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Hiding a Drift

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Abstract: In this article we consider a Brownian motion with drift of the form

$$dS_t = \mu_t dt + dB_t \qquad \text{for } t \geq 0,$$

with a specific non-trivial $(\mu_t)_{t\geq 0}$, predictable with respect to $\mathbb{F}^B$, the natural filtration of the Brownian motion $B = (B_t)_{t\geq 0}$. We construct a process $H = (H_t)_{t\geq 0}$, also predictable with respect to $\mathbb{F}^B$, such that $((H \cdot S)_t)_{t\geq 0}$ is a Brownian motion in its own filtration. Furthermore, for any $\delta > 0$, we refine this construction such that the drift $(\mu_t)_{t\geq 0}$ only takes values in $]\mu - \delta, \mu + \delta[$, for fixed $\mu > 0$.

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1. Introduction

Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a fixed constant $\mu > 0$, denote the Brownian motion with drift μ as $S = (S_t)_{t\geq 0}$, defined by

$$S_t = \mu t + B_t, \qquad t \geq 0. \tag{1.1}$$

Furthermore let $\mathbb{F}^B := (\mathcal{F}_t^B)_{t\geq 0}$ denote the right continuous, saturated filtration generated by B. Given a predictable, $\mathbb{F}^B$-adapted process $H = (H_t)_{t\geq 0}$ we

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consider the stochastic integral $(H \cdot S) = ((H \cdot S)_t)_{t \geq 0}$ in its right continuous, saturated filtration $\mathbb{F}^{(H \cdot S)} := (\mathcal{F}_t^{(H \cdot S)})_{t \geq 0}$. Marc Yor posed the following

> **Question 1:** Can we define an $\mathbb{F}^B$-predictable process H such that the resulting stochastic integral $(H \cdot S)$ is a Brownian motion (without drift) in its own filtration, i.e. an $\mathbb{F}^{(H \cdot S)}$-Brownian motion?

Clearly, the predictable integrand H can only take values in $\{-1, 1\}, \mathbb{P} \otimes \lambda$- a.s., λ denoting Lebesgue measure on $[0, \infty)$, in order to make sure that the process $(H \cdot S)$ has the quadratic variation of a Brownian motion.

In fact, at first glance it seems completely unlikely that an $\mathbb{F}^B$-predictable process H with the required property does exist. Indeed, intuitively speaking, it would have to start with $\mathbb{P}[H_0 = 1] = \mathbb{P}[H_0 = -1] = 1/2$, which seems absurd as H_0 is required to be $\mathcal{F}_0^B$- measurable and therefore $\mathbb{P}$-a.s. constant (the sigma-algebra $\mathcal{F}_0^B$ is trivial). Fortunately this intuitive argument is not quite correct, as the predictable process $H = (H_t)_{t \geq 0}$ is only defined modulo $\mathbb{P} \otimes \lambda$ null-sets, so that it does not really makes sense to speak about the random variable H_0. Nevertheless the preceding heuristics seem to indicate that we need some random sign ϵ with $\mathbb{P}[\epsilon = 1] = \mathbb{P}[\epsilon = -1] = 1/2$ which is *independent* of the Brownian motion B to be able to start a successful construction of the desired integrand $H = (H_t)_{t \geq 0}$ for t close to $t = 0$.

So, let us cheat for a moment and fix a random variable ϵ, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}[\epsilon = 1] = \mathbb{P}[\epsilon = -1] = 1/2$, and consider the enlarged filtration $\mathbb{F}^{B,\epsilon}$ defined by letting $\mathcal{F}_t^{B,\epsilon} = \sigma(\mathcal{F}_t^B, \epsilon)$ for $t \geq 0$.

Let us now try to construct an integrand $H = (H_t)_{t \geq 0}$ which is predictable in the enlarged filtration $\mathbb{F}^{B,\epsilon}$ and such that the stochastic integral $(H \cdot S)$ is a Brownian motion (without drift) in its own filtration $\mathbb{F}^{(H \cdot S)}$. We have an obvious way to start the construction of H at time $t = 0$ by letting

$$H_0 := \epsilon, \tag{1.2}$$

or rather, reasoning heuristically with infinitesimals,

$$H_u := \epsilon \qquad \text{for } 0 \leq u \leq dt.$$

This yields an integrand $(H_u)_{0 \leq u \leq dt}$ such that the stochastic integral $(H \cdot S)_{0 \leq u \leq dt}$ is a martingale for the infinitesimal time interval $[0, dt]$. Indeed

$$\mathbb{E}[d(H \cdot S)_0] = \mathbb{E}[\epsilon(S_{dt} - S_0)] = \mathbb{E}[\epsilon(B_{dt} - B_0) + \epsilon\mu dt] = 0dt.$$

But already an infinitesimal instant of time later we again are in trouble: after having observed the process $(H \cdot S)$ during the infinitesimal time interval $[0, dt]$ we have learned something (which turns out to be of the order $dt^{1/2}$) on the probability of ϵ equaling $+1$ or -1, conditionally on the process $(H \cdot S)_{0 \leq u \leq dt}$.

Hence the approach of defining $H_t = H_0 = \epsilon$ for $t \in [0, \Delta t]$ for a *finite* increment $\Delta t > 0$ yields a process $(H \cdot S)_{0 \leq t \leq \Delta t}$ which *fails to be* a martingale in its own filtration as one easily verifies.

At this stage we remembered Pólya's famous dictum:

"To every problem there is an easier problem".

Instead of asking Yor's original question for the process S with constant drift μ as in (1.1) we pose the same question, but with μ replaced by an appropriate predictable process $(\mu_t)_{t\geq 0}$, i.e.,

$$dS_t = \mu_t dt + dB_t, \tag{1.3}$$

where $(\mu_t)_{t\geq 0}$ is tailor-made such that, for the integrand $H_t = \epsilon$, for $t \geq 0$, we indeed obtain a process $(H \cdot S)_{t\geq 0}$ which is a Brownian motion in its own filtration $\mathbb{F}^{(H\cdot S)}$. This program indeed turns out to be doable as summarized in the subsequent statement.

Proposition 1.1. *Suppose that on $(\Omega, \mathcal{F}, \mathbb{P})$ there is a standard Brownian motion $B = (B_t)_{t\geq 0}$ and a random variable ϵ with $\mathbb{P}[\epsilon = 1] = \mathbb{P}[\epsilon = -1] = 1/2$, independent of B. Denote by $\mathbb{F}^B$ the filtration generated by B.*

For each $\mu > 0$, there is an $\mathbb{F}^B$-predictable process μ_t taking values in $]0, 2\mu[$ such that defining $S = (S_t)_{t\geq 0}$ by $S_0 = 0$ and

$$dS_t = \mu_t dt + dB_t, \qquad t \geq 0, \tag{1.4}$$

we have that

$$Y_t = \epsilon S_t, \qquad t \geq 0,$$

is a Brownian motion in its own filtration.

The preceding result is a preliminary step towards a satisfactory answer to Yor's question. It has two deficiencies: firstly we had to replace the constant μ by a process $(\mu_t)_{t\geq 0}$ fluctuating in $]0, 2\mu[$, and secondly we had to enlarge the filtration $\mathbb{F}^B$ to $\mathbb{F}^{\epsilon,B}$ in order to be able to define our predictable integrand $H_t \equiv \epsilon$, for $t \geq 0$.

As regards the second issue, we can get completely rid of the necessity of introducing the additional source of randomness ϵ by applying the Lévy transform, see Section 3. We can indeed find an integrand H which is predictable with respect to $\mathbb{F}^B$ instead of $\mathbb{F}^{\epsilon,B}$ and still does the job. As regards the first issue we can refine the construction in such a way that the process $(\mu_t)_{t\geq 0}$ only takes values in $]\mu - \delta, \mu + \delta[$ instead of $]0, 2\mu[$, for given $\delta > 0$.

We summarize our findings in the subsequent theorem. Very roughly speaking, it states that the answer to Yor's question is positive, provided that we allow for a tailor-made drift process $(\mu_t)_{t\geq 0}$ instead of a constant drift μ, which may be chosen to satisfy $|\mu_t - \mu| < \delta$, for given $\delta > 0$.

Theorem 1.2. *Let $B = (B_t)_{t\geq 0}$ be a Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}^B, \mathbb{P})$, where $\mathbb{F}^B = (\mathcal{F}_t^B)_{t\geq 0}$ is the right-continuous saturated filtration generated by B.*

(i) *For each $\mu > 0$, there are $\mathbb{F}^B$- predictable processes $H = (H_t)_{t\geq 0}$, taking values in $\{-1, 1\}$, and $(\mu_t)_{t\geq 0}$, taking values in $]0, 2\mu[$, such that for $S = (S_t)_{t\geq 0}$ defined by $S_0 = 0$ and*

$$dS_t = \mu_t dt + dB_t$$

we have that the process $((H \cdot S)_t)_{t \geq 0}$ is a Brownian motion in its own filtration $\mathbb{F}^{(H \cdot S)}$.

(ii) Furthermore, for each $\delta > 0$ we can choose $(\mu_t)_{t \geq 0}$ such that it only takes values in $]\mu - \delta, \mu + \delta[$.

However, Question 1 of M. Yor in its original form above still remains an open and challenging problem. For recent work on the conservation of the martingale property under a change of filtration we refer to [1].

M. Émery asked us the following question: what about the discrete-time version of the problem? The proper discrete analogue is an i.i.d. sequence $(\epsilon_n)_{n \leq 0}$ in its natural filtration $(\mathcal{F}_n)_{n \leq 0}$ such that $\mathbb{P}[\epsilon_n = 1] = 1 - \mathbb{P}[\epsilon_n = -1] = p \in]0, 1[\setminus \{1/2\}$. The question now reads as follows: is there an $(\mathcal{F}_n)_{n \leq 0}$-predictable sequence $(h_n)_{n \leq 0}$ of $\{-1, 1\}$-valued random variables such that the sequence $(h_n \epsilon_n)_{n \leq 0}$ is i.i.d. with $\mathbb{P}[h_n \epsilon_n = 1] = \mathbb{P}[h_n \epsilon_n = -1] = 1/2$?

This discrete version turns out to be simpler than the continuous one and we shall give in the appendix a positive solution, even in a slightly more general setting.

The article is structured as follows: In Section 2 we describe the details of the construction of $(\mu_t)_{t \geq 0}$ with respect to $\mathbb{F}^{\epsilon, B}$. Next in Section 3 we prove the first part of Theorem 1.2 above. Finally, in Section 4 we use stopping techniques in order to show the second statement of Theorem 1.2.

2. Constructing the drift process

Fix a probability space $(\Omega, \mathcal{F}, P)$. Let B be a Brownian motion and let ϵ be an independent random sign with $P(\epsilon = 1) = P(\epsilon = -1) = 1/2$. Consider

$$S_t := \int_0^t \mu_s ds + B_t, \ t \geq 0, \tag{2.1}$$

with some bounded $\mathbb{F}^B$-predictable drift μ_t and set $Y_t := \epsilon S_t$. Our purpose is to find μ_t such that Y_t is a Brownian motion in its own filtration.

We imagine μ_t as being "glued together" from two $\mathbb{F}^Y$-predictable processes. Formally, let μ_t^+, μ_t^- be $\mathbb{F}^Y$-predictable bounded processes such that

$$\mu_t := 1_{\{\epsilon = +1\}} \mu_t^+ + 1_{\{\epsilon = -1\}} \mu_t^-. \tag{2.2}$$

We wish to derive conditions on μ_t^+, μ_t^- which ensure that Y_t is as required. To this end introduce the conditional probabilities

$$p_t := P[\epsilon = 1 | \mathcal{F}_t^Y], \ t \geq 0. \tag{2.3}$$

In the language of filtering theory p_t gives the distribution of the "signal" ϵ, conditionally on the "observations" Y.

Proposition 2.1. *Let S and Y be as above and $(\mu_t)_{t \geq 0}$ a bounded $\mathbb{F}^B$-predictable process of the form (2.2). The conditional probabilities p_t defined in (2.3) satisfy*

$p_0 = 1/2$ *and*

$$dp_t = \frac{(\mu_t^+ + \mu_t^-)^2}{2} \left[\epsilon p_t \left(1 - p_t\right) + p_t(1 - p_t)(1 - 2p_t)\right] dt + \epsilon p_t(1-p_t)(\mu_t^+ + \mu_t^-) dB_t. \tag{2.4}$$

Proof. For each $T > 0$ there exists a measure $\mathbb{Q}_T \sim \mathbb{P}|_{\mathcal{F}_T^{\epsilon,B}}$ such that $(Y_t)_{0\le t\le T}$ is a $(\mathbb{Q}_T, \mathbb{F}^{\epsilon,B})$- Brownian motion. By the Girsanov theorem, we know that the Radon-Nikodym derivative is given by

$$\frac{d\mathbb{Q}_T}{d\mathbb{P}} = \begin{cases} \exp\left(-\int_0^T \mu_t^+ dB_t - (1/2)\int_0^T (\mu_t^+)^2 dt\right), & \text{if } \epsilon = 1 \\ \exp\left(-\int_0^T \mu_t^- dB_t - (1/2)\int_0^T (\mu_t^-)^2 dt\right), & \text{if } \epsilon = -1. \end{cases} \tag{2.5}$$

It follows that for each $T > 0$, the $(\mathbb{Q}_T, \mathbb{F}^{\epsilon,B})$-martingale $Z_t := \frac{d\mathbb{P}}{d\mathbb{Q}_T}\Big|_{\mathcal{F}_t^{\epsilon,B}}$, $0 \le t \le T$ is of the form

$$Z_t = Z_t^+ 1_{\{\epsilon=1\}} + Z_t^- 1_{\{\epsilon=-1\}}, \qquad 0 \le t \le T, \tag{2.6}$$

where the processes $(Z_t^+)_{0\le t\le T}$ and $(Z_t^-)_{0\le t\le T}$ are given by $Z_0^+ = Z_0^- = 1$ and

$$\begin{aligned} dZ_t^+ &= \mu_t^+ Z_t^+ dY_t, \\ dZ_t^- &= -\mu_t^- Z_t^- dY_t, \end{aligned}$$

respectively. Note that μ_t^+, μ_t^- are assumed to be $\mathbb{F}^Y$-predictable and thus $(Z_t^+)_{0\le t\le T}$ and $(Z_t^-)_{0\le t\le T}$ are $\mathbb{F}^Y$-predictable, too. By the assumption on μ_t, Z_t is clearly $\mathbb{F}^B$-predictable.

We claim that, under $\mathbb{Q}_T$, ϵ is independent of Y and has the same law as under $\mathbb{P}$. Indeed, as Z_T is $\mathcal{F}_T^B$-measurable and ϵ is independent of B (and hence of S) under $\mathbb{P}$, for any bounded measurable functions h, j we have

$$\mathbb{E}_{\mathbb{Q}_T}[h(S)j(\epsilon)] = \mathbb{E}_{\mathbb{P}}[(1/Z_T)h(S)]\mathbb{E}_{\mathbb{P}}[j(\epsilon)] = \mathbb{E}_{\mathbb{Q}_T}[h(S)]\mathbb{E}_{\mathbb{Q}_T}[j(\epsilon)],$$

showing the $\mathbb{Q}_T$-independence of S and ϵ as well as $\mathbb{Q}_T[\epsilon = \pm 1] = 1/2$.

Now note that, under $\mathbb{Q}_T$, S is a Brownian motion. By symmetry of the Brownian motion we also get that $Y = \epsilon S$ and ϵ are $\mathbb{Q}_T$-independent as claimed above.

Clearly, we have $1_{\{\epsilon=1\}} = \frac{\epsilon+1}{2}$. Thus for calculating

$$\mathbb{E}_{\mathbb{P}}[1_{\{\epsilon=1\}} \mid \mathcal{F}_t^Y] = \frac{1}{2}(\mathbb{E}_{\mathbb{P}}[\epsilon \mid \mathcal{F}_t^Y] + 1), \qquad t \ge 0$$

we first calculate $\mathbb{E}_{\mathbb{P}}[\epsilon \mid \mathcal{F}_t^Y], t \ge 0$. Fix any $T > 0$ and consider that by Bayes' formula and the tower law applied to the $(\mathbb{Q}_T, \mathbb{F}^{\epsilon,B})$-martingale Z_T, it holds that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[\epsilon \mid \mathcal{F}_t^Y] &= \frac{\mathbb{E}_{\mathbb{Q}_T}[\epsilon Z_T \mid \mathcal{F}_t^Y]}{\mathbb{E}_{\mathbb{Q}_T}[Z_T \mid \mathcal{F}_t^Y]} = \\ &= \frac{\mathbb{E}_{\mathbb{Q}_T}[\epsilon \mathbb{E}_{\mathbb{Q}_T}[Z_T \mid \mathcal{F}_t^{\epsilon,B}] \mid \mathcal{F}_t^Y]}{\mathbb{E}_{\mathbb{Q}_T}[\mathbb{E}_{\mathbb{Q}_T}[Z_T \mid \mathcal{F}_t^{\epsilon,B}] \mid \mathcal{F}_t^Y]} = \\ &= \frac{\mathbb{E}_{\mathbb{Q}_T}[\epsilon Z_t \mid \mathcal{F}_t^Y]}{\mathbb{E}_{\mathbb{Q}_T}[Z_t \mid \mathcal{F}_t^Y]}, \qquad 0 \le t \le T.\end{aligned}$$

By independence of ϵ and Y (under $\mathbb{Q}_T$) and $\mathbb{F}^Y$-adaptedness of $(Z_t^+)_{0\le t\le T}$ and $(Z_t^-)_{0\le t\le T}$ we get

$$\begin{aligned}&\frac{\mathbb{E}_{\mathbb{Q}_T}[\epsilon Z_t \mid \mathcal{F}_t^Y]}{\mathbb{E}_{\mathbb{Q}_T}[Z_t \mid \mathcal{F}_t^Y]} = \frac{\mathbb{E}_{\mathbb{Q}_T}[Z_t^+ 1_{\{\epsilon=1\}} \mid \mathcal{F}_t^Y] - \mathbb{E}_{\mathbb{Q}_T}[Z_t^- 1_{\{\epsilon=-1\}} \mid \mathcal{F}_t^Y]}{\mathbb{E}_{\mathbb{Q}_T}[Z_t^+ 1_{\{\epsilon=1\}} \mid \mathcal{F}_t^Y] + \mathbb{E}_{\mathbb{Q}_T}[Z_t^- 1_{\{\epsilon=-1\}} \mid \mathcal{F}_t^Y]} = \frac{Z_t^+ - Z_t^-}{Z_t^+ + Z_t^-} \\ &= \frac{\exp\big(\int_0^t \mu_u^+ dY_u - (1/2)\int_0^t (\mu_u^+)^2 du\big) - \exp\big(-\int_0^t \mu_u^- dY_u - (1/2)\int_0^t (\mu_u^-)^2 du\big)}{\exp\big(\int_0^t \mu_u^+ dY_u - (1/2)\int_0^t (\mu_u^+)^2 du\big) + \exp\big(-\int_0^t \mu_u^- dY_u - (1/2)\int_0^t (\mu_u^-)^2 du\big)} \\ &= \frac{\exp\big(\int_0^t (\mu_u^+ + \mu_u^-) dY_u - (1/2)\int_0^t [(\mu_u^+)^2 - (\mu_u^-)^2] du\big) - 1}{\exp\big(\int_0^t (\mu_u^+ + \mu_u^-) dY_u - (1/2)\int_0^t [(\mu_u^+)^2 - (\mu_u^-)^2] du\big) + 1}, \qquad 0 \le t \le T.\end{aligned}$$

So we have

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[1_{\{\epsilon=1\}} \mid \mathcal{F}_t^Y] &= \frac{1}{2}\left(\frac{\mathbb{E}_{\mathbb{Q}_T}[\epsilon Z_t \mid \mathcal{F}_t^Y]}{\mathbb{E}_{\mathbb{Q}_T}[Z_t \mid \mathcal{F}_t^Y]} + 1\right) = \\ &= \frac{\exp\big(\int_0^t (\mu_u^+ + \mu_u^-) dY_u - (1/2)\int_0^t [(\mu_u^+)^2 - (\mu_u^-)^2] du\big)}{\exp\big(\int_0^t (\mu_u^+ + \mu_u^-) dY_u - (1/2)\int_0^t [(\mu_u^+)^2 - (\mu_u^-)^2] du\big) + 1},\end{aligned} \tag{2.7}$$

for $0 \le t \le T$. Define the process $(U_t)_{t\ge0}$ given by $U_0 = 0$ and

$$dU_t = (\mu_u^+ + \mu_u^-)dY_t - \frac{(\mu_u^+)^2 - (\mu_u^-)^2}{2} dt.$$

Applying the Itô formula to (2.7) and recalling the expression for $(Y_t)_{t\ge0}$ we get

$$\begin{aligned}d\frac{\exp(U_t)}{\exp(U_t)+1} = \frac{\exp(U_t)}{(\exp(U_t)+1)^2}&\left(\epsilon\frac{(\mu_t^+ + \mu_t^-)^2}{2}dt + \epsilon(\mu_t^+ + \mu_t^-)dB_t\right) \\ &+ \frac{1}{2}\frac{\exp(U_t) - \exp(2U_t)}{(\exp(U_t)+1)^3}(\mu_t^+ + \mu_t^-)^2 dt.\end{aligned}$$

Using $p_t = \frac{\exp(U_t)}{\exp(U_t)+1}, t \ge 0$ we get (2.4). □

We also have

Proposition 2.2. *Under the assumptions of Proposition 2.1 suppose, in addition, that for all $u \geq 0$,*

$$p_u \mu_u^+ - (1-p_u)\mu_u^- = 0 \ a.s., \tag{2.8}$$

then the process Y is an $\mathbb{F}^Y$-Brownian motion.

Proof. Obviously $(Y_t)_{t \geq 0}$ is $\mathbb{F}^Y$-adapted, continuous and has the right quadratic variation as the drift is bounded. In order to fulfill Lévy's characterization theorem of Brownian motion we need to check the martingale condition. Therefore fix $s \leq t < \infty$ and consider that

$$\mathbb{E}[\epsilon\big(S_t - S_s\big) \mid \mathcal{F}_s^Y] = \mathbb{E}[\int_s^t \epsilon \mu_u du \mid \mathcal{F}_s^Y] + \mathbb{E}[\epsilon(B_t - B_s) \mid \mathcal{F}_s^Y].$$

The second conditional expectation is 0. The martingale property is thus equivalent to

$$\mathbb{E}[\int_s^t \epsilon \mu_u du 1_A] = 0, \tag{2.9}$$

for all $A \in \mathcal{F}_s^Y$. Note that the Fubini theorem applies as $|\epsilon \mu_u 1_A|$ is bounded. Furthermore using the tower law we get that (2.9) holds iff

$$\mathbb{E}[\int_s^t \epsilon \mu_u du 1_A] = \int_s^t \mathbb{E}[\mathbb{E}[\epsilon \mu_u \mid \mathcal{F}_u^Y] 1_A] du = 0,$$

for all $A \in \mathcal{F}_u^Y$. Recall that μ_u^+ and μ_u^- are assumed to be $\mathcal{F}_u^Y$-measurable for $u \geq 0$. It follows from the hypotheses of this Proposition that

$$\mathbb{E}[\epsilon \mu_u \mid \mathcal{F}_u^Y] = p_u \mu_u^+ - (1-p_u)\mu_u^- = 0,$$

concluding the proof. ◻

Formula (2.8) shows that it is reasonable to choose μ_t^+ proportional to $(1-p_t)$ and μ_t^- proportional to p_t. This will guarantee the validity of (2.8) as the next Proposition shows.

Proposition 2.3. *Let $\mu > 0$ be an arbitrary constant. Let g_t be a solution of the equation*

$$dg_t = 2\mu^2 \left[\epsilon g_t(1-g_t) + g_t(1-g_t)\left(1-2g_t\right)\right] dt + 2\mu\epsilon g_t(1-g_t) dB_t, \ g_0 = 1/2, \tag{2.10}$$

adapted to the filtration $\mathbb{F}^{\epsilon,B}$ and satisfying $0 \leq g_t \leq 1$ for $t \geq 0$. Set

$$\mu_t^+ = 2\mu(1-g_t), \quad \mu_t^- = 2\mu g_t, \quad t \geq 0, \tag{2.11}$$

and define μ_t, S_t, Y_t, p_t accordingly. If g_t is $\mathbb{F}^Y$-predictable and μ_t is $\mathbb{F}^B$-predictable then p_t equals g_t and Y is a Brownian motion in its own filtration.

Proof. First note that the coefficients of the autonomous SDE (2.10) are Lipschitz-continuous and bounded when restricted to the interval $[0,1]$ hence g_t is the *unique strong* solution of (2.10) adapted to $\mathbb{F}^{\epsilon,B}$. Thus it suffices to prove that p_t is also a solution of (2.10). Obviously, $p_0 = \mathbb{P}[\epsilon = 1] = 1/2 = g_0$.

If g_t is $\mathbb{F}^Y$-predictable then so are μ_t^+, μ_t^-. Proposition 2.1 shows that p_t is a solution of (2.10) hence indeed $p_t = g_t$. With this choice of μ_t^+, μ_t^- equation (2.8) is satisfied and Proposition 2.2 allows us to conclude. □

It remains to solve the stochastic differential equation (2.10).

Proof of Proposition 1.1. In section 4.2 of [4] a different filtering problem leads to almost the same equation as (2.10). That equation has an explicit solution (see (4.55) on page 180) from which it is easy to make the guess

$$g_t := \begin{cases} \frac{\exp(2\mu B_t + 2\mu^2 t)}{1+\exp(2\mu B_t+2\mu^2 t)}, & \text{if } \epsilon = 1 \\ \frac{1}{1+\exp(2\mu B_t+2\mu^2 t)}, & \text{if } \epsilon = -1. \end{cases} \tag{2.12}$$

Applying Itô's formula we may check that this indeed gives a (strong) solution to (2.10) which trivially stays in $(0,1)$. Define $\mu_t^+, \mu_t^-, \mu_t, S_t, Y_t, p_t$ as in Proposition 2.3. One may check that

$$dg_t = 2\mu g_t(1-g_t)dY_t, \tag{2.13}$$

showing that g_t is $\mathbb{F}^Y$-predictable. We find that the dynamics of μ_t is

$$d\mu_t = -\mu_t^2(2\mu - \mu_t)dt - \mu_t(2\mu - \mu_t)dB_t, \tag{2.14}$$

hence μ_t is $\mathbb{F}^B$-predictable. For later use we note that, substituting in to (2.11) we get the following formula for μ_t:

$$\mu_t = \frac{2\mu}{1+\exp(2\mu B_t + 2\mu^2 t)}. \tag{2.15}$$

Proposition 2.3 now implies that $p_t = g_t$ and Y_t is indeed as required. □

3. Passing to the Lévy transform

In this section we describe how to get rid of the enlargement of the filtration $\mathbb{F}^B$ by the sign ϵ. We will make use of the Lévy transform which arises naturally in the famous Tanaka formula for the SDE of $(|B_t|)_{t\geq 0}$ for some Brownian motion B (for the derivation of the Tanaka formula see e.g. [3]).

Recall that the Lévy transform $(M_t^0)_{t\geq 0}$, of a Brownian motion $(B_t)_{t\geq 0}$ is defined by

$$M_t^0 = \int_0^t \operatorname{sign}(B_s)dB_s, \qquad t \geq 0, \tag{3.1}$$

where we use the sign function in the following left-continuous form

$$\operatorname{sign}(x) = 1_{\{x>0\}} - 1_{\{x\le 0\}}, \qquad \text{for } x \in \mathbb{R}.$$

Among the properties of the Lévy transform we mention that $(M_t^0)_{t\ge 0}$ is a Brownian motion in its own filtration and that the filtration generated by $(M_t^0)_{t\ge 0}$ equals the one generated by $(|B_t|)_{t\ge 0}$ which is strictly smaller than the filtration generated by $(B_t)_{t\ge 0}$.

Proof of i) in Theorem 1.2. We come back to the setting of section 2 and consider the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}^{\epsilon,B}, \mathbb{P})$. Let us take μ_t, S_t as constructed in the proof of Proposition 1.1. Introduce $(Y_t)_{t\ge 0} = (\epsilon S_t)_{t\ge 0}$, this is a Brownian motion in its own filtration, by Proposition 1.1.

Now consider the Lévy transform $(M_t)_{t\ge 0}$ of the $\mathbb{F}^Y$-Brownian motion $Y = (Y_t)_{t\ge 0}$. It is defined by $M_0 = 0$ and

$$\begin{aligned} dM_t &= \operatorname{sign}(Y_t)dY_t = \operatorname{sign}(\epsilon S_t)\epsilon dS_t \\ &= \operatorname{sign}(S_t)dS_t, \qquad t \ge 0. \end{aligned} \tag{3.2}$$

This is again a Brownian motion (in $\mathbb{F}^Y$ as well as in $\mathbb{F}^M$) by the properties of the Lévy-transform. It follows that with the choice $H_t = \operatorname{sign}(S_t)$, the process $(H\cdot S)_t, t \ge 0$ is a Brownian motion in its own filtration. □

4. L^∞-approximation of a constant drift

The aim of this section is to show that we can in fact define a process $(S_t)_{t\ge 0}$ such that the drift is close to a constant drift μ with respect to the norm in L^∞. The strategy is that we stop whenever the drift $(\mu_t)_{t\ge 0}$ has moved by some small fixed number. After that we will restart the construction. A somewhat delicate point is that the stopping has to be done in a way adapted to $\mathbb{F}^M$. Lemma 4.1 below shows that this is indeed possible.

The distance of the drift process μ_t from μ is proportional to the distance of p_t from one half. Namely, by (2.2) and (2.11),

$$\begin{aligned} |\mu_t - \mu| &= |2\mu(1-p_t) - \mu| 1_{\{\epsilon=1\}} + |2\mu p_t - \mu| 1_{\{\epsilon=-1\}} \\ &= 2\mu \left| \frac{1}{2} - p_t \right|, \qquad \text{for } t \ge 0. \end{aligned} \tag{4.1}$$

In the following lemma we show that one can define a stopping time in the filtration generated by the Lévy transform $(M_t)_{t\ge 0}$ of the Brownian motion $(Y_t)_{t\ge 0}$, i.e. $\mathbb{F}^M := (\mathcal{F}_t^M)_{t\ge 0}$, such that we have a control over the distance of p from $1/2$.

Lemma 4.1. *Take p_t, μ_t, S_t, Y_t as constructed in the proof of Proposition 1.1 in section 2 and consider the Lévy transform $M = (M_t)_{t\ge 0}$ of $Y = (Y_t)_{t\ge 0}$, defined by $M_0 = 0$ and*

$$dM_t = \operatorname{sign}(Y_t)dY_t.$$

For each $\delta > 0$ we define the stopping time $\rho_\delta := \inf\{t : |M_t| \geq \delta\} \wedge \delta$. The following estimate holds:

$$\left| p_t - \frac{1}{2} \right| \leq \delta \frac{(2\mu + 3\mu^2)}{2} \qquad \text{for } 0 \leq t \leq \rho_\delta. \tag{4.2}$$

Proof. We present the proof in two steps.
Step 1: We show that $|Y_t| \leq 2\delta$, for $0 \leq t \leq \rho_\delta$.

Let $\tau := \inf\{t : |Y_t| \geq 2\delta\}$ and let $\sigma := \max\{t \leq \tau : Y_t = 0\}$, i.e. the time of the last zero of Y preceding τ. We note in passing that τ is a stopping time in the filtration $\mathbb{F}^Y$, while σ fails to be a stopping time. Observe that by Tanaka's formula (see for instance [3])

$$M_t = |Y_t| - L_t,$$

where L is the local time of Y at zero. By definition of σ the local time L does not grow on $[\sigma, \tau]$ and thus a.s. $L_\sigma = L_\tau$. For the process M this gives

$$|M_\sigma - M_\tau| = 2\delta,$$

so that $\sup_{0 \leq t \leq \tau} |M_t| \geq \delta$, which shows that a.s. $\rho_\delta \leq \tau$, i.e. $|Y_t| \leq 2\delta$ for $0 \leq t \leq \rho_\delta$.

Step 2: By straightforward calculation, $|p_t - (1/2)| = (1/2)|\mathrm{th}(\mu^2 t + \mu B_t)|$, where th denotes the hyperbolic tangent. As $|\mathrm{th} x| \leq |x|$, $dY_t = \epsilon \mu_t dt + \epsilon dB_t$ and $\mu_t \in]0, 2\mu[$ we may write, for $t \leq \rho_\delta$,

$$\left| p_t - \frac{1}{2} \right| \leq \mu^2 t/2 + \mu |B_t|/2 \leq \mu^2 t/2 + \mu |Y_t|/2 + \mu \left| \int_0^t \mu_s ds \right| /2 \leq \mu^2 \delta/2 + \mu(2\delta)/2 + \mu^2 \delta,$$

using Step 1 and $\rho_\delta \leq \delta$. □

Using the previous Lemma we can refine the construction by stopping and restarting, when we are too far away from a constant drift, considering the information of $\mathbb{F}^M$ only. Fix the constant $\mu > 0$. For the goal of controlling the L^∞ distance of μ and the drift process μ_t to be constructed fix also a constant $\delta > 0$.

The strategy is straightforward. We start at $t = 0$ using the drift $(\mu_t^1)_{t \geq 0}$ which is given by $\mu_0^1 = \mu$ and (2.14). Define the process S^1 by $S_0^1 = 0$,

$$dS_t^1 = \mu_t^1 dt + dB_t, \qquad t \geq 0.$$

We perform the Lévy transform which results in a process $(M_t^1)_{t \geq 0}$. Introducing the $\mathbb{F}^{M^1}$-stopping time

$$\tau_1 := \inf\{t > 0 : |M_t^1| \geq \delta\} \wedge \delta$$

we can assure by Lemma 4.1 and (4.1) that

$$|\mu_t^1 - \mu| \leq \delta(3\mu^3 + 2\mu^2) \qquad \text{for } 0 \leq t \leq \tau_1. \tag{4.3}$$

Then after τ_1 we restart the construction by defining the drift $(\mu_t^2)_{t\geq\tau_1}$ where $\mu_{\tau_1}^2 = \mu$ and $(\mu_t^2)_{t\geq\tau_1}$ fulfills (2.14). Set $S_{\tau_1}^2 = 0$ and

$$dS_t^2 = \mu_t^2 dt + dB_t, \qquad t \geq \tau_1.$$

Furthermore we perform the Lévy transform resulting in $(M_t^2)_{t\geq\tau_1}$ and we define the stopping time

$$\tau_2 := \inf\{t > \tau_1 : |M_t^2| \geq \delta\} \wedge (\delta + \tau_1).$$

By this construction we have that the estimate (4.3) holds for $(\mu_t^2)_{\tau_1\leq t\leq\tau_2}$, and we may continue the construction in the same fashion.

Now we proceed formally:

Set $\tau_0 = 0$ and define recursively for $l \geq 1$ $(\tilde{S}_t^l)_{t\geq 0}$ by $\tilde{S}_0^l = 0$ and

$$d\tilde{S}_t^l = \tilde{\mu}_t^l dt + dW_t^l, \qquad t \geq 0,$$

where the Brownian motion $(W_t^l)_{t\geq 0}$ is given by

$$W_t^l := B_{\tau_{l-1}+t} - B_{\tau_{l-1}}, \qquad t \geq 0,$$

and the drift process $(\tilde{\mu}_t^l)_{t\geq 0}$ is given by

$$\tilde{\mu}_t^l = \frac{2\mu}{1 + \exp(2\mu W_t^l + 2\mu^2 t)}, \tag{4.4}$$

(compare to (2.15)).

The integrand $(H_t^l)_{t\geq 0}$ is defined analogously to Section 3 by

$$H_t^l = \operatorname{sign}(\tilde{S}_t^l), \qquad t \geq 0,$$

and the stopping time γ_l is defined by

$$\gamma_l := \inf\{t : |(H^l \cdot \tilde{S}^l)_t| \geq \delta\} \wedge \delta. \tag{4.5}$$

Then we set

$$\tau_l = \tau_{l-1} + \gamma_l \tag{4.6}$$

and go on with the recursive definition.

Finally for $l \geq 1$ we introduce the processes $(\tilde{N}_t^l)_{t\geq 0}$ and $(N_t^l)_{t\geq 0}$ by

$$\tilde{N}_t^l := (H^l \cdot \tilde{S}^l)_t, \qquad t \geq 0$$

and

$$N_t^l := \tilde{N}_{t\wedge\gamma_l}^l, \qquad t \geq 0.$$

Note that by the considerations of sections 2 and 3 $(\tilde{N}_t^l)_{t\geq 0}$ as well as its stopped version $(N_t^l)_{t\geq 0}$ are martingales in their own filtrations for $l \geq 1$.

Remark 4.2. It is evident that $(\gamma_l)_{l\geq 1}$ as well as $\{(W_t^l)_{0\leq t\leq\gamma_l}\}_{l\geq 1}$ are i.i.d. sequences. By $\sigma(N_t^l, t \geq 0) \subseteq \mathcal{F}_{\gamma_l}^{W^l}$ it holds that $(N_t^l)_{l\geq 1}$ are independent (and identically distributed) processes and that $\mathcal{F}_{\tau_{l-1}}^B$ is independent of $(W_t^l)_{t\geq 0}$ for $l \geq 1$. By these observations it follows that $\mathcal{F}_{\tau_{l-1}}^B$ is independent of $(N_t^l)_{t\geq 0}$ for $l \geq 1$.

We need to show that the union of the stochastic intervals $\bigcup_{l\geq 1} [\![\tau_{l-1}, \tau_l]\!]$ equals the whole real line.

Lemma 4.3. *Let $(\tau_l)_{l=0}^\infty$ be defined by $\tau_0 = 0$ and (4.6) for $l \geq 1$. Then*

$$\mathbb{P}[\tau_l \to \infty,\ l \to \infty] = 1.$$

Proof. We already noticed in Remark 4.2 that the interval lengths $\tau_l - \tau_{l-1} = \gamma_l$ are positive and identically distributed. A well-known result (see e.g. [2, Proposition 4.14]) tells us that

$$\lim_{l\to\infty} \tau_l = \sum_{l=1}^{\infty} (\tau_l - \tau_{l-1}) = \infty$$

almost surely. □

The last step is to show that the process which is given by the concatenation of the Lévy transforms $N^l, l \geq 1$ on the respective stochastic intervals is a Brownian motion in its own filtration. We want to apply Lévy 's criterion and first concentrate on proving the martingale property. We need three lemmas:

Lemma 4.4. *Let $(G_t)_{t\geq 0}$ be a martingale in its own filtration. Then*

$$(G_{(t-x)^+})_{t\geq 0}$$

is also a martingale in its own filtration for each fixed number $x \geq 0$.

Proof. Obvious. □

Lemma 4.5. *Let η be an $\mathcal{F}^B$-stopping time and $(G_t)_{t\geq 0}$ be a continuous martingale in its own filtration such that $\mathcal{F}_\eta^B \perp\!\!\!\perp (G_t)_{t\geq 0}$. Define the filtration $\mathcal{F}_t := \mathcal{F}_\eta^B \vee \mathcal{G}_t$, where $\mathcal{G}_t := \sigma(G_{(u-\eta)^+}, 0 \leq u \leq t)$, then $(G_{(t-\eta)^+})_{t\geq 0}$ is a martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$.*

Proof. We want to show that

$$\mathbb{E}[G_{(t-\eta)^+} \mid \mathcal{F}_\eta^B \vee \sigma(G_{(u-\eta)^+}, 0 \leq u \leq s)] = G_{(s-\eta)^+}.$$

Note that

$$\mathcal{F}_\eta^B \vee \sigma(G_{(u-\eta)^+}, 0 \leq u \leq s) \subseteq \mathcal{F}_\eta^B \vee \sigma(G_{(u-\cdot)^+}, 0 \leq u \leq s)$$

and consider an event in the latter sigma-algebra given by

$$A := \{B(\cdot \wedge \eta) \in C, L(\cdot) \in D\},$$

with Borel sets C, D of $C[0,\infty)$ where $C[0,\infty)$ is the space of continuous functions on $[0,\infty)$ equipped with the topology of uniform convergence on compacts. We regard $B(\cdot\wedge\eta)$ as a random function

$$B(\cdot\wedge\eta):\Omega\to C[0,\infty),$$

and

$$L(\cdot):=G_{(u_1-\cdot)^+}:\Omega\to C[0,\infty)$$

for some $0\le u_1\le s$. Furthermore, we define the random functions

$$H_1(\cdot):=G_{(t-\cdot)^+}:\Omega\to C[0,\infty)\qquad\text{and}$$
$$H_2(\cdot):=G_{(s-\cdot)^+}:\Omega\to C[0,\infty).$$

Now consider that the law ν of $(H_1,H_2,B(\cdot\wedge\eta),L,\eta)$ on the space

$$\Theta:=(C[0,\infty))^4\times(\mathbb{R}_+\cup\{\infty\}),$$

can be decomposed as

$$d\nu(x_1,x_2,x_3,x_4,x_5)=d\nu(x_1,x_2,x_4\mid x_3,x_5)d\mu(x_3,x_5),$$

where μ is the law of $(B(\cdot\wedge\eta),\eta)$ and $\nu(\cdot,\cdot,\cdot\mid x_3,x_5)$ is the conditional law of (H_1,H_2,L) knowing $B(\cdot\wedge\eta)=x_3$ and $\eta=x_5$. The martingale property of $G_{(t-x)^+}$ for each $x\ge 0$ (Lemma 4.4) implies

$$\int_\Theta x_1(x)1_D(x_4(x))d\vartheta(x_1,x_2,x_4)=\int_\Theta x_2(x)1_D(x_4(x))d\vartheta(x_1,x_2,x_4),\tag{4.7}$$

for each $x\ge 0$ where ϑ is the (unconditional) law of (H_1,H_2,L).

Furthermore, the hypotheses of the Lemma entail the independence of H_1,H_2,L from $B(\cdot\wedge\eta)$ and η, so it follows that $\nu(x_1,x_2,x_4\mid x_3,x_5)$ does not depend on (x_3,x_5) and thus

$$d\nu(x_1,x_2,x_4\mid x_3,x_5)=d\vartheta(x_1,x_2,x_4).\tag{4.8}$$

By the decomposition of ν and (4.8) we can write

$$\begin{aligned}\mathbb{E}[G_{(t-\eta)^+}1_A]&=\int_\Theta x_1(x_5)1_C(x_3)1_D(x_4(x_5))d\nu(x_1,x_2,x_3,x_4,x_5)\\&=\int_\Theta x_1(x_5)1_D(x_4(x_5))d\vartheta(x_1,x_2,x_4)1_C(x_3)d\mu(x_3,x_5)\end{aligned}$$

which by (4.7) equals

$$\int_\Theta x_2(x_5)1_D(x_4(x_5))d\vartheta(x_1,x_2,x_4)1_C(x_3)d\mu(x_3,x_5)=\mathbb{E}[G_{(s-\eta)^+}1_A].$$

For more general sets A of the form

$$A:=\{B(\cdot\wedge\eta)\in C,G_{(u_1-\eta)^+}\in D_1,\dots,G_{(u_n-\eta)^+}\in D_n\}$$

the equality

$$\mathbb{E}[G_{(t-\eta)^+}1_A]=\mathbb{E}[G_{(s-\eta)^+}1_A]$$

holds by the same argument, which proves the lemma. □

Lemma 4.6. *Let X be a random variable, $\mathcal{G}$ and $\mathcal{H}$ be sigma-algebras in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If $\mathcal{G} \perp\!\!\!\perp \mathcal{H}$ and $X \perp\!\!\!\perp \mathcal{H}$, then*

$$\mathbb{E}[X \mid \mathcal{G} \vee \mathcal{H}] = \mathbb{E}[X \mid \mathcal{G}].$$

Proof. Obvious. □

Finally define the process $(S_t)_{t\geq 0}$ using $\tilde{S}^l, l \geq 1$ and $(H_t)_{t\geq 0}$ using $H^l, l \geq 1$:

$$S_t := \sum_{j=1}^{l-1} \tilde{S}^j_{\gamma_j} + \tilde{S}^l_{t-\tau_{l-1}} \qquad \text{for } \tau_{l-1} \leq t \leq \tau_l,$$

thus the drift $(\mu_t)_{t\geq 0}$ of $(S_t)_{t\geq 0}$ is given by

$$\mu_t := \tilde{\mu}^l_{t-\tau_{l-1}} \qquad \text{for } \tau_{l-1} \leq t \leq \tau_l, \tag{4.9}$$

and the integrand is defined as

$$H_t := H^l_{t-\tau_{l-1}} \qquad \text{for } \tau_{l-1} \leq t \leq \tau_l.$$

We obviously have

$$S_t = \int_0^t \mu_s ds + B_t.$$

Then the stochastic integral $(M_t)_{t\geq 0}$ is defined by

$$M_t := (H \cdot S)_t = \sum_{l=1}^{k-1} (H^l \cdot \tilde{S}^l)_{\gamma_l} + (H^k \cdot \tilde{S}^k)_{t-\tau_{k-1}} \qquad \text{for } \tau_{k-1} \leq t \leq \tau_k. \tag{4.10}$$

Note that, by construction, S_t and M_t are continuous processes.

Proposition 4.7. *The process $(M_t)_{t\geq 0}$ as defined in (4.10) satisfies*

$$M_t = \sum_{l=1}^{\infty} N^l_{(t-\tau_{l-1})^+}, \tag{4.11}$$

where the sum converges in L^2. $(M_t)_{t\geq 0}$ is a martingale in its own filtration. That is, for $0 \leq s < t < \infty$

$$\mathbb{E}[M_t | \mathcal{F}^M_s] = M_s.$$

Proof. First we show that the sum on the right-hand-side of (4.11) converges in L^2. Note that

$$\begin{aligned}
\sum_{l=1}^{k} N^l_{(t-\tau_{l-1})^+} &= \int_0^{\tau_1 \wedge t} \operatorname{sign}(\tilde{S}^1_{s-\tau_0}) \tilde{\mu}^1_{s-\tau_0} ds + \cdots + \int_{\tau_{k-1}\wedge t}^{\tau_k \wedge t} \operatorname{sign}(\tilde{S}^k_{s-\tau_{k-1}}) \tilde{\mu}^k_{s-\tau_{k-1}} ds \\
&\quad + \int_0^{\tau_1 \wedge t} \operatorname{sign}(\tilde{S}^1_{s-\tau_0}) dB_s + \cdots + \int_{\tau_{k-1}\wedge t}^{\tau_k \wedge t} \operatorname{sign}(\tilde{S}^k_{s-\tau_{k-1}}) dB_s \\
&= \int_0^{\tau_k \wedge t} H_s \mu_s ds + \int_0^{\tau_k \wedge t} H_s dB_s,
\end{aligned}$$

which by Lemma 4.3 a.s. converges to

$$M_t = \int_0^t H_s \mu_s ds + \int_0^t H_s dB_s \qquad \text{as } k \to \infty. \tag{4.12}$$

Furthermore, we know from Proposition 2.3 that

$$|H_t \mu_t| \leq 2\mu \qquad \text{for } t \geq 0.$$

By the Doob inequality and the Itô isometry we get

$$\mathbb{E}[\sup_{0 \leq u \leq t} |\int_0^u H_s dB_s|^2] \leq 4\mathbb{E}[(\int_0^t H_s dB_s)^2] = 4t,$$

as $|H_s| = 1$. The L^2 convergence of the infinite sum follows.

Now we prove that $(M_t)_{t \geq 0}$ is a martingale in its own filtration. Define the filtrations $(\mathcal{G}_t^l)_{t \geq 0}$ for $l \geq 1$ by

$$\mathcal{G}_t^l := \sigma(N^l_{(u-\tau_{l-1})^+}, 0 \leq u \leq t),$$

and consider that

$$\mathbb{E}[M_t \mid \mathcal{F}_s^M] = \mathbb{E}[\sum_{l=1}^{\infty} \mathbb{E}[N^l_{(t-\tau_{l-1})^+} \mid \bigvee_{j=1}^{\infty} \mathcal{G}_s^j] \mid \mathcal{F}_s^M], \tag{4.13}$$

where L^2-convergence allows us to exchange summation and expectation and we used that $\mathcal{F}_s^M \subseteq \bigvee_{j=1}^{\infty} \mathcal{G}_s^j$ for $s \geq 0$. Furthermore, notice that

$$\bigvee_{j=1}^{\infty} \mathcal{G}_s^j \subseteq \bigvee_{j=1}^{l} \mathcal{G}_s^j \vee \sigma(N^{l+1}, N^{l+2}, \ldots).$$

To see this, note that for $m \geq l+1$ we have that $\mathcal{G}_s^m \subseteq \sigma(N^m, \tau_{m-1} \wedge s)$ and that by definition $\sigma(\tau_{m-1} \wedge s) \subseteq \sigma(N^{m-1}, \tau_{m-2} \wedge s)$. Continuing this inductively we get $\mathcal{G}_s^m \subseteq \sigma(N^m, \ldots, N^{l+1}, \tau_l \wedge s)$ and finally, $\sigma(\tau_l \wedge s) \subseteq \mathcal{G}_s^l$.

Thus define $\mathcal{H}_s^l := \bigvee_{j=1}^{l} \mathcal{G}_s^j \vee \sigma(N^{l+1}, N^{l+2}, \ldots)$ and recall that $\bigvee_{j=1}^{l} \mathcal{G}_s^j \subseteq \mathcal{F}^B_{\tau_l \wedge s}$ which is independent of $N^k, k \geq l+1$, recall Remark 4.2. By Lemma 4.6 and the tower law the inner conditional expectation in (4.13) is given by

$$\begin{aligned}\mathbb{E}[\mathbb{E}[N^l_{(t-\tau_{l-1})^+} \mid \mathcal{H}_s^l] \mid \bigvee_{j=1}^{\infty} \mathcal{G}_s^j] &= \mathbb{E}[\mathbb{E}[N^l_{(t-\tau_{l-1})^+} \mid \bigvee_{j=1}^{l} \mathcal{G}_s^j] \mid \bigvee_{j=1}^{\infty} \mathcal{G}_s^j] \\ &= \mathbb{E}[N^l_{(t-\tau_{l-1})^+} \mid \bigvee_{j=1}^{l} \mathcal{G}_s^j].\end{aligned}$$

Note that $\bigvee_{j=1}^{l}\mathcal{G}_s^j \subseteq \mathcal{F}_s^l := \mathcal{F}_{\tau_{l-1}}^B \vee \mathcal{G}_s^l$. By applying Lemma 4.5 with $(G_t)_{t\geq 0} = (N_t^l)_{t\geq 0}$ we get that

$$\mathbb{E}[N_{(t-\tau_{l-1})^+}^l \mid \bigvee_{j=1}^{l}\mathcal{G}_s^j] = N_{(s-\tau_{l-1})^+}^l.$$

Substituting these results into (4.13) and using the representation (4.11) for M_s we get that

$$\mathbb{E}[M_t \mid \mathcal{F}_s^M] = \mathbb{E}[\sum_{l=1}^{\infty} N_{(s-\tau_{l-1})^+}^l \mid \mathcal{F}_s^M] = \mathbb{E}[M_s \mid \mathcal{F}_s^M] = M_s.$$

□

Proof of ii) in Theorem 1.2. By construction and by Proposition 4.7 $(M_t)_{t\geq 0}$ is a continuous martingale and its bracket is $\langle M\rangle_t = t$ by (4.12) and by $|H_s| = 1, s \geq 0$, hence $(M_t)_{t\geq 0}$ is a Brownian motion (in its own filtration).

For $(\mu_t)_{t\geq 0}$, the drift of S, we conclude that, due to (4.1), Lemma 4.1, (4.5), (4.9) and Lemma 4.3

$$\sup_{t\geq 0} |\mu_t - \mu| \leq \delta(3\mu^3 + 2\mu^2) \qquad \text{a.s.,} \tag{4.14}$$

which can be made arbitrarily small. □

Appendix A: Michel Émery's question

We now take up again the question discussed at the end of the introduction. We adopt the notation from there but assume, slightly more generally, that the independent sequence $(\epsilon_n)_{n\leq 0}$ of $\{-1,1\}$- valued random variables fulfill the condition

$$\mathbb{P}[\epsilon_n = 1] = 1 - \mathbb{P}[\epsilon_n = 1] = p_n, \qquad n \leq 0,$$

for some sequence $(p_n)_{n\leq 0}$ in $]0,1[$ satisfying

$$\sum_{n=-\infty}^{0} \min(p_n, 1-p_n) = \infty. \tag{A.1}$$

In the sequel we call $\{-1,1\}$-valued random variables *Bernoulli* variables and let $\mathbb{N}_-$ denote the integers less than or equal to zero.

The role of the regularity condition (A.1) is explained in the following lemma.

Lemma A.1. *The law of the $\{-1,1\}^{\mathbb{N}_-}$-valued random variable $(\epsilon_n)_{n\leq 0}$ is diffuse iff (A.1) holds true.*

Proof. First assume that there exists an atom $A = (a_0, a_{-1}, \dots)$ with $\mathbb{P}[A] > 0$, then $\prod_{n=0}^{-\infty} p_n^{(1+a_n)/2}(1-p_n)^{(1-a_n)/2} > 0$ which is equivalent to

$$\sum_{n=-\infty}^{0} \left(1 - p_n^{(1+a_n)/2}(1-p_n)^{(1-a_n)/2}\right) = \sum_{n=-\infty}^{0} \left((1-p_n)^{(1+a_n)/2} p_n^{(1-a_n)/2}\right) < \infty,$$

which implies that the sum in (A.1) is finite. On the other hand if the sum in (A.1) is finite, this is equivalent to $\prod_{n=0}^{-\infty} p_n^{(1-a_n)/2}(1-p_n)^{(1+a_n)/2} > 0$ for a sequence $(a_n)_{n\le 0}$ such that

$$p_n^{(1+a_n)/2}(1-p_n)^{(1-a_n)/2} = \min(p_n, 1-p_n) \qquad \text{for } n \le 0,$$

and we find $A = (a_0, a_{-1}, \dots)$ with $\mathbb{P}[A] > 0$. $\square$

We call a $\{-1, 1\}$- valued random variable X *symmetric Bernoulli* if

$$\mathbb{P}[X = 1] = \mathbb{P}[X = -1] = \frac{1}{2}.$$

Lemma A.2. *Let $(\epsilon_n)_{n\le 0}$ be a sequence of Bernoulli random variables, and $(h_n)_{n\le 0}$ an i.i.d. sequence of symmetric Bernoulli variables independent of $(\epsilon_n)_{n\le 0}$. Then*

(a) $(h_n\epsilon_n)_{n\le 0}$ is an i.i.d. sequence of symmetric Bernoulli random variables and

(b)
$$\mathrm{law}[(\epsilon_n)_{n\le 0} | (h_n\epsilon_n)_{n\le 0}] = \mathrm{law}[(\epsilon_n)_{n\le 0}] \qquad a.s.$$

Proof. Fix $N \ge 1$ and consider signs $x_1, \dots, x_N$ as well as indices $i_1, \dots, i_N$. Then by independence of $(h_n)_{n\le 0}$ and $(\epsilon_n)_{n\le 0}$ combined with the i.i.d. property we get

$$\begin{aligned} \mathbb{P}[h_{i_1}\epsilon_{i_1} = x_1, \dots, h_{i_N}\epsilon_{i_N} = x_N] &= \\ \sum_{y_1,\dots,y_n} \mathbb{P}[h_{i_1} = x_1/y_1, \dots, h_{i_N} = x_N/y_N]\mathbb{P}[\epsilon_{i_1} = y_1, \dots, \epsilon_{i_N} = y_N] &= \\ \left(\frac{1}{2}\right)^N \sum_{y_1,\dots,y_n} \mathbb{P}[\epsilon_{i_1} = y_1, \dots, \epsilon_{i_N} = y_N] &= \left(\frac{1}{2}\right)^N, \end{aligned}$$

which proves (a). For proving (b), we fix again $N, M \ge 1$ and consider signs $x_1, \dots, x_N$ and $y_1, \dots, y_N$ as well as indices $i_1, \dots, i_N$ such that $\mathbb{P}[h_{i_1}\epsilon_{i_1} = x_1, \dots, h_{i_N}\epsilon_{i_N} = x_N] > 0$. By independence of $(h_n)_{n\le 0}$ and $(\epsilon_n)_{n\le 0}$ and the previous argument we can calculate that

$$\begin{aligned} &\mathbb{P}[\epsilon_{i_1} = y_1, \dots, \epsilon_{i_N} = y_N \mid h_{i_1}\epsilon_{i_1} = x_1, \dots, h_{i_N}\epsilon_{i_N} = x_N] \\ =&\frac{\mathbb{P}[\epsilon_{i_1} = y_1, \dots, \epsilon_{i_N} = y_N, h_{i_1}\epsilon_{i_1} = x_1, \dots, h_{i_N}\epsilon_{i_N} = x_N]}{\mathbb{P}[h_{i_1}\epsilon_{i_1} = x_1, \dots, h_{i_N}\epsilon_{i_N} = x_N]} \\ =&(1/2)^N \frac{\mathbb{P}[\epsilon_{i_1} = y_1, \dots, \epsilon_{i_N} = y_N]}{(1/2)^N} \\ =&\mathbb{P}[\epsilon_{i_1} = y_1, \dots, \epsilon_{i_N} = y_N], \end{aligned}$$

which proves (b). ☐

Assuming (A.1) we can find disjoint, infinite subsets $(I_n)_{n\leq 0}$ of $\mathbb{N}_-$ such that $i > n$, for all $i \in I_n$, and

$$\sum_{i\in I_n} \min(p_i, 1-p_i) = \infty. \tag{A.2}$$

For these sets we define the following infinite sequence $\left(I^{(l)}\right)_{l=0}^{\infty}$ of subsets of $\mathbb{N}_-$ by

$$\begin{aligned} I^{(0)} &:= I_0 \\ I^{(1)} &= \bigcup_{n\in I^{(0)}} I_n \\ I^{(2)} &= \bigcup_{n\in I^{(1)}} I_n \\ &\vdots \qquad \text{etc.} \end{aligned} \tag{A.3}$$

Additionally to the sequence $\left(I^{(l)}\right)_{l=0}^{\infty}$ from (A.3) we furthermore introduce

$$J = \mathbb{N}_- \setminus \left(\{0\} \cup \bigcup_{l=0}^{\infty} I^{(l)} \right).$$

In the following lemma we summarize three properties of these sets.

Lemma A.3. *For the sequence $\left(I^{(l)}\right)_{l=0}^{\infty}$ defined in (A.3) we have*

(a) $I^{(l)} \subset \{\dots, -l-2, -l-1\}$ for $l \geq 0$.
(b) The sets $\left(I^{(l)}\right)_{l=0}^{\infty}$ are mutually disjoint.
(c) For each $m \in J$ we have $I_m \subset J$.

Proof. Proof of (a): We prove the statement by induction. $0 \notin I^{(0)}$ by construction. Thus assume that the statement holds for $I^{(0)}, \dots, I^{(n)}$. For $I^{(n+1)}$ consider that

$$I^{(n+1)} = \bigcup_{x\in I^{(n)}} I_x,$$

and by the induction hypothesis $x \leq -n-1$ for $x \in I^{(n)}$. But then also $y \leq x-1$, for all $y \in I_x$, thus it follows that $y \leq -n-2$ and $I_x \subset \{\dots, -n-3, -n-2\}$ for each $x \in I^{(n)}$, which proves (a).
Proof of (b): Again by induction: Let us assume that $I^{(0)}, \dots, I^{(n)}$ are pairwise disjoint. We want to prove that $I^{(0)}, \dots, I^{(n+1)}$ are also pairwise disjoint. Take $m \in I^{(n+1)} = \bigcup_{x\in I^{(n)}} I_x$. If we had $m \in I^{(j)}$ for some $1 \leq j \leq n$ then $m \in I_y$ for some $y \in I^{(j-1)}$ and also $m \in I_w$ for some $w \in I^{(n)}$. But as the $I_i, i \in \mathbb{N}_-$ are disjoint this implies $y = w$ so that $I^{(n)} \cap I^{(j-1)} \neq \emptyset$ which is a contradiction. Finally if $m \in I^{(0)}$, then $w = 0$, but $0 \notin I^{(i)}$ for $i \geq 1$ by (a).

Proof of (c): Let $m \in J$ and $x \in I_m$. Let $I^{(-1)} := \{0\}$. Assume that there is a $k \geq 0$ such that $x \in I^{(k)}$, this implies that there is $y \in I^{(k-1)}$ such that $x \in I_y$. Then $m = y$ and $m \in I^{(k-1)}$ which is a contradiction. □

We now can prove a positive answer for M. Émery's question.

Theorem A.4. *Let $(\epsilon_n)_{n\leq 0}$ be a sequence of independent $\{-1,1\}$- valued random variables such that,*

$$\sum_{n=-\infty}^{0} \min(\mathbb{P}[\epsilon_n = 1], \mathbb{P}[\epsilon_n = -1]) = \infty. \tag{A.4}$$

Then there is a predictable process $(h_n)_{n\leq 0}$ of $\{-1,1\}$-valued random variables, such that $(h_n\epsilon_n)_{n\leq 0}$ is an i.i.d. sequence of symmetric Bernoulli random variables.

Proof. Consider the disjoint, infinite subsets $(I_n)_{n\leq 0}$ of $\mathbb{N}_-$ verifying (A.2).

By Lemma A.1, we may find Borel-functions

$$f_n : \{-1,1\}^{\mathbb{N}_-} \to \{-1,1\},$$

such that

$$h_n = f_n((\epsilon_i)_{i\in I_n}) \tag{A.5}$$

satisfies

$$\mathbb{P}[h_n = 1] = \mathbb{P}[h_n = -1] = \frac{1}{2}.$$

We claim that these $(h_n)_{n\leq 0}$ do the job.

For this aim we show that

$$\mathbb{P}[h_0\epsilon_0 = 1|(h_n\epsilon_n)_{n\leq -1}] \overset{a.s.}{=} \mathbb{P}[h_0\epsilon_0 = -1|(h_n\epsilon_n)_{n\leq -1}] \overset{a.s.}{=} 1/2. \tag{A.6}$$

To see the dependence structure of the $(h_n)_{n\in\mathbb{N}_-}$ note that by (A.5) for $n \geq 0$ the $\{-1,1\}^{\mathbb{N}_-}$-valued random variable $(h_i)_{i\in I^{(n)}}$ does depend on $(\epsilon_i)_{i\in I^{(n+1)}}$ but it is independent of $(\epsilon_i)_{i\in I^{(n)}}$ as $I^{(n)} \cap I^{(n+1)} = \emptyset$, by Lemma A.3(b).

The $(h_i)_{i\in I_0}$ are an i.i.d. sequence of symmetric Bernoulli random variables independent of $(\epsilon_i)_{i\in I_0}$ and $(\epsilon_i)_{i\in J}$ hence by Lemma A.2 (b)

$$\mathrm{law}[(\epsilon_i)_{i\in I_0} \mid (h_i\epsilon_i)_{i\in I_0}, (\epsilon_i)_{i\in J}] \overset{a.s.}{=} \mathrm{law}[(\epsilon_i)_{i\in I_0}]. \tag{A.7}$$

It follows that

$$\begin{aligned} \mathbb{P}[h_0 = 1 \mid (h_i\epsilon_i)_{i\in I_0}, (\epsilon_i)_{i\in J}] &\overset{a.s.}{=} \mathbb{P}[f_0((\epsilon_i)_{i\in I_0}) = 1 \mid (h_i\epsilon_i)_{i\in I_0}, (\epsilon_i)_{i\in J}] \\ &\overset{a.s.}{=} \mathbb{P}[f_0((\epsilon_i)_{i\in I_0}) = 1] = \frac{1}{2}. \end{aligned} \tag{A.8}$$

Similarly we get for $(h_i)_{i\in I_0} = (f_i(\epsilon_n, n \in I_i))_{i\in I_0}$ that

$$\mathrm{law}[(h_i)_{i\in I_0} \mid (h_i\epsilon_i)_{i\in I^{(1)}}, (\epsilon_i)_{i\in J}] \qquad \text{is a.s. i.i.d. symmetric Bernoulli.} \tag{A.9}$$

Now we claim that

$$\mathrm{law}[h_0 \mid (h_i\epsilon_i)_{i\in I_0\cup I^{(1)}}, (\epsilon_i)_{i\in J}] \qquad \text{is a.s. symmetric Bernoulli.} \tag{A.10}$$

Indeed, fix finite index sets $K \subset I_0$, $L \subset I^{(1)}$, $M \subset J$ and signs $(x_i)_{i\in K}$ as well as $(y_i)_{i\in L}$, $(z_i)_{i\in M}$. Then

$$\begin{aligned}&\mathbb{P}[h_0 = 1 \mid (h_i\epsilon_i)_{i\in K} = (x_i)_{i\in K}, (h_i\epsilon_i)_{i\in L} = (y_i)_{i\in L}, (\epsilon_i)_{i\in M} = (z_i)_{i\in M}]\\ &= \frac{\mathbb{P}[h_0 = 1, (h_i\epsilon_i)_{i\in K} = (x_i)_{i\in K} \mid (h_i\epsilon_i)_{i\in L} = (y_i)_{i\in L}, (\epsilon_i)_{i\in M} = (z_i)_{i\in M}]}{\mathbb{P}[(h_i\epsilon_i)_{i\in K} = (x_i)_{i\in K} \mid (h_i\epsilon_i)_{i\in L} = (y_i)_{i\in L}, (\epsilon_i)_{i\in M} = (z_i)_{i\in M}]}.\end{aligned}$$

For the denominator consider that by (A.9) together with independence of $(\epsilon_i)_{i\in I_0}$ from $(\epsilon_i)_{i\in I^{(1)}\cup I^{(2)}\cup J}$ and by Lemma A.2 (a)

$$\mathbb{P}[(h_i\epsilon_i)_{i\in K} = (x_i)_{i\in K} \mid (h_i\epsilon_i)_{i\in L} = (y_i)_{i\in L}, (\epsilon_i)_{i\in M} = (z_i)_{i\in M}] = 2^{-|K|}.$$

By (A.8) and (A.9) we get for the numerator

$$\begin{aligned}\mathbb{P}[h_0 = 1, (h_i\epsilon_i)_{i\in K} = (x_i)_{i\in K} \mid (h_i\epsilon_i)_{i\in L} = (y_i)_{i\in L}, (\epsilon_i)_{i\in M} = (z_i)_{i\in M}]&\\ = \mathbb{P}[h_0 = 1, (h_i\epsilon_i)_{i\in K} = (x_i)_{i\in K}] = \frac{1}{2}2^{-|K|}&,\end{aligned}$$

and (A.10) follows as the same conclusion passes to infinite index sets.

Continuing analogously we get by induction that

$$\mathrm{law}[h_0 \mid (h_i\epsilon_i)_{i\in \bigcup_{l=0}^{\infty} I^{(l)}}, (\epsilon_i)_{i\in J}] \qquad \text{is a.s. symmetric Bernoulli,}$$

which gives the claim (A.6) since $\sigma(h_n\epsilon_n, n \leq -1)$ is contained in $\sigma(\epsilon_n h_n, n \in \bigcup_{l=0}^{\infty} I^{(l)}, \epsilon_n, n \in J)$ by Lemma A.3 (c).

Analogous arguments show that

$$\mathbb{P}[h_i\epsilon_i = 1 \mid (h_n\epsilon_n)_{n\leq i-1}] \overset{a.s.}{=} \mathbb{P}[h_i\epsilon_i = -1 \mid (h_n\epsilon_n)_{n\leq i-1}] \overset{a.s.}{=} 1/2,$$

for any $i \leq -1$, which proves the theorem. □

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