N_t} = \|\gamma_t^{(m)}\|^2 dt + \|\gamma_t^{(m)}\| dW_t^{(m)}.$$

If the process $\|\gamma^{(m)}\|$ is adapted to the filtration generated by the Brownian Motion $W^{(m)}$ then Corollary 3.3 applies again. In particular, this happens when the minimal market price of risk is a stochastic vector, but its Euclidean norm is *deterministic*.

If we depart from Corollary 3.3, we can apply Corollary 3.2 to situations when all European options on the numéraire portfolio expiring at time T can be replicated by trading in N (because of the very special form on N_T), but, however, the numéraire portfolio does not generate a complete market (with respect to either its natural filtration or a larger filtration). This situation is described in Example 4.7.

We would also like to emphasize here that *no completeness assumption* for the whole market is made in either Corollary 3.2, 3.3 or Theorem 3.1. As a matter of fact, some form of completeness (the condition (**WC**) below) is a part of the conclusion of these results.

There is no direct converse to Theorem 3.1, i.e. the replicability condition (**R**) is not necessary for the mutual fund theorem to hold true. Indeed, if the financial model S is one-dimensional then the mutual fund theorem trivially holds true (the only stock may serve as mutual fund) while the replicability condition (**R**) may fail (this rather obvious fact is illustrated in Example 4.5). Therefore, in order to get an implication in the opposite direction, we need to impose some extra assumption on the model. The assumption is actually a weak form of completeness, requiring that any European option on the numéraire portfolio N can be replicated by trading in all stocks and the risk-free asset; formally it can be expressed as:

$$(\mathbf{WC}) \qquad\qquad L^\infty(\mathcal{G}_T^N) \subset \mathcal{R}(S).$$

Remark 3.5 Taking into account the definition of the replicability condition (**R**), it is obvious that (**R**) $\Longrightarrow$ (**WC**). Combining this trivial implication with Theorem 3.1 we conclude that, if the financial market S satisfies Assumptions I , IV and V, then we have the implication:

$$(\mathbf{R}) \Longrightarrow (\mathbf{MFT}) + (\mathbf{WC}).$$

We can now formulate a precise converse to Theorems 3.1:

Theorem 3.6 *Let the financial market S satisfy Assumptions I , IV and V as well as the weak completeness condition* (**WC**). *If S satisfies the (MFT) with respect to the class $\mathcal{A}(S)$ of all utility functions satisfying Assumptions II and III then the replicability condition* (**R**) *holds true. Formally:*

$$(\mathbf{MFT}) + (\mathbf{WC}) \Longrightarrow (\mathbf{R}).$$

In view of Remark 3.5, the message of Theorems 3.1 and 3.6 together can be be formally summarized by the following “if and only if” result:

$$(\mathbf{R}) \Longleftrightarrow (\mathbf{MFT}) + (\mathbf{WC}).$$

Remark 3.7 Theorems 3.1 and 3.6 can be easily generalized to take into account ”multi-fund separation theorems”. In order to do this, we just have to consider the process M in Definition 2.3 to be K-dimensional: $M = (M^1, \ldots, M^K)$, where M^i, $i = 1, \ldots, K$ are mutual funds and $K \leq d$ (usually much smaller), and see that all proofs follow identically. We have decided to consider the ”one-fund separation” case since, in practice, the main application of Theorem 3.1 is Corollary 3.3.

Remark 3.8 Assumption IV can be removed from the formulation of our main results, Theorem 3.1 and 3.6, Corollaries 3.2 and 3.3 and even from Theorem 3.10 below. However, without this assumption, the numéraire portfolio N (which can still be defined, as already pointed out) does not have the important economic meaning of the wealth process corresponding to the logarithmic maximizer, as the expected utility of the logarithmic maximizer then equals infinity.

Remark 3.9 The hypothesis (**WC**) of Theorem 3.6 is certainly satisfied if $S = (S_t)_{0 \leq t \leq T}$ is a complete model with respect to the original filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$. Theorem 3.6 can also be applied in some incomplete markets using the idea of completion by “fictitious securities” described in [19]. For example, in the Brownian framework of Remark 2.2, if the filtration is generated by the Brownian motion W, we can add a finite number of securities to create a complete market that admits as the unique martingale measure the measure $\mathbf{Q}^{(m)}$. The enlarged model is complete, so we can apply Theorem 3.6 to the new model. If no utility maximizer (for some $U \in \mathcal{A}(S)$) chooses to invest in the "fictitious securities", then Theorem 3.6 applies to the original incomplete model. As a matter of fact, the possibility of completing the market in such a peculiar way is characterized in Theorem 3.10 below as being equivalent to (**WC**), and means that the dual optimizer is the same for all utility maximizing agents, as described in Remark 4.1.

In this spirit we give a dual characterization of the weak completeness condition (**WC**). We recall that for two positive random variable ξ and ζ we say that ξ dominates ζ stochastically in the second order and write $\xi \succeq_2 \zeta$ if

$$\int_0^t \mathbf{P}[\xi \geq u]du \geq \int_0^t \mathbf{P}[\zeta \geq u]du \text{ for each } t \geq 0.$$

Proposition 3.10 *Assume that the financial model S satisfies Assumptions I, IV and V. Then the following statements are equivalent:*

(i) the weak completeness condition (**WC**) *holds true*

(ii) for each $\mathbf{Q} \in \mathcal{M}^e$ we have

$$\frac{d\mathbf{Q}^{(m)}}{d\mathbf{P}} = \mathbf{E}[\frac{d\mathbf{Q}}{d\mathbf{P}}|\mathcal{G}_T^N] \tag{3.2}$$

(iii) for each $\mathbf{Q} \in \mathcal{M}^e$ we have

$$\frac{d\mathbf{Q}^{(m)}}{d\mathbf{P}} \succeq_2 \frac{d\mathbf{Q}}{d\mathbf{P}}. \tag{3.3}$$

Remark 3.11 Instead of the optimal investment problem (2.6) one can consider the more general problem of *optimal investment with random endowment*:

$$u(x, q; f) = \sup_{X \in \mathcal{X}^b(x)} \mathbf{E}[U(X_T + qf)], \tag{3.4}$$

where f is a bounded random variable on $(\Omega, \mathbf{P}, \mathcal{F})$ having the economic meaning of a contingent claim (option), $q \in \mathbb{R}$ represents the number of options and $\mathcal{X}^b(x)$ is the set of *bounded* wealth processes starting at x. We refer the reader to [5] and [14] for a detailed treatment of the duality theory related to this optimization problem. For fixed (x, q) such that $-\infty < u(x, q; f) < \infty$ one can define the *utility-based price* $p(x, q; f)$ of the claim f as any price p such that

$$u(x + \tilde{q}p, q - \tilde{q}; f) \leq u(x, q; f) \text{ for any } \tilde{q} \in \mathbb{R}.$$

In other words, $p(x, q)$ is a price set in such a way that, if the investor having x initial capital and q contingent claims is allowed to trade at time zero the claims for the price $p(x, q; f)$, the optimal strategy is to keep the q claims and trade only in stocks. The price $p(x, 0; f)$, analyzed extensively in the literature (for example in [6], [10], [13], [29]) and sometimes called Davis price, represents the partial equilibrium price of the claim corresponding to zero demand. Using refined duality arguments, it is proved in [15] that the price $p(x, 0; f)$ is uniquely defined and it can be computed as the expectation of the claim f under the *dual minimizer measure* in case that such minimizer actually exists. In other words

$$p(x, 0; f) = \mathbf{E}_{\mathbf{Q}(x,U)}[f],$$

where $\mathbf{Q}(x,U)$ (the *pricing measure* of the agent with utility U, initial capital x and zero demand for the claims) is the minimizer in (4.5) below, in case there is a minimizer. Taking the above discussion into account, the completeness condition (**WC**) is equivalent to each of the following conditions, which we formulate in economic terms (without elaborating on the appropriate technical assumptions):

- all economic agents, independent of their utility function U and initial capital x have the same pricing measure:

$$\mathbf{Q}(x,U) = \mathbf{Q}^{(m)} \text{ for all } U \text{ and } x$$

- all economic agents, independent of their utility function U and initial capital x assign the same utility-based prices to infinitesimally small quantities of contingent claims i.e. $p(x,0;f)$ does not depend on U and x, for each $f \in L^\infty(\Omega,\mathcal{F},\mathbf{P})$.

In addition, it was shown in [22] that the *second order stochastic domination* condition (3.3) is equivalent to the validity of some important qualitative properties of first order approximations of utility based prices

$$p(x,q;f) = p(x,0;f) + q\frac{\partial p}{\partial q}|_{q=0} + o(q),$$

so the completeness condition (**WC**) is directly related not only to the zeroth order behavior of utility-based prices (i.e. the Davis price $p(x,0;f)$) but also to the first order behavior, which relates to the *nonlinearity* of the pricing rule derived by the classical principle of *marginal rate of substitution.*

Remark 3.12 Taking into account the obvious implication $(\mathbf{R^N}) \Longrightarrow (\mathbf{R})$ (which is actually the proof of Corollary 3.2), following Remark 3.5 we have $(\mathbf{R^N}) \Longrightarrow (\mathbf{MFT}) + (\mathbf{WC})$. A natural question is whether the reverse implication, namely $(\mathbf{MFT}) + (\mathbf{WC}) \Longrightarrow (\mathbf{R^N})$, holds true. This amount to asking whether Theorem 3.6 can be reformulated *only* in terms of the numéraire portfolio N, without any need of a (possibly different) mutual fund M. The answer is no, in general. The rather simple Example 4.4 below shows that we can have a *complete* market, where the mutual fund theorem holds true, but the numéraire portfolio cannot be chosen to be the mutual fund. However, under a mild technical assumption, we can show that the numéraire portfolio *can be* chosen as mutual fund. In the Brownian framework of Remark 2.2, the technical hypothesis can be formulated as "the minimal market price of risk never vanishes", i.e.

$$\gamma_t^{(m)} \neq 0 \text{ whenever } \sigma_t \neq 0 \text{ for } \lambda \otimes \mathbf{P} \text{ a.e. } (t,\omega) \in [0,T] \times \Omega. \tag{3.5}$$

In the general semimartingale framework, a similar condition, informally written as "$dN_t \neq 0$ whenever $dS_t \neq 0$", has to be imposed. The rigorous version of such a condition involves technical details on semimartingales (see, for example [17]) which we choose not to give here. Instead, we formulate our result only in the Setting of Remark 2.2:

Proposition 3.13 *Consider a Brownian Financial market S as in* (2.11)*, such that Assumptions I , IV and V are satisfied. Assume, in addition, that the weak-completeness condition* (**WC**) *and condition* (3.5) *are satisfied. If the financial model S satisfies the Mutual Fund Theorem with respect to the set of all utility functions $\mathcal{A}(S)$, then condition* $(\mathbf{R^N})$ *holds true, i.e., formally:*

$$(\mathbf{MFT}) + (\mathbf{WC}) + (3.5) \Longrightarrow (\mathbf{R^N}).$$

A second natural question, related to *information* and motivated by Corollary 3.3, is the following: is the condition $L^\infty(\mathcal{G}_T^N) \subset \mathcal{R}(N)$ equivalent to the stronger one

$$L^\infty(\mathcal{G}_T^N) \subset \mathcal{R}(N, \mathcal{F}^N)?$$

In other words, if any bounded European option on N can be hedged by trading in N only, but using the full information $\mathcal{F}$, can we do this by just using the information $\mathcal{F}^N$? In terms of mutual funds, the question amounts to: if any rational investor will only invest in the mutual fund corresponding to the logarithmic maximizer, can she do so only observing the evolution of this mutual fund and having no other information about the market? The answer is negative in general, and will be summarized in the following result:

Proposition 3.14 *There exists a complete financial market S generated by a two-dimensional Brownian Motion W as in* (2.11) *and satisfying Assumptions I, IV and V such that*

(i) the original filtration $(\mathcal{F}_t)_{0\le t\le T}$ is the filtration generated by the driving Brownian Motion $(\mathcal{F}_t^W)_{0\le t\le T}$, and, in addition, it equals the filtration generated by the stock price process S, i.e.

$$(\mathcal{F}_t)_{0\le t\le T} = (\mathcal{F}_t^W)_{0\le t\le T} = (\mathcal{F}_t^S)_{0\le t\le T}$$

(ii) any bounded path-dependent option on the numéraire portfolio N can be replicated by only trading in N and observing the "full" information $(\mathcal{F}_t)_{0\le t\le T}$, i.e.

$$L^\infty(\mathcal{F}_T^N) \subseteq \mathcal{R}(N), \text{ hence, in particular } L^\infty(\mathcal{G}_T^N) \subseteq \mathcal{R}(N)$$

(iii) there exists an European option on the numéraire portfolio N with maturity T that cannot be replicated by trading in N and using only the information obtained by observing the process N. In other words

$$L^\infty(\mathcal{G}_T^N) \nsubseteq \mathcal{R}(N, \mathcal{F}^N).$$

Also on the negative side, we show that, if we allow for general Brownian market models as described in Remark 2.2, we cannot hope for a reasonable Mutual Fund Theorem to hold true. In fact, Proposition 3.15 below gives an analogous result to the theorem which Cass and Stiglitz [3] obtained for processes in discrete time.

Let us first recall that the optimization problem (2.6) for either $U(x) = \ln x$ or $U(x) = \frac{x^\alpha}{\alpha}$, where $\alpha \in]-\infty, 1[\,\setminus\{0\}$ has an obvious scaling property, namely

$$\hat{X}(x, U) = x\hat{X}(1, U), \ x > 0, \tag{3.6}$$

provided that the utility function U satisfies Assumption III . As a matter of fact, the only C^2 utility function, for which the scaling property (3.6) holds true, are those obtained by the transformation

$$U \to A + BU, \tag{3.7}$$

where $A \in \mathbb{R}$, $B > 0$ and U is either the logarithmic or power utility.

In order to state the next result, which is related to the seminal papers [3] and [12], we need a technical strengthening of the regularity conditions on the utility function U:

Assumption VI *The utility function U is C^3 on its domain $(0, \infty)$.*

Theorem 3.15 *Let $\mathcal{U}$ be a class of utility functions satisfying Assumptions II and VI. Assume that every complete Brownian financial market S (as described in Remark 2.2) with values in $\mathbb{R}^2$, satisfying Assumption I and also Assumption III with respect to all utilities $U \in \mathcal{U}$, satisfies (MFT) with respect to the class $\mathcal{U}$.*

Then the family $\mathcal{U}$ consists only of a single utility function U (modulo affine transformations of the type (3.7)) which is either

(i) $U(x) = \log(x),\ x > 0$ *or*

(ii) $U(x) = \frac{x^\alpha}{\alpha},\ x > 0,$ *for some* $\alpha \in\,]-\infty, 1[\ \setminus\{0\}$.

Remark 3.16 The above Theorem shows that a class of investors would all invest in the same mutual fund, independent of their initial wealth, and in *any* complete Brownian two-dimensional financial model, if and only if they all have the *same constant relative risk-aversion coefficient* (in the case when preferences of investors are described by utility functions which are finite for $x > 0$). If $\mathcal{U} = \{U\}$ (singleton), where the utility function U is of logarithmic or power type, then by the scaling argument (3.6) the mutual fund theorem trivially holds true for any model S. The message of Theorem 3.15 is that this trivial case is the only possible one, once we allow for general financial models of the form described in Theorem 3.15.

A similar result to Theorem 3.15 holds true if we consider utilities $U : (a, \infty) \to \mathbb{R}$, or even $U : \mathbb{R} \to \mathbb{R}$. In the former case, the conclusion of Theorem 3.15 is just a translation on the x-axis, and in the latter, the class $\mathcal{U}$ is generated by affine transformations of

$$U(x) = -\exp(-\alpha x),\ x \in \mathbb{R},$$

for some $\alpha > 0$. The proofs require slightly different technical details, as pointed out in Remark 2.1.

4 Proofs and examples

A well known tool in studying the optimization problem (2.6) is the use of duality relations in the spaces of convex functions and semimartingales. Following [21], for a fixed utility function U, we define the dual optimization problem to (2.6) by

$$v(y) = \inf_{Y \in \mathcal{Y}(y)} \mathbf{E}[V(Y_T)], \quad y > 0. \tag{4.1}$$

Here V is the convex conjugate function to U, that is

$$V(y) = \sup_{x>0} \{U(x) - xy\}, \quad y > 0,$$

and $\mathcal{Y}(y)$ is the family of nonnegative supermartingales Y that are dual to $\mathcal{X}(1)$ in the following sense

$$\mathcal{Y}(y) = \{Y \geq 0 : Y_0 = y \text{ and } XY \text{ is a supermartingale for all } X \in \mathcal{X}(1)\}. \tag{4.2}$$

Note that the set $\mathcal{Y}(1)$ contains the density processes of all $\mathbf{Q} \in \mathcal{M}^e$ and $Z \in \mathcal{Y}(1)$.

The optimization problems (2.6) and (4.1) are well studied. For example, it was shown in [21] that under Assumption I on the model and Assumptions II, III on the utility function U, the value functions u and v in (2.6) and (4.1) are continuously differentiable on $(0, \infty)$ and they are conjugate

$$v(y) = \sup_{x>0}\{u(x) - xy\}, \quad y > 0. \tag{4.3}$$

In addition, there exist unique optimizers $\hat{X}(x,U)$ and $\hat{Y}(y,V)$ to (2.6) and (4.1) for all $x>0$ and $y>0$. If $y=u'(x)$ then

$$\hat{X}_T(x,U) = -V'(\hat{Y}_T(y,V)) \tag{4.4}$$

and the product $\hat{X}(x,U)\hat{Y}(y,V)$ is a martingale, not only a supermartingale. The value function in (4.1) can also be represented as a supremum over the smaller set of densities of equivalent martingale measures

$$v(y) = \inf_{\mathbf{Q}\in\mathcal{M}^e} \mathbf{E}[V(y\frac{d\mathbf{Q}}{d\mathbf{P}})]. \tag{4.5}$$

Before we prove our main results, we prove Proposition 3.10:

Proof of Proposition 3.10. (i) $\Longrightarrow$ (ii) If $f\in\mathcal{R}(S)$ then $f=p(f)+(H\cdot S)_T$ where $H\cdot S$ is a bounded martingale under $\mathbf{Q}^{(m)}$ and under each measure $\mathbf{Q}\in\mathcal{M}^e$. Hence,

$$\mathbf{E}_{\mathbf{Q}}[f] = \mathbf{E}_{\mathbf{Q}^{(m)}}[f] = p(f) \text{ for each } \mathbf{Q}\in\mathcal{M}^e.$$

In particular, if $L^\infty(\mathcal{G}_T^N)\subset\mathcal{R}(S)$ then for each $f\in L^\infty(\mathcal{G}_T^N)$ we have $\mathbf{E}_{\mathbf{Q}}[f]=\mathbf{E}_{Q^{(m)}}[f]$ or $\mathbf{E}[\frac{d\mathbf{Q}}{d\mathbf{P}}f]=\mathbf{E}[\frac{d\mathbf{Q}^{(m)}}{d\mathbf{P}}f]$. Taking into account that $\frac{d\mathbf{Q}^{(m)}}{d\mathbf{P}}$ is measurable with respect to $\mathcal{G}_T^N$, we obtain

$$\mathbf{E}[\frac{d\mathbf{Q}}{d\mathbf{P}}|\mathcal{G}_T^N] = \frac{d\mathbf{Q}^{(m)}}{d\mathbf{P}}.$$

(ii) $\Longrightarrow$ (iii) is a well known property of second order stochastic domination, based on Jensen's inequality. We include the short argument here for the sake of completeness. For any martingale measure $\mathbf{Q}\in\mathcal{M}^e$ we have

$$\mathbf{E}[\psi(\frac{d\mathbf{Q}^{(m)}}{d\mathbf{P}})] = \mathbf{E}\left[\psi(\mathbf{E}[\frac{d\mathbf{Q}}{d\mathbf{P}}|\mathcal{G}_T^N])\right] \le \mathbf{E}\left[\mathbf{E}[\psi(\frac{d\mathbf{Q}}{d\mathbf{P}})|\mathcal{G}_T^N]\right] = E[\psi(\frac{d\mathbf{Q}}{d\mathbf{P}})], \tag{4.6}$$

for any convex function ψ such that all the expectations above are well defined. This is equivalent to the second order stochastic domination relation (3.3).

(iii) $\Longrightarrow$ (i): As just pointed out, the second order stochastic domination relation (3.3) is equivalent to inequality (4.6) between the first and the last term, inequality that holds for all convex "test" functions ψ. If we consider the "test" function $\psi(\cdot)=V(y\cdot)$ for any $y>0$, where V is the dual conjugate of an utility U we obtain that

$$\mathbf{E}[V(y\frac{d\mathbf{Q}^{(m)}}{d\mathbf{P}})] \le \mathbf{E}[V(y\frac{d\mathbf{Q}}{d\mathbf{P}})], \text{ for each } \mathbf{Q}\in\mathcal{M}^e,$$

which means that the infimum in (4.5) is attained by the martingale measure $\mathbf{Q}^{(m)}$. Comparing the optimization problems (4.1) and (4.5), it is clear that *once* the infimum in (4.5) is *actually attained*, it also has to be the minimizer in the optimization problem (4.1). Therefore, the minimizer $\hat{Y}(y,V)$ is (up to the multiplicative constant y) equal to the density of the minimal martingale measure:

$$\hat{Y}_t(y,V) = yE[\frac{d\mathbf{Q}^{(m)}}{d\mathbf{P}}|\mathcal{F}_t] = yZ_t, \quad 0\le t\le T. \tag{4.7}$$

Let $h:(0,\infty)\to(0,\infty)$ be a continuous and strictly increasing function such that

$$\lim_{y\to 0} h(y)=0, \ \lim_{y\to\infty} h(y)=\infty \text{ and } \int_0^\infty h(y^{-1})dy<\infty. \tag{4.8}$$

We consider $V : (0,\infty) \to \mathbb{R}$, defined by $V(y) = -\int_0^y h(u^{-1})du$. The function V is the conjugate of a *bounded* convex utility function U, so that Assumption III is satisfied. Taking $y = 1$ in the optimization problem (4.1), and taking into account (4.4) together with (4.7) we conclude that the random variable $h(N_T) = h((\frac{d\mathbf{Q}^{(m)}}{d\mathbf{P}})^{-1}) = -V'(\frac{d\mathbf{Q}^{(m)}}{d\mathbf{P}})$ is the optimal terminal wealth for the investor having utility U and initial wealth $x = -v'(1)$, i.e. $h(N_T) = X_T(-v'(1), U)$. In addition, the wealth process $X(-v'(1), U)$ is a martingale under $\mathbf{Q}^{(m)}$.

Let now $g : (0,\infty) \to \mathbb{R}$ be a smooth function with compact support in $(0,\infty)$. We can represent $g = g_1 - g_2$ as the difference between two nondecreasing functions $g_1, g_2 : (0,\infty) \to \mathbb{R}$ which are both constant outside the support of g (in particular, they vanish on the left side of the support). If we consider a function h^* strictly increasing and satisfying (4.8), we observe that the functions $h_i = g_i + h^*$, for $i = 1, 2$, are strictly increasing and also satisfy (4.8). Therefore, $h_1(N_T)$ and $h_2(N_T)$ are both replicable by wealth processes which are martingales under $\mathbf{Q}^{(m)}$. Since $g = h_1 - h_2$, we conclude that the *bounded* random variable $h(N_T)$ is replicable, (and the wealth process that replicates it is a bounded martingale under $\mathbf{Q}^{(m)}$), therefore $h(N_T) \in \mathcal{R}(S)$. Taking into account that any bounded continuous function can be approximated in the topology of uniform convergence on compact subsets of $(0,\infty)$ (by cut-off and regularization by convolution) by a sequence of smooth functions with compact support which is uniformly bounded, we obtain that for any bounded continuous function $f : (0,\infty) \to \mathbb{R}$ we have $f(N_T) \in \mathcal{R}(S)$.

In order to finish the proof, we denote by $\mathcal{H}$ the vector space of bounded functions $f : (0,\infty) \to \mathbb{R}$ such that $f(N_T) \in \mathcal{R}(S)$ and by $\mathcal{C}$ the set of bounded continuous functions. Since $\mathcal{H}$ contains all constant functions and is a monotone class, $\mathcal{C}$ is closed under point-wise multiplication and $\mathcal{C} \subseteq \mathcal{H}$ we obtain from a version of the Monotone Class Theorem (see, e.g. Theorem 2.2 on page 3 in [27]), that $\mathcal{H}$ contains all the bounded $\sigma(\mathcal{C})$-measurable functions. Since $\sigma(\mathcal{C})$, the smallest sigma-field with respect to each all bounded continuous functions are measurable is exactly the Borel sigma-field on $(0,\infty)$, we conclude that for each bounded Borel measurable function $f : (0,\infty) \to \mathbb{R}$ we have $f(N_T) \in \mathcal{R}(S)$ which is the same thing as $L^\infty(\mathcal{G}_T^N) \subseteq \mathcal{R}(S)$, so the proof is complete. ■

Remark 4.1 From the proof of Proposition 3.10 it is easy to see that items (i)-(iii) are also equivalent to relation (4.7), which is one of the main tools for the proofs of our main results, namely Theorems 3.1 and 3.6.

Proof of Theorem 3.1.

Let us consider the optimizer $\hat{X}(x, U)$ in (2.6). Under the hypotheses of Theorem 3.1 we know from Remark 3.5, Proposition 3.10 and Remark 4.1 that relation (4.7) holds true. According to (4.4), we have

$$\hat{X}_T(x, U) = -V'(y\frac{d\mathbf{Q}^{(m)}}{d\mathbf{P}}), \text{ for } \ y = u'(x).$$

This implies that $\hat{X}_T(x, U)$ is a positive, $\mathcal{G}_T^N$-measurable and $\mathbf{Q}^{(m)}$-integrable random variable. The process $\hat{X}(x, U)$ is a martingale under $\mathbf{Q}^{(m)}$. Since $\hat{X}_T(x, U)$ is $\mathcal{G}_T^N$-measurable and $L^\infty(\mathcal{F}_T^N) \subset \mathcal{R}(M)$ (where M is the mutual fund which appears in condition (**R**)), there exists a sequence of 1-dimensional integrands k^n such that

$$\hat{X}_T(x, U) \wedge n = x_n + \int_0^T k_u^n dM_u,$$

for each n, and $\int k^n dM$ is uniformly bounded in t and ω. Taking into account that the set of stochastic integrals which are martingales under $\mathbf{Q}^{(m)}$ (identified with their last element) is closed in $L^1(\Omega, \mathcal{F}, \mathbf{Q}^{(m)})$, we can let now $n \to \infty$ to conclude that there exists an $\mathcal{F}$-predictable k such that

$$\hat{X}(x,U)_t = x + \int_0^t k_u dM_u, \ 0 \le t \le T,$$

which finishes the proof. ■

Proof of Corollary 3.2. Since $(\mathbf{R^N}) \Longrightarrow (\mathbf{R})$, the result follows from Theorem 3.1 ■

Proof of Corollary 3.3. The space $L^\infty(\Omega, \mathcal{F}_T^N, \mathbf{P})$ or, briefly, $L^\infty(\mathcal{F}_T^N)$ represents the set of all bounded path-dependent options on the numéraire portfolio N. Therefore, the assumption that the numéraire portfolio N generates a complete market with respect to its own filtration $(\mathcal{F}_t^N)_{0 \le t \le T}$ can be rewritten as $L^\infty(\mathcal{F}_T^N) \subseteq \mathcal{R}(N, \mathcal{F}^N)$. Since, obviously, $\mathcal{R}(N, \mathcal{F}^N) \subset \mathcal{R}(N)$ we obtain

$$L^\infty(\mathcal{F}_T^N) \subseteq \mathcal{R}(N).$$

The above condition is a stronger condition than the replicability condition $(\mathbf{R})$, so, from Theorem 3.1 it follows that the (MFT) holds true for the model S and the class of utilities $\mathcal{A}(S)$, and, in addition, the numéraire portfolio may serve as the mutual fund. We would like to point out that, under the hypotheses of Corollary 3.3, all utility maximizing agents will only invest in the numéraire porfolio by *only* considering the information obtained by observing the evolution of the numéraire portfolio. In other words, the one-dimensional integrand in (2.14) is $(\mathcal{F}_t^N)_{0 \le t \le T}$-predictable. ■

Remark 4.2 The hypotheses of Corollary 3.3 may be replaced by assuming that the numéraire portfolio N generates a complete market with respect to a filtration $(\mathcal{H}_t)_{0 \le t \le T}$ such that

$$\mathcal{F}_t^N \subseteq \mathcal{H}_t \subseteq \mathcal{F}_t, \quad 0 \le t \le T.$$

As presented in the Proof of Proposition 3.14 below, the situation when N generates a complete market with respect to a larger filtration $(\mathcal{H}_t)_{0 \le t \le T}$ but *not* with respect to its natural filtration $(\mathcal{F}_t^N)_{0 \le t \le T}$ may, indeed, occur, so the Remark extends the range of applicability of Corollary 3.3.

Proof of Theorem 3.6. The proof is very similar to the proof of (iii) $\Longrightarrow$ (i) in Theorem 3.10 above. Namely, consider the same kind of strictly increasing function $h : (0, \infty) \to \mathbb{R}$ as in (4.8) and the associated dual function $V(y) = -\int_0^y h(z^{-1})dz$, as well as the convex conjugate utility function U. Since the weak completeness condition $(\mathbf{WC})$ is satisfied, according to Theorem 3.10 and Remark 4.1 we know that (4.7) holds true for any y, in particular when $y = 1$. Using (4.4) we have that the random variable $h(N_T) = h((\frac{d\mathbf{Q}^{(m)}}{d\mathbf{P}})^{-1}) = -V'(\frac{d\mathbf{Q}^{(m)}}{d\mathbf{P}})$ is actually the terminal wealth of the investor with initial capital $-v'(1)$ and utility U, i.e. :

$$h(N_T) = \hat{X}_T(-v'(1), U).$$

Since the (MFT) holds true for the model S with respect to the set of all utilities $\mathcal{A}(S)$ and a mutual fund M we conclude that the random variable $h(N_T)$ can actually be replicated by trading in M only, and the replication process (which is, in fact, $\hat{X}(-v'(1), U)$) is a martingale under $\mathbf{Q}^{(m)}$. We can use now a similar argument (based on approximation and Monotone Class Theorem) to the one used in the the proof of (iii) $\Longrightarrow$ (i) in Proposition 3.10 to conclude that for any *bounded Borel measurable* function $f : (0, \infty) \to \mathbb{R}$ we have that the random variable $f(N_T)$ is replicable by trading in M only (and the stochastic integral is bounded). This means that

$$L^\infty(\mathcal{G}_T^N) \subseteq \mathcal{R}(M),$$

so the proof is complete. ■

Remark 4.3 Taking into account the proof of Theorem 3.10, we see that it is enough to require the existence of an equivalent martingale measure $\mathbf{Q} \in \mathcal{M}^e$ such that $L^\infty(\sigma(\frac{d\mathbf{Q}}{d\mathbf{P}})) \subset \mathcal{R}(S)$ to conclude that the measure $\mathbf{Q}$ dominates stochastically in the second order all other equivalent martingale measures, so, in particular, $\mathbf{Q} = \mathbf{Q}^{(m)}$. Therefore, since $\frac{d\mathbf{Q}^{(m)}}{d\mathbf{P}} = 1/N_T$, it is apparent that conditions ($\mathbf{R}$) and ($\mathbf{WC}$) can be, respectively, weakened to

($\mathbf{R}'$) there exists some $\mathbf{Q} \in \mathcal{M}^{\mathrm{e}}$ and a mutual fund $\mathbf{M}$ such that $\mathbf{L}^\infty(\sigma(\dfrac{\mathbf{dQ}}{\mathbf{dP}})) \subset \mathcal{R}(\mathbf{M})$,

($\mathbf{WC}'$) there exists some $\mathbf{Q} \in \mathcal{M}^e$ such that $L^\infty(\sigma(\dfrac{d\mathbf{Q}}{d\mathbf{P}})) \subset \mathcal{R}(S)$.

In particular, this means that in both conditions ($\mathbf{R}$) and ($\mathbf{WC}$), the set of bounded European options written on the numéraire portfolio can be replaced by the set of bounded European options written on the wealth process of any investor with utility function U and initial capital $x > 0$, namely the set $L^\infty(\mathcal{G}_T^{\hat{X}(x,U)})$, provided that the dual optimizer is a probability measure, which, by the results of [21], is equivalent to the condition

$$\mathbf{E}[U'(\hat{X}_T(x,U))] = u'(x).$$

Proof of Proposition 3.13. Let M be the mutual fund. The condition (3.5) ensures that the numéraire portfolio N can actually be chosen as mutual fund, instead of M. In order to finish the proof, we need to follow the steps of the proof of Theorem 3.6 to conclude that condition ($\mathbf{R^N}$) holds true. ■

Example 4.4 This example is a complement to Remark 3.12 and shows that one can construct a complete market where (MFT) holds true (so condition ($\mathbf{R}$) holds true, according to Theorem 3.6) but the numéraire portfolio N cannot be chosen as the mutual fund (or, in other words, condition ($\mathbf{R^N}$) is not satisfied, according to Theorem 3.1). The economic idea is the following: while the logarithmic investor is *myopic*, always maximizing the expected logarithmic utility over the *next* (infinitesimal) time period and therefore never investing in a martingale, some other investors may be willing to invest for some periods of time in martingales, in order to take advantage of the later benefits of being in a "better state" from the point of view of their expected utility at the final time-horizon.

The construction goes as follows: let $\varepsilon_1, \varepsilon_2$ be two independent $\{-1, 1\}$-valued random variables with $\mathbf{P}[\varepsilon_1 = 1] = \frac{1}{2}$ while $\mathbf{P}[\varepsilon_2 = 1] = p \in]0,1[\setminus \{\frac{1}{2}\}$. Define $S = (S_t)_{t=0}^2$ by

$$\begin{aligned} S_0 &= 2, \quad \Delta S_1 = \varepsilon_1 \text{ and} \\ \Delta S_2 &= \begin{cases} \varepsilon_2 & \text{if } \varepsilon_1 = 1 \\ 0 & \text{if } \varepsilon_1 = -1, \end{cases} \end{aligned}$$

where $\Delta S_n = S_n - S_{n-1}$ are the increments of S. This defines a one-dimensional complete market S for which (MFT) obviously holds true (taking, for example, the stock process S as mutual fund).

If we denote by $\widehat{H}^N = (\widehat{H}_1^N, \widehat{H}_2^N)$ the numbers of shares of the stock S that the logarithmic maximizer holds between time 0 and 1, and time 1 and 2, respectively, we can see (either by direct computation or a simple qualitative argument) that $\widehat{H}_1^N = 0$. On the other hand, for a generic utility function U and endowment $x > 0$, the utility maximizing agent will take a non-trivial position over the first time period. In other words, using the same notation $\widehat{H}(x,U) = (\widehat{H}_1(x,U), \widehat{H}_2(x,U))$ for the predictable process representing the optimal number of shares, we may find a utility function U (and an initial capital $x > 0$) such that $\widehat{H}_1(x,U) \neq 0$. This obviously means that the numéraire portfolio N cannot be chosen as mutual fund.

The fact that $\widehat{H}_1(x,U) \neq 0$ for some utility function U can again be verified by direct computations (e.g. for the case $U(x) = \frac{x^\alpha}{\alpha}$) or by a more qualitative argument reproduced below.

Fix x and U and let $\widehat{X}(U,x) = x + \widehat{H} \cdot S$ where $\widehat{H} = (\widehat{H}_1, \widehat{H}_2)$ is predictable. Let us first look at the choice of $\widehat{H}_2$, which depends on ε_1 and on the wealth $x + (\widehat{H} \cdot S)_1 = x + \widehat{H}_0 \Delta S_1$ of the investor at time 1. Given this information the investor solves the optimization problem for the second period to determine $\widehat{H}_2$. The point is that we thus obtain a conditional indirect utility function $u_1(x, \varepsilon_1)$ which is defined as the optimal expected utility at time $T = 2$, conditionally on the value of ε_1 and wealth x at time $t = 1$. (For more details on conditional indirect utilities, see, e.g., [31].) The optimization problem at time $t = 0$, i.e., the determination of $\widehat{H}_1$, then may be viewed as a one period optimization problem with horizon $t = 1$ and with respect to the random utility function $u_1(x, \varepsilon_1)$. The case of $U(x) = \log(x)$ is special, as in this case we have

$$u_1(x, \varepsilon_1) = \log(x) + \text{constant}(\varepsilon_1), \tag{4.9}$$

i.e., $u_1(x, \varepsilon_1)$ is a vertical shift of the logarithm, the constant depending on ε_1 (but not on x). This is exactly the *myopic* property of the logarithmic investor described in the beginning of the example, and can also be reformulated saying that the marginal indirect utility $\frac{\partial}{\partial x} u_1(x, \varepsilon_1) = x^{-1}$ does *not* depend on ε_1.

However, if we take a generic utility function U (e.g., power or exponential), then $\frac{\partial}{\partial x} u_1(x, \varepsilon_1)$ will depend on ε_1. Therefore, by a first moment argument, it will now be optimal to take a gamble on the random variable ε_1, contrary to the case of the logarithm. More precisely, fix the endowment x_0 at time $t = 0$. If $\frac{\partial}{\partial x} u_1(x, \varepsilon_1 = 1)|_{x=x_0} \neq \frac{\partial}{\partial x} u_1(x, \varepsilon_1 = -1)|_{x=x_0}$ then the strategy $H_1 = 0$ cannot be optimal as one may exploit the difference of the marginal utilities.

We still Remark that it is easy to reformulate Example 4.4 in terms of continuous processes driven by a Brownian motion whose increments replace the independent random variables ε_1 and ε_2.

Example 4.5 In this example we will show that the "completeness" assumption (**WC**) is essential in Theorem 3.6. The basic idea of the (counter)example is that in a one dimensional market, (MFT) always holds true in a trivial way, but the replicability condition $L^\infty(\mathcal{G}_T^N) \subset \mathcal{R}(M)$ may fail, for any choice of the mutual fund M.

More precisely, consider a two dimensional Brownian motion $W = (W^{(1)}, W^{(2)})$ on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbf{P})$, where $(\mathcal{F}_t)_{0 \le t \le T}$ is the filtration generated by the Brownian Motion W, and define the one-dimensional stock price process by

$$dS_t = S_t\{\text{arctg}(W_t^{(2)})^2 dt + \text{arctg}(W_t^{(2)}) dW_t^{(1)}\},\ S_0 = 1.$$

In this case, the numéraire portfolio equals the stock price process, i.e. $N = S$. Let us consider the random variable $f = \ln(N_T)$ and use Ito's formula to obtain

$$f = \int_0^T \frac{dS_t}{S_t} - \frac{1}{2}\int_0^T (\text{arctg}(W_t^{(2)})^2 dt.$$

Changing the measure $\mathbf{P}$ to the minimal martingale measure $\mathbf{Q}^{(m)}$ (defined by (2.13) for $\gamma_t^{(m)} = (\text{arctg}(W_t^{(2)}, 0), 0 \leq t \leq T)$ under which the law of the second coordinate of the Brownian Motion, $W^{(2)}$, is left unchanged, we can easily see that the random variable $\int_0^T (\text{arctg}(W_t^{(2)})^2 dt$ cannot be replicated by trading in S, so, a fortiori, the random variable f cannot be replicated by trading in S. Using now a simple density argument, we obtain that there exists a positive integer n such that

$$\mathbb{I}_{\{-n \leq N_T \leq n\}} \ln(N_T) \notin \mathcal{R}(S).$$

Since the random variable above is an element of $L^\infty(\mathcal{G}_T^N)$, the (counter)example is complete.

Proof of Proposition 3.14. The construction is based on the well known idea that *shrinking* the filtration may destroy the *predictable representation property* (c.f. Problem 4.22 in [27]). As a matter of fact, the argument relies on a classical example of Tanaka, described below.

Let $W = (W^{(1)}, W^{(2)})$ be a two-dimensional Brownian motion with its natural (right continuous, saturated) filtration $(\mathcal{F}_t)_{0 \leq t \leq T} = (\mathcal{F}_t^W)_{0 \leq t \leq T}$ on the probability space$(\Omega, \mathcal{F}_T, \mathbf{P})$. Define now the Levy transform of the first coordinate of the Brownian Motion

$$B_t^{(1)} = \int_0^t sgn(W_s^{(1)}) dW_s^{(1)},$$

which is a one-dimensional Brownian Motion. At the risk of being somewhat pedantic, $B^{(1)}$ is a Brownian Motion with respect to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$, and, of course, also a Brownian Motion with respect to its own natural filtration $\mathcal{F}^{B^{(1)}} = (\mathcal{F}_t^{B^{(1)}})_{0 \leq t \leq T}$ (which is right continuous and saturated). The crucial point is the loss of information that occurs when passing from $W^{(1)}$ to $B^{(1)}$: for each $0 < t_0 \leq T$, the random variable $sgn(W_{t_0}^{(1)})$ and the process $(B_t^{(1)})_{0 \leq t \leq T}$ are independent under the probability measure $\mathbf{P}$, and the Brownian Motion $(B_t^{(1)})_{0 \leq t \leq T}$ is adapted to the filtration $(\mathcal{F}_t^{|W^{(1)}|})_{0 \leq t \leq T} = (\mathcal{F}_t^{B^{(1)}})_{0 \leq t \leq T}$.

Fix $t_0 \in (0, T)$ and consider the positive process

$$h(t) = 2 + sgn(W_{t_0}^{(1)}) \mathbb{I}_{(t_0 < t)}, \ 0 \leq t \leq T.$$

Note that the process h is predictable with respect to the filtration $(\mathcal{F}_t^{W^{(1)}})_{0 \leq t \leq T}$ (so, of course, predictable with respect to $(\mathcal{F}_t)_{0 \leq t \leq T}$). On the triplet $(\Omega, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$ we consider the two-dimensional stock price

$$\begin{cases} dS_t^{(1)} = S_t^{(1)}[h^2(t)dt + h(t)dB_t^{(1)}] = S_t^{(1)}[h^2(t)dt + h(t)sgn(W_t^{(1)})dW_t^{(1)}], \ 0 \leq t \leq T \\ dS_t^{(2)} = S_t^{(2)}(\pi + \text{arctg}(W_t^{(1)}))dW_t^{(2)}, \quad 0 \leq t \leq T. \end{cases}$$

In other words, the first stock evolves like Geometric Brownian Motion driven by the Brownian Motion $B^{(1)}$ and has volatility 2 up to time t_0 and after that has either volatility 1 or 3, chosen independently of $B^{(1)}$, but adapted to $W^{(1)}$. We would like to point out that, written in terms on W, all coefficients in the equation describing the evolution of the stock price are adapted to W. We also Remark that $\mathcal{F}^{S^{(1)}} \subset \mathcal{F}^{W^{(1)}}$.

The unique equivalent martingale measure (for the two-dimensional stock price $S = (S^{(1)}, S^{(2)})$ with respect to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$) has density $Z = 1/S^{(1)}$, therefore, the numéraire portfolio process is the first stock, i.e.

$$N = S^{(1)}.$$

The second stock $S^{(2)}$ is not traded by the logarithmic maximizer. It serves to encode the information of both $W^{(1)}$ and $W^{(2)}$. As a matter of fact,

$$\mathcal{F}^W = \mathcal{F}^{S^{(2)}} \subset \mathcal{F}^S \subset \mathcal{F}^W,$$

so item (i) in the Proposition is satisfied.

Since $N = S^{(1)}$, we have $\mathcal{F}^N \subset \mathcal{F}^{W^{(1)}}$. Using Girsanov's Theorem (based on the predictable representation property of $W^{(1)}$ with respect to $\mathcal{F}^{W^{(1)}}$), we can show that the only equivalent martingale measure for N with respect to $\mathcal{F}^{W^{(1)}}$ has density $Z = 1/N = 1/S^{(1)}$ (the same as the martingale measure for the two-dimensional stock S). This means that the reduced market $(N, \mathcal{F}^{W^{(1)}})$ is a complete market, which implies

$$L^\infty(\mathcal{F}_T^N) \subset \mathcal{R}(N, \mathcal{F}^{W^{(1)}}) \subset \mathcal{R}(N, \mathcal{F}),$$

so item (ii) is also satisfied.

For a fixed strike price $K > 0$, let us consider $f = (K - N_T)^+$, a European put on the asset N with maturity T. From the previous considerations ($(N, \mathcal{F}^{W^{(1)}})$ is a complete market) we conclude that there exist a price $p(f) > 0$ and a $\mathcal{F}^{W^{(1)}}$-predictable process k such that the integral

$$I_t = p(f) + \int_0^t k_u dN_u, \quad 0 \le t \le T,$$

is a bounded martingale under $\mathbf{Q}^{(m)}$, and $f = I_T$. Assume that we can choose an integrand k above which is $\mathcal{F}^N$-predictable. From time t_0 to maturity T, the process $N = S^{(1)}$ evolves (except for the drift) like a Geometric Brownian Motion with volatility D_{t_0}, where we denote by $D_{t_0} = 2 + sgn(W^1_{t_0})$. We emphasize that D_{t_0} is a $\mathcal{F}^N_{t_0}$-measurable random variable, since $\mathcal{F}^N$ is the *saturated* filtration generated by $N = S^{(1)}$. If we denote by $BS(t, T, s, \sigma)$ the Black-Merton-Scholes price of the European put at time t, with maturity T, when the stock price is s and volatility σ (and, furthermore the interest rate is $r = 0$ and call price is K), then the price of the above call f at time t_0 is

$$I_{t_0} = BS(t_0, T, N_{t_0}, D_{t_0}).$$

The process I is continuous and adapted to the filtration $\mathcal{F}^N$, so I_{t_0} is measurable with respect to the sigma-algebra $\mathcal{F}^N_{t_0-} = \mathcal{F}^{B^{(1)}}_{t_0}$. However, this is absurd as D_{t_0} and $\mathcal{F}^{B^{(1)}}_T$ are independent and the function $BS(t_0, T, s, \cdot)$ is strictly monotone. Therefore, we obtained by contradiction that the European option f written on the asset N cannot be replicated using integrands predictable with respect to $\mathcal{F}^N$. In other words, $f \in L^\infty(\mathcal{G}^N_T)$ but $f \notin \mathcal{R}(N, \mathcal{F}^N)$, so the proof is complete. ■

Remark 4.6 Proposition 3.14 is sharp in two directions: the coefficients in (2.11) are adapted to the driving Brownian Motion, i.e. $\mathcal{F} = \mathcal{F}^W$, and the filtration is not larger than the one generated by the stock itself, which means $\mathcal{F} = \mathcal{F}^S$. If we do not insist in constructing a sharp example in the sense that $\mathcal{F}^S = \mathcal{F}^W$ as in item (i), then a one dimensional market following the dynamics of S^1 is sufficient for our purposes.

Proof of Theorem 3.15. Fix two utility functions $U_1, U_2 \in \mathcal{U}$. The proof is based on the construction of a (double indexed) sequence of financial markets. The price process will be fixed, along with the risk-neutral measure, the only variable quantity being the objective probability measure.

Consider a two dimensional Brownian motion $W = (W^{(1)}, W^{(2)})$ on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{Q})$, where $(\mathcal{F}_t)_{0\le t\le T}$ is the saturated filtration generated by the Brownian

Motion. We define the two-dimensional price process by $S = (S^1, S^2)$ where $S^1 = \mathcal{E}(W^{(1)})$ and $S^2 = \mathcal{E}(W^{(2)})$ are the stochastic exponentials of the two components of the Brownian motion W. This two-dimensional model $S, (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{Q})$ is complete, and $\mathbf{Q}$ is the unique equivalent martingale measure. The (sequence of) financial markets $S, (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{P})$ described above will be constructed by considering appropriate *objective* measures $\mathbf{P} \approx \mathbf{Q}$.

Consider the $(\mathbf{Q}, \mathcal{F}_t)$ square-integrable martingale

$$L_t = \int_0^t \operatorname{arctg}(W_s^{(2)}) dW_s^{(1)}, \quad 0 \le t \le T,$$

and let

$$\tau_n = \inf\{t, |L_t| = n \text{ or } t = T\}, \quad n = 1, 2, \dots \quad \text{and } \alpha_n = L_{\tau_n}, \quad \alpha = L_T.$$

The main idea is to construct the objective measure such that, for fixed $x > 0, n \in \mathbb{N}$ and fixed $\varepsilon \in \mathbb{R}$ satisfying $|\varepsilon| \le x/2n$ the optimal wealth process of the investor with utility U_1 starting at x is

$$X(x) = x + \varepsilon L^{\tau_n}, \text{ i.e } X_T(x) = x + \varepsilon\alpha_n > 0.$$

This can be done by choosing the objective measure $\mathbf{P}^{x,n,\varepsilon}$ such that

$$\frac{d\mathbf{Q}}{d\mathbf{P}^{x,n,\varepsilon}} = c(n, \varepsilon) U_1'(x + \varepsilon\alpha_n),$$

where the normalizing constant $c(x, n, \varepsilon)$ is chosen such that

$$\mathbf{E}_{\mathbf{Q}}[\frac{d\mathbf{P}^{x,n,\varepsilon}}{d\mathbf{Q}}] = 1.$$

Note that there are no integrability problems since $U_1'(x+\varepsilon\alpha_n)$ is bounded above and bounded away from zero. In addition, we would like to point out the the market S defined with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{P}^{x,n,\varepsilon})$ can easily be represented in the form (2.11), using Girsanov's Theorem. We denote by $\mathcal{H}^2(\mathbf{Q})$ the set of $(\mathbf{Q}, (\mathcal{F}_t)_{0\le t\le T})$ square-integrable martingales and by $\mathcal{H}^2_n(\mathbf{Q})$ the closed subspace of elements $M \in \mathcal{H}^2(\mathbf{Q})$ such that

$$M_t = M_0 + \int_0^t H_s^1 dW_s^{(1)} + \int_0^t H_s^2 dW_s^{(2)} \text{ and } H_t^2 1_{\{t\le\tau_n\}} = 0, \quad 0 \le t \le T.$$

In other words, up to an additive constant, $\mathcal{H}^2_n(\mathbf{Q})$ consists of those stochastic integrals where the second component of the integrand is identical zero before time τ_n. As a limiting case, we denote by $\mathcal{H}^2_\infty(\mathbf{Q})$ the closed subspace of $M \in \mathcal{H}^2(\mathbf{Q})$ such that

$$M_t = M_0 + \int_0^t H_s^1 dW_s^{(1)}, \ 0 \le t \le T.$$

We identify spaces of martingales with their terminal values in the usual way, so we may say that $\alpha = L$ and $\alpha_n = L^{\tau_n}$.

Fix $y > 0$. Since the random variable $yU_1'(x + \varepsilon\alpha_n)$ is a multiple of the density of the unique martingale measure for the financial market $S, (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{P}^{x,n,\varepsilon})$ we conclude that $I_2(yU_1'(x + \varepsilon\alpha_n))$ is the terminal wealth $\hat{X}_T(x, U_2)$ of an optimal investment strategy for the investor with utility U_2 and some initial endowment $x > 0$ (here I_2 is the inverse of U_2'). If we suppose that MFT applies to the model S defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{P}^{x,n,\varepsilon})$ with respect to the class $\mathcal{U}$ which contains U_1 and U_2 (and $\mathbf{Q}$ is the unique martingale measure), we conclude that

$$I_2(yU_1'(x + \varepsilon\alpha_n)) \in \mathcal{H}^2_n(\mathbf{Q}).$$

Indeed, to obtain the above relation, we compare the optimal strategy of the investor with utility U_2 which has terminal wealth $I_2(yU_1'(x+\varepsilon\alpha_n))$ with the optimal investment strategy of the investor with utility U_1 starting at x. Letting $f(x,y) = I_2(yU_1'(x))$, we have obtained that, for fixed $y > 0$,$n \in \mathbb{N}$, $x > 0$ and $|\varepsilon| \le x/2n$ we have

$$f(x+\varepsilon\alpha_n, y) \in \mathcal{H}_n^2(\mathbf{Q}).$$

For $x > 0$ and $\varepsilon \le x/2n$ we define the two-variable $\mathcal{H}^2(\mathbf{Q})$-valued function

$$F(x,\varepsilon) := f(x+\varepsilon\alpha_n, y) \in \mathcal{H}_n^2(\mathbf{Q}) \subset \mathcal{H}^2(\mathbf{Q}).$$

Using the bounded convergence theorem (since α_n is bounded and $f(\cdot, y)$ is C^2), we obtain that the function F is two-times differentiable with respect to ε as a $\mathcal{H}^2(\mathbf{Q})$-valued function and

$$\frac{\partial^2}{\partial\varepsilon^2}F(x,\varepsilon)|_{\varepsilon=0} = \frac{\partial^2 f}{\partial x^2}(x,y)(\alpha_n)^2 \in \mathcal{H}^2(\mathbf{Q}).$$

Since $\mathcal{H}_n^2(\mathbf{Q})$ is a closed subspace of $\mathcal{H}^2(\mathbf{Q})$, we conclude that

$$\frac{\partial^2 f}{\partial x^2}(x,y)(\alpha_n)^2 \in \mathcal{H}_n^2(\mathbf{Q}).$$

Taking into account that $\frac{\partial^2 f}{\partial x^2}(x,y)$ is a scalar (real number) we can let $n \to \infty$ to obtain

$$\frac{\partial^2 f}{\partial x^2}(x,y)\alpha^2 \in \mathcal{H}_\infty^2(\mathbf{Q}). \tag{4.10}$$

However, according to Itô's formula

$$\alpha^2 = (L_T)^2 = 2\int_0^T L_t \mathrm{arctg}(W_t^{(2)})dW_t^{(1)} + \int_0^T (\mathrm{arctg}(W_t^{(2)}))^2 dt,$$

so

$$\alpha^2 \notin \mathcal{H}_\infty^2(\mathbf{Q}).$$

The above relation, together with (4.10), shows that

$$\frac{\partial^2 I_2(yU_1'(x))}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}(x,y) = 0, \quad \text{for all } x > 0, y > 0.$$

This means that the function $I_2(yU_1'(\cdot))$ is linear, or $I_2(yU_1'(x)) = a(y)x + b(y)$ for any $x, y > 0$. Since $I_2, U_1' : (0,\infty) \to (0,\infty)$ are strictly decreasing bijective functions, we conclude that $x \to I_2(yU_1'(x))$ is a strictly increasing and bijective function from $(0,\infty)$ to $(0,\infty)$, so $b(y) = 0$ and $a(y) > 0$ for each $y > 0$. Choosing $x = I_1(z)$, we obtain $I_2(yz) = a(y)I_1(z)$, for all $y, z > 0$, which can be rewritten as

$$\frac{I_2(yz)}{I_1(z)} = a(y), \text{ for all } y, z > 0.$$

Differentiating with respect to z, we obtain

$$\frac{I_2'(yz)}{I_2(yz)} = \frac{1}{y}\frac{I_1'(z)}{I_1(z)} \text{ for all } y, z > 0.$$

Now, we first consider $z = 1$ and vary y over $(0,\infty)$ and then we consider $y = 1$ and vary z over $(0,\infty)$ to obtain

$$\frac{I_2'(y)}{I_2(y)} = \frac{I_1'(1)}{I_1(1)}\frac{1}{y} = \frac{I_1'(y)}{I_1(y)} \text{ for each } y > 0.$$

Taking into account that $\frac{I_1'(1)}{I_1(1)} < 0$ is a constant, the conclusion of the Theorem follows by integration. ∎

Example 4.7 In the market model (2.11), let $N = d \geq 2$ and let σ_t be the identity matrix. Let $A \in \mathcal{F}_T$ be any set with $0 < \mathbf{P}(A) < 1$ and let $a \in \,]0,1[$. Define the probability measure $\mathbf{Q}$ on $\mathcal{F}_T$, such that $\frac{d\mathbf{Q}}{d\mathbf{P}}$ takes the constant value a on A as well as the constant value $b = (1 - a\mathbf{P}(A))/(1 - \mathbf{P}(A))$ on the complement A^c. Since $a \neq 1$, these two values are different. We define $Z_t = \mathbf{E_P}[\frac{d\mathbf{Q}}{d\mathbf{P}} \,|\, \mathcal{F}_t]$, for $t \in \mathbb{T}$. By uniform boundedness from above and from below it easily follows that Z is an exponential martingale. Thus it can be represented as in formula (2.13) with a unique $\gamma^{(m)}$, since the kernel of σ_t is trivial. Defining $\gamma = \gamma^{(m)}$ in the price model (2.11), it follows that the thus obtained market model S is complete. The e.l.m.m. $\mathbf{Q}$ is then unique, so $\mathbf{Q}^{(m)} = \mathbf{Q}$ satisfies Assumption I.

Solving the utility maximization problem (2.6), for any utility function $U \in \mathcal{A}$, yields according to (4.4) that

$$X_T(x) = a_1 \mathbf{1}_A + a_2 \mathbf{1}_{A^c}, \tag{4.11}$$

for some $a_1, a_2 \in \mathbb{R}$ depending on x and U. This formula applies to the logarithmic case as well, so that $N_T(x) = b_1 \mathbf{1}_A + b_2 \mathbf{1}_{A^c}$ for some $b_1 \neq b_2$. Hence, just regarded from time zero to maturity, the numéraire portfolio behaves like a one-period binomial model, so $L^\infty(\mathcal{G}_T^N) \subseteq \mathcal{R}(N)$ and (MFT) holds true with respect to $\mathcal{A}$. However, since the choice of A is completely free, it is easy to give examples (following, e.g., the lines of Example 4.5) where the condition $L^\infty(\mathcal{F}_t^N) \subset \mathcal{R}(N)$ is not satisfied for some $0 \leq t \leq T$.

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References

[1] Back, K. *Asset Pricing and Portfolio Choice Theory*, Oxford University Press, forthcoming.

[2] Becherer, D. *The Numéraire Portfolio for Unbounded Semimartingales*, Finance and Stochastics **5**, 327–341 (2001).

[3] Cass, D. and Stiglitz, J. E.: *The Structure of Investor Preferences and Asset Returns, and Separability in Portfolio Allocation: A Contribution to the Pure Theory of Mutual Funds*, Journal of Economic Theory, **2**, 122–160 (1970).

[4] Chamberlain, G. : *Asset Pricing in Multiperiod Security Markets*, Econometrica, **56**(6), 1283–1300 (1988).

[5] Cvitanic, J., Schachermayer, W. and Wang, H. *Utility Maximization in Incomplete Markets with Random Endowment*, Finance and Stochastics, **5**, 259-272 (2001).

[6] M.H.A. Davis. Option Pricing in Incomplete Markets. In M.A.H. Dempster and S.R. Pliska, editors, *Mathematics of Derivative Securities*, pages 216–226. Cambridge University Press, 1997.

[7] Delbaen, F. and Schachermayer, W.: *A General Version of the Fundamental Theorem of Asset Pricing*, Mathematische Annalen, Vol. 300 (1994), pp. 463-520.

[8] Delbaen, F. and Schachermayer, W.: *The Mathematics of Arbitrage*, Springer Finance Collection, Springer-Verlag, 2006.

[9] Ekeland, I. and Taflin, E.: *A Theory of Bond Portfolios*, The Annals of Applied Probability, **15**, 1260–1305 (2005) (See also `http://arxiv.org/abs/math.OC/0301278`).

[10] Foldes, L., *Valuation and Martingale Properties of Shadow Prices: An exposition*, Journal of Economic Dynamics and Control, **24**, 1641-1701 (2000).

[11] Föllmer, H. and Schweizer, M.: *Hedging of Contingent Claims under Incomplete Information*, in: M. H. A. Davis and R. J. Elliott (eds.), "Applied Stochastic Analysis", Stochastics Monographs, Vol. 5, Gordon and Breach, London/New York (1991), 389–414.

[12] Hakansson, N.H.: *Risk Disposition and the Separation Property in Portfolio Selection*, Journal of Financial and Quantitative Analysis, **4**, 401–416 (1969).

[13] Hicks, J.R. *A Revision of Demand Theory*, Oxford University Press, 1956.

[14] Hugonnier, J. and Kramkov, D.: Optimal Investment with Random Endowments in Incomplete Markets. *The Annals of Applied Probability*, **14**(2):845–864, 2004.

[15] Hugonnier, J., Kramkov, D. and Schachermayer, W.: On Utility-Based Pricing of Contingent Claims in Incomplete Markets, *Mathematical Finance*, **15**: 203–212, 2005.

[16] Karatzas, I. and Shreve, S.E.: *Methods of Mathematical Finance*, Applications of Mathematics, Volume 9, Springer-Verlag 1999.

[17] Jacod, J. and Shiryaev, A.N.: *Limit Theorems for Stochastic Processes*, second edition, Springer-Verlag, A Series of Comprehensive Studies in Mathematics, Vol. 288.

[18] Karatzas, I. and Kardaras, K..: *The Numéraire Portfolio in Semimartingale Financial Models*, Finance and Stochastics **11** 447-493 (2007).

[19] Karatzas, I., Lehoczky, J., Shreve, S.E. and Xu, G.L.: *Martingale and Duality Methods for Utility Maximization in an Incomplete Market*, SIAM Journal on Control and Optimization **29** 707-730 (1991)

[20] Khanna, A. and Kulldorff, M.: *A Generalization of Mutual Fund Theorem*, Finance and Stochastics **3**, 167–185 (1999).

[21] Kramkov, D. and Schachermayer W.: *The Asymptotic Elasticity of Utility Functions and Optimal Investment in Incomplete Markets*, The Annals of Applied Probability **9**, 904–950 (1999).

[22] Kramkov, D. and Sîrbu M.: *Sensitivity Analysis of Utility-Based Prices and Risk-Tolerance Wealth Processes*, The Annals of Applied Probability **16**, 2140-2194 (2006).

[23] Markowitz, H.: *Portfolio Selection*, Journal of Finance, **7**, 77–91 (1952).

[24] Markowitz, H.: *Portfolio Selection: Efficient Diversification of Investments*, 1959, New-York, Wiley & Sons, Inc. (see also `http://cowles.econ.yale.edu/`)

[25] Merton, R.: *Optimum Consumption and Portfolio Rules in a Continuous Time Model*, Journal of Economic Theory, **3**, 373–413 (1971).

[26] Merton, R.: *An Intertemporal Capital Asset Pricing Model*, Econometrica, **41**, 867–887 (1973).

[27] Revuz, D. and Yor, M.: *Continuous Martingales and Brownian Motion*, Grundlehren der mathematischen Wissenschaften, Band 293, Springer-Verlag 199 (third edition: 1999, corrected third printing: 2005)

[28] Ross, S.: *Mutual Fund Separation in Financial Theory-The Separating Distributions*, Journal of Economic Theory, **17**, 254–286 (1978).

[29] Rubinstein M.: The Valuation of Uncertain Income Streams and the Pricing of Options. *Bell Journal of Economics*, 7:407–425, 1976.

[30] Schachermayer, W.: *Optimal Investment in Incomplete Markets when Wealth may Become Negative*, The Annals of Applied Probability **11**, 694–734 (2001).

[31] Schachermayer, W.: A Super-Martingale Property of the Optimal Portfolio Process, Finance and Stochastics, **7**, 433–456 (2003).

[32] Tobin, J.: *Liquidity Preference as Behavior Towards Risk*, The Reviev of Economic Studies, Nr. 58, 65–86 (1958).