A Rotational Invariant Technique for Rare Event Simulation

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Abstract

This paper introduces a new importance sampling technique to increase the efficiency of rare event simulation. By using a multidimensional rotational invariant auxiliary density, this method can be applied to estimate risk measures for credit risk portfolios where the method of importance sampling via mean shifting is not suitable. The loss distribution of the portfolio is obtained by simulating scenarios that are constituted by macroeconomic factors. Especially in inhomogeneous portfolios, i.e. portfolios with multiple areas of factors resulting in high losses, rotational invariant importance sampling is advantageous. Furthermore, rotational invariant importance sampling allows to calculate contributions to risk measures of different customer clusters in credit portfolios quicker and more precisely. The new method is applied to a real world credit portfolio with a high number of obligors. A comparison between standard Monte Carlo simulation and simulation using rotational invariant importance sampling in terms of computation time is given.

1 Introduction

The estimation of credit default risk is often based on credit portfolio models, e.g., [GFB97, BHRS02], which try to capture dependencies between obligors in the portfolio. These models should reflect diversification and concentration effects accurately. The most important output of such credit portfolio models is the portfolio loss distribution, from which very popular credit risk management measures like *value at risk*(VaR) or *expected shortfall*(ES) can be derived.
Portfolio models can be split into two categories - analytic and simulation-based models.

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Analytic models have mathematical elegance but are often too restrictive in their assumptions for practical purposes. Hence one has to pass to simulation techniques. The idea of simulation-based models is to approximate the true portfolio loss distribution by an empirical distribution generated by a large number of simulations, so called *scenarios*, which consist of one simulated value for each macroeconomic factor. This process enables gains in flexibility and can accommodate complex distributions for risk factors.
However, in credit risk management - at least when the underlying distributions are assumed to be Gaussian - one often focuses on VaR or ES figures at extremely high confidence levels, e.g. 99.95%, which are defined by very high portfolio losses only. Since such losses are very scarce, one has to run a prohibitively high number of simulations, and consequently use more computation time. Therefore, standard Monte Carlo techniques are inefficient for getting correct VaR or ES values. The calculation of correct obligor contributions to these risk measures is even more sensible. Therefore, *importance sampling* techniques for Monte Carlo simulations come into focus.

The classic idea of importance sampling (see, e.g., [KM53]) is to sample more frequently in those areas of the underlying space which are *important* for the problem at hand. The technique is to pass from the original measure $\mathbb{P}$ to a measure $\mathbb{Q}$ such that the Radon-Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$ takes high values in the "important", and low ones in the "unimportant" areas.

The mathematically challenging part is to find a good change of measure for the importance sampling procedure, i.e. a measure under which the scenarios can be sampled more efficiently and whose Radon-Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$ has a computationally tractable form. The classical approach in the context of Gaussian distributions is to make a mean shift in which case the above Radon-Nikodym derivative simply is an exponential function. This was suggested by, e.g., [KLO04], [Gla05] or [GL05]. However, portfolios are often quite inhomogeneous in the sense that there are several "regions" of scenarios in the underlying space $\mathbb{R}^d$ causing a high portfolio loss.

We suggest a new - rotational invariant - importance sampling procedure to take into account inhomogeneity of credit risk portfolios. The presented technique of rotational invariant importance sampling is not only applicable to credit portfolios, but rather to arbitrary probability distributions on high dimensional spaces which are essentially rotational invariant. In this case the presented method of rotational invariant importance sampling is better adapted than mean shifting.

Importance sampling is not the only means to calculate risk contributions. For example, [TM06] calculate the default probability of an obligor based on the Ensemble method for rather general distributions of the macroeconomic factors. Their method seems to be particularly suitable for VaR contributions (when the VaR is known); see also the references in [TM06] for other methods which are not based on importance sampling.

The paper is structured as follows: The required technical definitions and the underlying credit risk model are given in Section 1. Section 2 contains a detailed problem description and the main part of finding a suitable importance sampling measure $\mathbb{Q}$. The model is fitted to real data in Section 3, Section 4 gives an explicit example of the use of the method. Section 5 concludes.

1.1 Foundations

A credit portfolios loss distribution is characterized by risk measures like *value at risk* or *expected shortfall*. The *value at risk* of our portfolio at the confidence level $\alpha \in (0,1)$ is given by the smallest number l such that the probability that the loss L exceeds l is no larger than $(1-\alpha)$; see e.g. [EFM05]. An alternative to VaR is ES. It is often preferred over VaR due to conceptual deficiencies related to sub additivity of the latter; see [EFM05]. The *expected shortfall* at $\alpha\%$ level is the expected return on the portfolio in the worst $\alpha\%$ of the cases. Both quantities depend only on the tail of the loss distribution of the credit portfolio.

Moreover, contributions of different clients to the VaR and ES of the credit portfolios loss distribution can be calculated via the *Euler capital allocation principle* (also known as *allocation by gradient*). The VaR and ES contributions of client i are essentially given by $E[L_i|L = VaR_\alpha]$ and $E[L_i|L \geq VaR_\alpha]$, respectively. For further definitions, characteristics and technical conditions see [EFM05] chapter 6.3 and [Kal05].

We will assume the loss distribution is obtained via a fitting to *macroeconomic factors*, e.g. stock market indices and foreign exchange rates, that constitute a scenario. Then we need to calculate expectations of the form $E_{\mathbb{P}}[h(Z)]$ for some random variable Z and a (measurable) function h. If there is no analytical formula, one way to calculate this quantity is by Monte Carlo simulation. However, for rare events, Monte Carlo does not work very well. A well-known technique to improve rare event simulation is *importance sampling*. It relies on the fact that for any $\mathbb{Q} \sim \mathbb{P}$

$$E_{\mathbb{P}}[h(Z)] = E_{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\mathbb{Q}}h(Z)\right] ,$$

where $d\mathbb{P}/d\mathbb{Q}$ denotes the Radon-Nikodym derivative of $\mathbb{P}$ with respect to $\mathbb{Q}$. If Z has density f under $\mathbb{P}$ and density g under $\mathbb{Q}$, a Monte Carlo estimate for $E[h(Z)]$ is given by

$$\frac{1}{M}\sum_{m=1}^{M} h(z_m)\frac{f(z_m)}{g(z_m)} , \tag{1}$$

where $z_1, z_2, \ldots z_M$ are M independent samples of Z generated under $\mathbb{Q}$ (instead of $\mathbb{P}$), i.e., according to the density g. If $\mathbb{Q}$ is chosen appropriately, estimator (1) has a small variance (under $\mathbb{Q}$) so that convergence becomes faster. In what follows we will be interested in estimating extreme events. Therefore we will choose $\mathbb{Q}$ such that most of its mass is concentrated on the tail.

1.2 Model

The client correlations, i.e. the systematic dependencies of the default risks, are calculated through a Merton model by means of macroeconomic factors. For the sake of simplicity homogeneous clients are pooled together in clusters according to region or economic sectors. Those clusters then form a portfolio. By N we denote the number of customer clusters in the portfolio. The quality of a credit cluster i, where $i = 1, ..., N$ and N is e.g. of the order 2000, is modeled by a normal random variable Y_i of the form

$$Y_i = \sum_{j=1}^{d} a_{ij}Z_j + \sigma_i\varepsilon_i \tag{2}$$

The random variables $((Z_j)_{j=1}^{d}, (\varepsilon)_{i=1}^{N})$ are independent standard normal. Here, d is the number of economic factors in consideration (e.g. d of the order 10) and the numbers $(a_{ij})_{i,j}$

have been determined from previous regression of the dependence of the credit clusters on the d macroeconomic factors. The numbers ε_i model the influence of the idiosyncratic risk.

The model assumption is that the credit cluster i defaults if the random variable Y_i surpasses a critical value $\chi_i = \Phi^{-1}(p_i)$, for given unconditional default probabilities p_i where Φ is the cumulative standard normal distribution function.
In practice the N vectors $(a_{ij})_{j=1}^d$ "point into quite different directions" of $\mathbb{R}^d$. This corresponds economically to the fact that the credit quality of different credit clusters depend on different macroeconomic influences. Even the sign of the influence may depend on the credit cluster.

We assume that the density of the (unconditional) loss distribution of the entire portfolio is of the form

$$f_{L(x)}(y) = E_{\mathbb{P}}\left[\varphi\left(\mu(Z), \sigma(Z), y\right)\right] ,$$

where $\varphi(\mu, \sigma, y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$, $Z \in I\!R^d$ is a vector of i.i.d. standard normal random variables,

$$\mu(z) = \sum_{i=1}^{N} x_i \mathrm{Sev}_i \mathrm{Edf}_{i|z} \qquad \sigma(z) = \sqrt{\sum_{i=1}^{N} x_i^2 \mathrm{Sev}_i^2 \frac{\mathrm{Edf}_{i|z}(1-\mathrm{Edf}_{i|z})}{N_i}},$$

and $\mathrm{Edf}_{i|z} = \Phi\left(\frac{\chi_i - a_i^T z}{\sigma_i}\right)$. The dependence of $\mu(z)$ and $\sigma(z)$ on the vector of exposures $x = (x_1, \ldots, x_N)$ as well as the severity Sev_i is suppressed for the time being. For cluster C_i we denote by x_i its exposure, by Sev_i its loss severity, by N_i the number of customers in this cluster, by σ_i its idiosyncratic component and by a_i its weights vector for the macroeconomic factors.

We assume that the macroeconomic factors $Z \in I\!R^d$ have a standard normal distribution under $\mathbb{P}$ and denote their density by $f(z)$. In order to increase the rate of scenarios which yield high portfolio losses, a change of measure is applied to the distribution of the macroeconomic factors $Z \in I\!R^d$. It is reasonable to choose the dimension d, which corresponds to the number of macroeconomic factors included in the model. The number of macroeconomic factors is in the order of 10, the precise number depending on the size and structure of the portfolio.

2 New Measure $\mathbb{Q}$ for Rotational Invariant Importance Sampling

In the following paragraphs we suggest a definition for the importance sampling measure $\mathbb{Q}$. As mentioned in the introduction, the difficulty is that there are several 'regions' for the macroeconomic factors $Z \in I\!R^d$, which yield high (conditional) portfolio losses. Almost all of the existing articles on importance sampling for credit portfolios, suggest to make one mean shift only. There, one chooses $\mathbb{Q}$ such that Z is still a vector of independent normal random variables, but with a non-zero vector of means; see, e.g., [KLO04], [Gla05] or [GL05]. This approach thus assumes, that the region of macroeconomic factor combinations which yields high portfolio losses can be separated from the origin of the $\mathbb{R}^d$ by a hyperplane, which is relatively far away from the origin. However, there are several regions of $Z \in I\!R^d$ yielding high portfolio losses. In particular for $d = 8$, i.e. consideration of eight macroeconomic factors, Bank Austria's credit portfolio was analyzed. The only importance sampling based article we know, which takes into account the existence of several

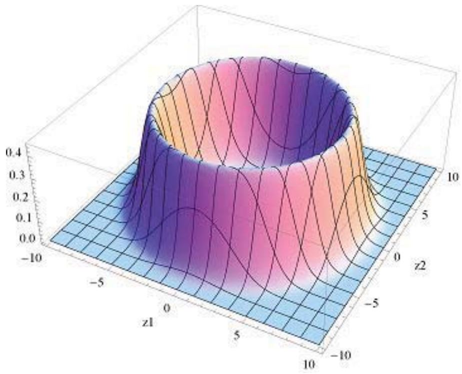

Figure 1: Density $g(z)$ for $d = 2$

such regions is [GKS07]. However, their approach seems inapplicable to a portfolio which has a big number of clusters. Based on real data, we therefore suggest a different approach which is based on the following observations:

Since the vector of macroeconomic factors $Z \in I\!R^d$ is assumed to be Gaussian, its squared norm $\|Z\|^2 = \sum_{i=1}^{d} Z_i^2$ is χ^2-distributed with d degrees of freedom. From this, the smooth dependence of the conditional expected losses on the macroeconomic factors, and since the level α for VaR and ES is quite extreme (e.g. $99,95\%$), it follows that high losses occur only for scenarios z with $\|z\|$ big.

This already implies that most of the samples generated by standard Monte Carlo cannot yield losses in the relevant area, as the value of $\|z\|$ is too small. On the other hand, if $\|z\|$ is chosen too big, the loss is too unlikely to play a significant role for value at risk or expected shortfall.

Since it is difficult to appropriately specify the regions of macroeconomic factor combinations in $I\!R^d$ which yield high portfolio losses, we do not try to define several mean shifts to capture all relevant regions, but rather stick to a rotational invariant distribution for Z. A density function $\varphi(z)$ is rotational invariant if it depends on z only via $\|z\|$. We change the distribution of $\|Z\|^2$ from a chi-squared distribution to one which has most of the mass in the potential area for high losses. To do so, we explicitly define $\mathbb{Q}$ by its density $g(z)$ with respect to Lebesgue measure on $\mathbb{R}^d$. We specify the density of Z under $\mathbb{Q}$ as

$$g(z) := \left(\int_{I\!R^d} e^{-\frac{(\|z'\|-c)^2}{2s}} \, dz' \right)^{-1} \cdot e^{-\frac{(\|z\|-c)^2}{2s}} \quad \text{for } z \in I\!R^d, \tag{3}$$

with $c \geq 0$ and $s > 0$ appropriately fitted to real data (see Section 5 for further discussion); see Figure 1.

Note that for $c = 0$ and $s = 1$, g is the density of a standard normal distribution, so that $\mathbb{P} = \mathbb{Q}$, i.e., this case corresponds to standard Monte Carlo simulation.
The random variable Z under $\mathbb{P}$ (resp. $\mathbb{Q}$) will be represented by $Z =$identity on $\mathbb{R}^d$ and $\mathbb{P}$ (resp. $\mathbb{Q}$) defined via the density function f (resp. g). The density of $\|Z\|$ under $\mathbb{Q}$ can be deduced from the relation

$$\int_{\left\{ z \in I\!R^d \,\middle|\, 0 \leq \|z\| \leq R \right\}} e^{-\frac{(\|z\|-c)^2}{2s}} dz = d \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} \int_0^R e^{-\frac{(r-c)^2}{2s}} r^{d-1} \, dr; \tag{4}$$

for $R > 0$ see [For84], Section 8, Theorem 1. Here $\Gamma(\,.\,)$ denotes the gamma function and $\pi^{\frac{d}{2}}/\Gamma(d/2+1)$ is the volume of a d-dimensional unit sphere. The density of $\|Z\|$ under $\mathbb{Q}$ is

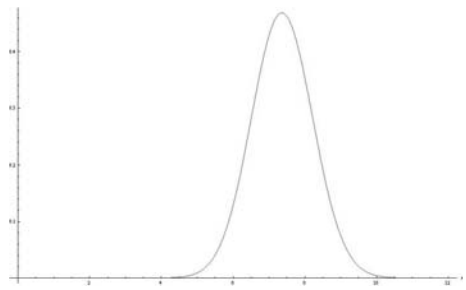

Figure 2: Function $\tilde{g}(r)$ for $d = 8$, $c = 6.6$ and $s = 0.8$

then given by

$$\tilde{g}(r) := \left(\int_0^\infty e^{-\frac{(\tilde{r}-c)^2}{2s}} \tilde{r}^{d-1}\, d\tilde{r} \right)^{-1} \cdot e^{-\frac{(r-c)^2}{2s}} r^{d-1} \mathbf{I}_{\{r>0\}}\,; \tag{5}$$

see Figure (2). The above density has a simple form and is easy to simulate. In fact, (5) can be used to simulate scenarios under $\mathbb{Q}$. The simulation is straightforward and easy to implement.

From the above it becomes obvious that it is straightforward to extend the method to an elliptical distribution for Z, as such a distribution is just a linear transformation of a rotational invariant distribution; see Chapter 3.3 in [EFM05].

3 Choice of Parameters

We want to find a pair (c, s) of parameters such that most of the mass of the measure $\mathbb{Q}$ sits on those $z \in \mathbb{R}^d$ such that $\mathbb{E}[L(x)|z]$ is of the order of $VaR_\alpha(L)$. Note that at this stage it is only necessary to have a crude approximation of the value of $VaR_\alpha(L)$.

In practice we proceed as follows. Fix $\eta > 0$ of the order 5% of $VaR_\alpha(L)$ and make a crude estimate $\widehat{VaR}_\alpha(L)$ of $VaR_\alpha(L)$ through either importance sampling with starting values (c_0, s_0) or brief Monte Carlo simulation. We generate samples of $z \in \mathbb{R}^d$ (unter $\mathbb{P}$ or under $\mathbb{Q}$) and then we single out those z such that $\mu(z)$, the expected loss of the portfolio, is in the interval $[\widehat{VaR}_\alpha(L) - \eta, \widehat{VaR}_\alpha(L) + \eta]$. In this class of $z \in \mathbb{R}^d$ we determine an average of $\|z\|$, e.g., by taking the arithmetic mean. A plausible way to make a good choice of the parameters c and s is to match the mode m of the radial density $\tilde{g}(r)$ of $\|z\|$ with the obtained average of $\|z\|$. We use the mode rather than the expectation because it is computationally more tractable.

The mode of $\tilde{g}(r)$ is given by

$$m(c, s) = \frac{c + \sqrt{c^2 + 4(d-1)s}}{2}. \tag{6}$$

We compute

$$T := \left| m(c, s) - E\left[\|z\| \middle| \mu(z) \in [\widehat{VaR}_\alpha(L) - \eta, \widehat{VaR}_\alpha(L) + \eta] \right] \right|. \tag{7}$$

The idea is that a low value of T indicates a good fit. In this case the average of $\|z\|$, where z ranges through the scenarios which create exponential losses close to $VaR_\alpha(L)$, is close to the mode $m(c,s)$. By varying over different c and s a good choice of (c,s) is obtained. For the present credit portfolio of Bank Austria these parameters were found to be $c = 6.6, s = 0.8$, see Figure 3.

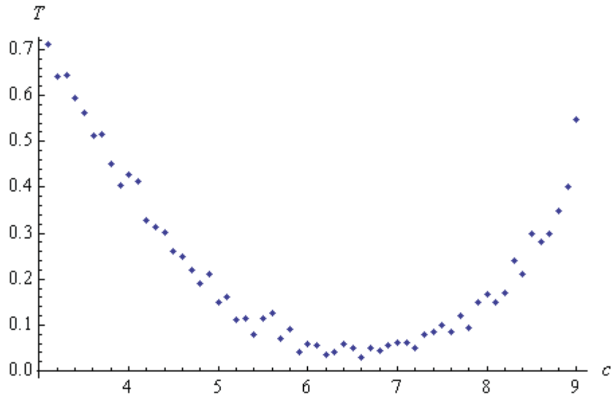

Figure 3: T as a function of the parameter c. For the discussed portfolio the parameter c was found by the above procedure at $c = 6.6$ with $s = 0.8$.

4 Application

The presented new technique of rotational invariant importance sampling was applied to a real world credit portfolio provided by Bank Austria involving several thousand obligors. In this section, an explicit example of the method is given.

In order to calculate the value at risk (and expected shortfall, respectively) as well as corresponding cluster contributions for a given portfolio the procedure is as follows:

Step 1: Conduct a short standard Monte Carlo simulation to roughly frame the area of $VaR_\alpha(L)$. In order to obtain such a rough estimate it is not necessary to invest time in extensive simulations. Alternatively, one can use importance sampling with starting values (c_0, s_0).

Step 2: Having obtained a crude approximation $\widehat{VaR}_\alpha(L)$ of $VaR_\alpha(L)$ in Step 1, define an interval around $\widehat{VaR}_\alpha(L)$ given by $[\widehat{VaR}_\alpha(L) - \eta, \widehat{VaR}_\alpha(L) + \eta]$ as discussed in Section 3. Based on starting values (c_0, s_0), e.g. $c_0 = 6.6$ and $s_0 = 0.8$ as obtained for Bank Austria's main credit portfolio, where $d = 8$, conduct the optimization procedure given in Section 3. It should be stated that the corresponding optimization criterion for ES leads to a different pair of (c,s). But as indicated by Table 1, the parameter s has only little impact on the convergence of optimization. Therefore, only the parameter c would be slightly higher for the calculation of ES and ES contributions, respectively. However, this can be disregarded and the parameters (c,s) obtained by the above procedure can be used for calculating both VaR and ES contributions.

Note: For practical purposes the parameters c and s do not need to be refitted unless the considered portfolio changes significantly. The provided parameter optimization procedure was conducted for several credit portfolios of Bank Austria varying, e.g., in exposure, number of clusters and average number of customers per clusters. However, the optimal

c\ s	0.6	0.7	0.8	0.9	1
3	0.726	0.694	0.711	0.674	0.668
4	0.411	0.444	0.426	0.375	0.431
5	0.324	0.317	0.301	0.282	0.351
6	0.137	0.214	0.108	0.129	0.164
7	0.049	0.071	0.062	0.041	0.176
8	0.070	0.140	0.168	0.182	0.254
9	0.352	0.359	0.448	0.440	0.433

Table 1: *Values for T as a function of the parameters c and s according to Eq. (7). As the optimization criterion is to find the minimal T the optimal pair of parameters was found to be* $(c, s) = (6.6, 0.8)$ *for the considered portfolio. It turned out that s has only very little impact on the value of T.*

parameters were found to be virtually the same.

Step 3: Let X be a $\mathbb{R}^d$-valued random variable distributed uniformly on the unit sphere and Y a $\mathbb{R}_+$-valued independent random variable distributed according to the density $\tilde{g}(r)$, then $Z = XY$ has the law of Z under $\mathbb{Q}$. Split into two steps, this easy observation can be implemented in the simulation.

i) Simulate a sample of $(X_1, ..., X_N) \in \mathbb{R}^d$ which is uniformly distributed on the unit sphere. This can be obtained by generating a vector $\widetilde{X} \in I\!R^d$ of i.i.d. standard normal random variables and setting $X := \widetilde{X}/\|\widetilde{X}\|$.

ii) Sample independently of X a random variable $Y \in I\!R_+$ using the density given in (5). In programs like R one can sample directly from (5). In Mathematica this is not possible, but one can exploit that if

$$F_g(R) := \left(\int_0^\infty e^{-\frac{(\tilde{r}-c)^2}{2s}} \tilde{r}^{d-1}\, d\tilde{r} \right)^{-1} \int_0^R e^{-\frac{(r-c)^2}{2s}} r^{d-1}\, dr$$

denotes the cumulative density function corresponding to (5), and if F_g^{-1} denotes its inverse and if U is uniformly distributed on the interval $[0, 1]$, then $F_g^{-1}(U)$ is distributed according to (5). Unfortunately, there are no closed form expressions for F_g and F_g^{-1} so that they have to be computed numerically given c and s. However, this numerical computation does not increase the computation time significantly.

Then the sample $(Z_1, ...Z_N) = (Y_1X_1, ..., Y_NX_N)$ is a simulation of the random variable $Z = YX$. Moreover, if f is the standard normal density of Z under $\mathbb{P}$, then the quotient with respect to g is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(z) = \frac{f(z)}{g(z)} = \frac{d}{2^{\frac{d}{2}}\Gamma\left(1 + \frac{d}{2}\right)} \int_0^\infty \exp\left(-\frac{1}{2}\|z\|^2 + \frac{(\|z\| - c)^2}{2s} \right) e^{-\frac{(r-c)^2}{2s}} r^{d-1}\, dr\,.$$

Given the obtained scenarios one can now proceed with a calculation of $VaR_\alpha(L)$ and $ES_\alpha(L)$ and corresponding contributions, respectively.

5 Conclusion

A new importance sampling technique to increase the efficiency of rare event simulation was introduced. By using a multidimensional rotational invariant auxiliary density, risk measures for credit risk portfolios were estimated.

The existing literature suggests miscellaneous approaches to importance sampling in credit risk [Mor04, KLO04, Gla05, GL05]. For example, [Mor04] establishes a method of importance sampling relying on the introduction of a scalar parameter into the asset correlation model that may be adjusted to increase correlations. Thereby, a greater number of correlated defaults is induced and thus samples are produced further out in the loss tail. In contrast to rotational invariant importance sampling, Morokoff [Mor04] considers applying importance sampling to a single dimension, leaving the other dimensions unchanged. The goal is to find a single dimension associated with the correlated asset returns that has the largest impact on the portfolio value. By orthogonalizing the covariance matrix of asset returns, i.e. the correlation matrix, and working with eigenvector decomposition, the most dominant factor from the factor model is found.

Utilizing the infinite granularity approximation, [KLO04] introduce an importance sampling technique for Merton-type-models suggesting one mean shift only. As discussed in section 2, this approach assumes that the region of macroeconomic factor combinations which yields high portfolio losses can be separated from the origin of the $\mathbb{R}^d$ by a hyperplane. Rotational invariant importance sampling does not follow this approach but rather focuses on several regions of macroeconomic factors $Z \in \mathbb{R}^d$ yielding high portfolio losses.

Similarly, [GL05] and [Gla05] describe the technique of importance sampling through mean shifting. Based on the widely used normal copula model of portfolio credit risk, [GL05] provides a two step importance sampling procedure. In the normal copula framework, dependence between obligors is captured through a multivariate normal vector of latent variables. Conditional on the factors, obligors become independent. In the first step of the importance sampling procedure a change of distribution is applied to the conditional default probabilities, given the factors. The second step applies a shift in mean to the factors themselves. Shifting the mean of underlying factors and then applying exponential twisting to default probabilities conditional on the factors is specifically applicable to complex dependence between defaults of multiple obligors that results from a normal copula.

Furthermore, [Gla05] develop three techniques to address the difficulty of calculating expectations conditioned on rare events, in the setting of the Gaussian copula model of portfolio risk. After developing importance sampling estimators specifically designed for conditioning on large losses, a hybrid method that combines an approximation with Monte Carlo is introduced. Additionally, a rough but fast approximation that dispenses entirely with Monte Carlo is presented. However, both the Monte Carlo methods and the approximations developed in [Gla05] rely on identifying a single "most important" factor outcome. Concludingly, the authors state to expect that techniques based on mixing multiple mean shifts could be applied to the problem of marginal risk contributions.

To sum up, it is difficult to appropriately specify the regions of macroeconomic factor combinations in $\mathbb{R}^d$ which yield high portfolio losses. Therefore, as discussed in section 2, we do not try to define several mean shifts to capture all relevant regions, but rather stick to a rotational invariant distribution for Z.

As tested on real data provided by Bank Austria calculation of risk measures themselves and especially contributions to risk measures of different customer clusters in credit portfolios via rotational invariant importance sampling are quicker and more precise. For a comparison of computation time of standard Monte Carlo simulation and simulation using rotational invariant importance sampling, see Table 2.

TECHNIQUE	COMPUTATION TIME
Standard Monte Carlo Simulation	54h 47min 20sec
Rotational Invariant Importance Sampling	32min 47sec

Table 2: *Comparison of Computation Time using standard Monte Carlo simulation and simulation via rotational invariant importance sampling, corresponding to the simulation of 10 000 relevant scenarios, i.e. scenarios generating losses in the area of interest around value at risk. Comparison shows that computation time can be reduced by approximately* 99% *using rotational invariant importance sampling. Calculations were done with optimal parameters using real data provided by Bank Austria and executed in R (version 2.7.1) on a FSC Esprimo E5905 Intel Pentium 4 PC with 3.20GHz and 2.5 GB RAM.*

As pointed out in Section 2, rotational invariant importance sampling is not confined to Gaussian distributions but also fits perfectly to elliptic distributions. The presented technique of rotational invariant importance sampling is not only applicable to credit portfolios, but rather to any function which does not behave similarly to a linear functional on $\mathbb{R}^d$ but rather similarly to a radial function. In this case the presented method of rotational invariant importance sampling is better adapted than mean shifting.

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