AFFINE PROCESSES ARE REGULAR

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Abstract. We show that stochastically continuous, time-homogeneous affine processes on the canonical state space $\mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ are always regular. In the paper of Duffie, Filipovic, and Schachermayer (2003) regularity was used as a crucial basic assumption. It was left open whether this regularity condition is automatically satisfied, for stochastically continuous affine processes. We now show that the regularity assumption is indeed superfluous, since regularity follows from stochastic continuity and the exponentially affine behavior of the characteristic function. For the proof we combine classic results on the differentiability of transformation semigroups with the method of the moving frame which has been recently found to be useful in the theory of SPDEs.

1. Introduction

A Markov process X taking values in $D := \mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ is called *affine* if there exist $\mathbb{C}$-valued functions[1] $\Phi(t,u)$ and $\psi(t,u)$ such that

$$\mathbb{E}^x\left[e^{\langle X_t, u\rangle}\right] = \Phi(t,u)\exp\left(\langle x, \psi(t,u)\rangle\right) , \tag{1.1}$$

for all $x \in D$, and for all $(t,u) \in \mathbb{R}_{\geqslant 0} \times i\mathbb{R}^d$ with $d = m + n$. We also assume that X is stochastically continuous, i.e. $X_t \to X_s$ in probability as $t \to s$.
Stochastic processes of this type have been studied for the first time in the seventies, where (on the state space $D = \mathbb{R}_{\geqslant 0}$) they have been obtained as continuous-time limits of classic Galton-Watson branching processes with and without immigration (see Kawazu and Watanabe (1971)). More recently, affine processes have attracted renewed interest, due to several applications in mathematical finance, where they are used as flexible models for asset prices, interest rates, default intensities and other economic quantities (see Duffie et al. (2003) for a survey).

The process X is said to be *regular*, if the functions Φ and ψ are differentiable with respect to t, with derivatives that are continuous in (t,u). This technical condition is of crucial importance for the theory of affine processes, as developed in Duffie et al. (2003). Under the assumption of regularity, it is possible to show that an affine process X is a semi-martingale (possibly killed at a state-dependent rate), and in fact to completely characterize all affine processes in terms of necessary and sufficient conditions on their semi-martingale characteristics. Likewise, regularity

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[1]We are using upper case Φ but lower case ψ for consistency with the notation of Duffie et al. (2003); see Remark 2.3 for a detailed discussion.

allows to represent the characteristic function of X as the solution of so-called generalized Riccati differential equations which determine the fundamental functions Φ and ψ.

Without regularity it is – a priori – not clear how we can "locally" characterize the process (e.g. in terms of its infinitesimal generator or of its semi-martingale characteristics) and therefore such processes are not well-parameterized families of models – a situation which could be compared to the theory of Lévy processes without knowledge on the Lévy-Khintchine formula. Therefore in Duffie et al. (2003) the authors assume regularity at the very beginning of their classification of affine processes. Without the assumption of stochastic continuity there are simple examples of non-regular Markov processes with the affine property (1.1), based on introducing jumps at fixed (non-random) times; see e.g. Duffie et al. (2003, Remark 2.11). In contrast, if one assumes stochastic continuity for an affine process, to the best of our knowledge it was not now known in general whether such a process is regular or not. Regularity has been shown in several special cases: Kawazu and Watanabe (1971) show automatic regularity for a single-type continuous branching process with immigration, which corresponds to an affine process with state space $D = \mathbb{R}_{\geqslant 0}$. Dawson and Li (2006) show regularity of affine processes under moment conditions. In Keller-Ressel (2008b) regularity of an affine process under a mixed homogeneity and positivity condition is shown. The approaches in the two latter publications are all based on the general techniques from Montgomery and Zippin (1955) for continuous (global) flows of homeomorphisms, which have been extended in Filipović and Teichmann (2003) to continuous (local) semi-flows.

One of the most fascinating aspects of the regularity problem for affine processes is the close connection to the Hilbert's fifth problem, whose history and mathematical development can be found in Montgomery and Zippin (1955). The reason is that the functions Φ and ψ defined by (1.1) satisfy certain functional equations, namely

$$\psi(t+s,u) = \psi(t,\psi(s,u)), \quad \Phi(t+s,u) = \Phi(t,\psi(s,u)) \cdot \Phi(s,u), \text{ for } s,t \geq 0 \tag{1.2}$$

with initial conditions $\psi(0,u) = u$ and $\Phi(0,u) = 1$ and for all u in $\mathcal{Q}$, a large enough subset of $\mathbb{C}^d$ (see Section 2 for the exact definition of $\mathcal{Q}$). Such functional equations are (non-linear) relatives of the multiplicative Cauchy functional equation

$$A(t+s) = A(t)A(s), \qquad A(0) = \mathrm{id}_d \ . \tag{1.3}$$

formulated for instance in a set of $d \times d$ matrices $M_d(\mathbb{C})$, simply by defining $\psi(t,u) = A(t)u$. It is well-known that the only continuous solutions of the Cauchy equation are the exponentials $A(t) = \exp(t\beta)$, where β is a $d \times d$ matrix. In particular all continuous solutions of (1.3) are automatically differentiable (even analytic) with respect to t, with derivatives that are continuous in u.

Hilbert's fifth problem asks whether this assertion can be extended to more general functional equations such as (1.2): Assuming that $\psi(t,u)$ and $\Phi(t,u)$ satisfy (1.2), and are differentiable in u, are they necessarily differentiable in t? The problem has been answered positive; for a precise formulation and a proof in the more general context of transformation groups see Montgomery and Zippin (1955, Chapter V.5.2, e.g. Theorem 3). From the point of view of stochastics this means that moment conditions (roughly speaking the existence of a first moment of X means differentiability of ψ with respect to u) imply t-differentiability of ψ and

Φ, and thus regularity. This has already been observed in Dawson and Li (2006). However, moment conditions on the process X are not natural in the present context of affine processes, and the question remained open, whether the moment conditions in Dawson and Li (2006) can simply be dropped.

We show in this article that the answer is again positive in the general case, and that any stochastically continuous affine process is automatically regular. There is one well-known case where moment conditions can be dropped, namely homogeneous affine processes, i.e., affine processes with $\psi(t,u) = u$ for $(t,u) \in \mathbb{R}_{\geqslant 0} \times i\mathbb{R}^d$. In this case Φ simply satisfies Cauchy's functional equation $\Phi(t+s,u) = \Phi(t,u)\Phi(s,u)$ and therefore regularity, that is differentiability of Φ, follows by the classical result on (1.3). This is precisely the case of Lévy processes. Whence – in the light of differentiability of functional equations of the type (1.2) – the results of this paper truly extend Montgomery and Zippin (1955) since no differentiability in u is assumed. We introduce the basic definitions and some notation in Section 2 and show preliminary results in Section 3. In Section 4 we show regularity of an affine process subject to a condition of 'semi-homogeneity'. Finally we show in Section 5 that all stochastically continuous, affine processes defined on the domain $D = \mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ are regular by reducing the general question to regularity of semi-homogeneous processes of Section 4.

2. Affine processes

Definition 2.1 (Affine process)**.** An affine process is a time-homogeneous Markov process $(X_t, \mathbb{P}^x)_{t\geq 0, x\in D}$ with state space $D = \mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$, whose characteristic function is an exponentially-affine function of the state vector. This means that there exist functions $\Phi : \mathbb{R}_{\geqslant 0} \times i\mathbb{R}^d \to \mathbb{C}$ and $\psi : \mathbb{R}_{\geqslant 0} \times i\mathbb{R}^d \to \mathbb{C}^d$ such that

$$\mathbb{E}^x \left[e^{\langle X_t, u\rangle} \right] = \Phi(t,u) \cdot \exp\left(\langle x, \psi(t,u)\rangle\right) , \tag{2.1}$$

for all $x \in D$, and for all $(t,u) \in \mathbb{R}_{\geqslant 0} \times i\mathbb{R}^d$.

Remark 2.2. The set $i\mathbb{R}^d$ denotes the purely imaginary numbers in $\mathbb{C}^d$, that is $i\mathbb{R}^d = \{u \in \mathbb{C}^d : \operatorname{Re} u = 0\}$.

Remark 2.3. The above definition differs in one detail from the definition given in Duffie et al. (2003): In their article the right hand side of (2.1) is defined in terms of a function $\phi(t,u)$ as $\exp\left(\phi(t,u) + \langle x, \psi(t,u)\rangle\right)$, whereas we formulate the equation in terms of $\Phi(t,u) = \exp(\phi(t,u))$. In particular, we do not assume a priori that $\Phi(t,u) \neq 0$. Their difference is subtle, but will play a role in Lemma 2.5 below, where we extend $\Phi(t,u)$ to a larger subset $\mathcal{Q}$ of the complex numbers. Essentially, the advantage of using $\Phi(t,u)$ is the following: if $\Phi(t,u)$ is well-defined on a set that is not simply connected, its logarithm $\phi(t,u)$ might only be defined as a multivalued function. Note that our definition using $\Phi(t,u)$ is very close to that of Kawazu and Watanabe (1971).

Assumption 2.4. We will assume throughout this article that X is stochastically continuous, i.e. for $t \to s$, the random variables X_t converge to X_s in probability, with respect to all $(\mathbb{P}^x)_{x\in D}$.

Note that the existence of a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$, where the process $(X_t)_{t\geq 0}$ is defined, is already implicit in the notion of a *Markov process* (we largely

follow Rogers and Williams (1994, Chapter III) in our notation and precise definition of a Markov process). $\mathbb{P}^x$ represents the law of the Markov process $(X_t)_{t\geq 0}$ *started at* x, i.e. we have that $X_0 = x$, $\mathbb{P}^x$-almost surely.

Let us at this point introduce some additional notation: We write

$$I = \{1, \dots, m\} \quad \text{and} \quad J = \{m+1, \dots, m+n\}$$

for the index sets of the $\mathbb{R}^m_{\geqslant 0}$-valued component and the $\mathbb{R}^n$-valued component of X respectively. For some vector $x \in \mathbb{R}^d$ we denote by $x = (x_I, x_J)$ its partition in the corresponding subvectors, and similarly for the function $\psi(t,u) = (\psi_I(t,u), \psi_J(t,u))$. Also, x_i denotes the i-th element of x, and $(e_i)_{i\in\{1,\dots,d\}}$ are the unit vectors of $\mathbb{R}^d$. We will often write

$$f_u(x) := \exp\left(\langle u, x\rangle\right)$$

for the exponential function with $u \in \mathbb{C}^d$ and $x \in D$. A special role will be played by the set

$$\mathcal{U} = \left\{u \in \mathbb{C}^d : \operatorname{Re} u_I \leq 0, \quad \operatorname{Re} u_J = 0\right\} ; \tag{2.2}$$

note that $\mathcal{U}$ is precisely the set of all $u \in \mathbb{C}^d$, for which $x \mapsto f_u(x)$ is a bounded function on D. We also define

$$\mathcal{U}^\circ = \left\{u \in \mathbb{C}^d : \operatorname{Re} u_I < 0, \quad \operatorname{Re} u_J = 0\right\}. \tag{2.3}$$

Lemma 2.5. *Let $(X_t)_{t\geq 0}$ be an affine process. Then*

$$\mathcal{Q} = \left\{(t,u) \in \mathbb{R}_{\geqslant 0} \times \mathcal{U} : \mathbb{E}^0\left[f_u(X_t)\right] \neq 0\right\} , \tag{2.4}$$

is open in $\mathbb{R}_{\geqslant 0} \times \mathcal{U}$ and there exists a unique continuous extension of $\Phi(t,u)$ and $\psi(t,u)$ to $\mathcal{Q}$, such that (2.1) holds for all $(t,u) \in \mathcal{Q}$. If $(t,u) \in (\mathbb{R}_{\geqslant 0} \times \mathcal{U}) \setminus \mathcal{Q}$, then $\mathbb{E}^x\left[f_u(X_t)\right] = 0$ for all $x \in D$.

Remark 2.6. From the facts that $\{0\} \times \mathcal{U} \in \mathcal{Q}$ and that $\mathcal{Q}$ is open in $\mathbb{R}_{\geqslant 0} \times \mathcal{U}$ we can deduce the following: For every $u \in \mathcal{U}$ there exists $t_*(u) > 0$, such that $(t,u) \in \mathcal{Q}$ for all $t \in [0, t_*(u))$.

Proof. We adapt the proof of Duffie et al. (2003, Lemma 3.1): For $(t,u,x) \in \mathbb{R}_{\geqslant 0} \times \mathcal{U} \times D$ define $g(t,u,x) = \mathbb{E}^x\left[f_u(X_t)\right]$. We show that for fixed $x \in D$ the function $g(t,u,x)$ is jointly continuous in (t,u): Let (t_k, u_k) be a sequence converging in $\mathcal{U}$ to (t,u). By stochastic continuity of X it holds that $X_{t_k} \to X_t$ in probability $\mathbb{P}^x$, and thus also in distribution. By dominated convergence we may therefore conclude that

$$g(t_k, u_k, x) = \mathbb{E}^x\left[f_{u_k}(X_{t_k})\right] \to \mathbb{E}^x\left[f_u(X_t)\right] = g(t,u,x) ,$$

and thus that $g(t,u,x)$ is continuous in (t,u). It follows that $\mathcal{Q}$ is open in $\mathbb{R}_{\geqslant 0} \times \mathcal{U}$. Because of the affine property (2.1) it holds that

$$g(t,u,x)g(t,u,\xi) = g(t,u,x+\xi)g(t,u,0) \tag{2.5}$$

for all $(t,u) \in \mathbb{R}_{\geqslant 0} \times i\mathbb{R}^d$ and $x, \xi \in D$. But both sides of (2.5) are continuous functions of $u \in \mathcal{U}$, and moreover analytic in $\mathcal{U}^\circ$. (This follows from well-known properties of the Laplace transform and the extension to its strip of regularity, cf. Duffie et al. (2003, Lemma A.2).) By the Schwarz reflection principle, (2.5) therefore holds for all $u \in \mathcal{U}$. Assume now that $(t,u) \in (\mathbb{R}_{\geqslant 0} \times \mathcal{U}) \setminus \mathcal{Q}$, such that $g(t,u,0) = 0$. Then it follows from (2.5) that $\mathbb{E}^x\left[f_u(X_t)\right] = g(t,u,x) = 0$ for all $x \in D$, as claimed in the Lemma. On the other hand, for all $(t,u) \in \mathcal{Q}$ it holds that $\Phi(t,u) \neq 0$, such that we can define $h(x) = \Phi(t,u)^{-1} g(t,u,x)$. The function

$h(x)$ is measurable and satisfies $h(x)h(\xi) = h(x+\xi)$ for all $x, \xi \in D$. Moreover $h(0) \neq 0$ by definition of $\mathcal{Q}$. Using a standard result on measurable solutions of the Cauchy equation (cf. Aczél (1966, Sec. 2.2)) we conclude that there exists a unique continuous extension of $\psi(t,u)$ such that $\Phi(t,u)^{-1} g(t,u,x) = e^{\langle \psi(t,u), x\rangle}$, and the proof is complete. □

From this point on, $\Phi(t,u)$ and $\psi(t,u)$ are defined on all of $\mathcal{Q}$, and given by the unique continuous extensions of Lemma 2.5. We can now give a precise definition of regularity of an affine process:

Definition 2.7. An affine process X is called regular, if the derivatives

$$F(u) = \frac{\partial}{\partial t}\Phi(t,u)\Big|_{t=0}, \qquad R(u) = \frac{\partial}{\partial t}\psi(t,u)\Big|_{t=0} \tag{2.6}$$

exist and are continuous functions of $u \in \mathcal{U}$.

Remark 2.8. In Duffie et al. (2003) $F(u)$ is defined in a slightly different way, as the derivative of $\phi(t,u)$ at $t = 0$. However the definitions are equivalent, since $\Phi(t,u) = \exp(\phi(t,u))$ and thus

$$\frac{\partial}{\partial t}\Phi(t,u)\Big|_{t=0} = e^{\phi(0,u)} \cdot \frac{\partial}{\partial t}\phi(t,u)\Big|_{t=0} = \frac{\partial}{\partial t}\phi(t,u)\Big|_{t=0}.$$

Proposition 2.9. *Let X be a stochastically continuous affine process. The functions Φ and ψ have the following properties:*

(i) *Φ maps $\mathcal{Q}$ to the unit disc $\{u \in \mathbb{C} : |u| \leq 1\}$.*
(ii) *ψ maps $\mathcal{Q}$ to $\mathcal{U}$.*
(iii) *$\Phi(0,u) = 1$ and $\psi(0,u) = u$ for all $u \in \mathcal{U}$.*
(iv) *Φ and ψ enjoy the '**semi-flow property**': Suppose that $t, s \geq 0$ and $(t+s,u) \in \mathcal{Q}$. Then also $(t,u) \in \mathcal{Q}$ and $(s, \psi(t,u)) \in \mathcal{Q}$, and it holds that*

$$\begin{aligned} \Phi(t+s,u) &= \Phi(t,u) \cdot \Phi(s, \psi(t,u)), \\ \psi(t+s,u) &= \psi(s, \psi(t,u)). \end{aligned} \tag{2.7}$$

(v) *Φ and ψ are jointly continuous on $\mathcal{Q}$.*
(vi) *With the remaining arguments fixed, $u_I \mapsto \Phi(t,u)$ and $u_I \mapsto \psi(t,u)$ are analytic functions in $\{u_I : \operatorname{Re} u_I < 0; (t,u) \in \mathcal{Q}\}$.*
(vii) *Let $(t,u), (t,w) \in \mathcal{Q}$ with $\operatorname{Re} u \leq \operatorname{Re} w$. Then*

$$\begin{aligned} |\Phi(t,u)| &\leq \Phi(t, \operatorname{Re} w) \\ \operatorname{Re} \psi(t,u) &\leq \psi(t, \operatorname{Re} w)\,. \end{aligned}$$

Proof. Let $(t,u) \in \mathcal{Q}$. Clearly $|\mathbb{E}^x[f_u(X_t)]| \leq \mathbb{E}^x[|f_u(X_t)|] \leq 1$. On the other hand $\mathbb{E}^x[f_u(X_t)] = \Phi(t,u) f_{\psi(t,u)}(x)$ by Lemma 2.5. Since $\|f_u\|_\infty \leq 1$ if and only if $u \in \mathcal{U}$, we conclude that $|\Phi(t,u)| \leq 1$ and $\psi(t,u) \in \mathcal{U}$ for all $(t,u) \in \mathcal{Q}$ and have shown (i) and (ii). Assertion (iii) follows immediately from $\mathbb{E}^x[f_u(X_0)] = f_u(x)$. For (iv) suppose that $(t+s,u) \in \mathcal{Q}$, such that

$$\mathbb{E}^x[f_u(X_{t+s})] = \Phi(t+s,u) f_{\psi(t+s,u)}(x) \tag{2.8}$$

by Lemma 2.5. Applying the law of iterated expectations and the Markov property of X it holds that

$$\mathbb{E}^x[f_u(X_{t+s})] = \mathbb{E}^x\left[\mathbb{E}^x[f_u(X_{t+s}) | \mathcal{F}_s]\right] = \mathbb{E}^x\left[\mathbb{E}^{X_s}[f_u(X_t)]\right]. \tag{2.9}$$

If $(t,u) \notin \mathcal{Q}$ then the inner expectation (and consequently the whole expression) evaluates to 0, which is a contradiction to the fact that $(t+s,u) \in \mathcal{Q}$. It follows that $\mathbb{E}^{X_s}\left[f_u(X_t)\right] = \Phi(t,u) f_{\psi(t,u)}(X_s)$. If $(s,\psi(t,u)) \notin \mathcal{Q}$ then the outer expectation in (2.9) evaluates to 0, which is also a contradiction. Thus also $(s,\psi(t,u)) \in \mathcal{Q}$, as claimed, and we can write (2.9) as

$$\mathbb{E}^x\left[f_u(X_{t+s})\right] = \mathbb{E}^x\left[\Phi(t,u) f_{\psi(t,u)}(X_s)\right] = \Phi(s,u)\cdot\Phi(t,\psi(s,u)) f_{\psi(t,\psi(s,u))}(x)\ ,$$

for all $x \in D$. Comparing with (2.8) the semi-flow equations (2.7) follow. Assertions (v) and (vi) can be derived directly from the proof of Lemma 2.5. To show (vii) note that

$$\left|\mathbb{E}^x\left[f_u(X_t)\right]\right| \leq \mathbb{E}^x\left[\left|f_u(X_t)\right|\right] = \mathbb{E}^x\left[f_{(\operatorname{Re} u)}(X_t)\right] \leq \mathbb{E}^x\left[f_{(\operatorname{Re} w)}(X_t)\right]\ ,$$

for all $x \in D$. If (t,u) and (t,w) are in $\mathcal{Q}$, we deduce from the affine property (2.1) that

$$\left|\Phi(t,u)\right| \cdot \exp\left(\langle x, \operatorname{Re}\psi(t,u)\rangle\right) \leq \Phi(t,\operatorname{Re} w)\cdot\exp\left(\langle x,\psi(t,\operatorname{Re} w)\rangle\right)\ .$$

Inserting first $x=0$ and then Ce_i with $C>0$ arbitrarily large yields the assertion. □

Finally we show one additional technical property concerning the existence of derivatives of Φ and ψ with respect to u, on the interior of $\mathcal{U}$.

Lemma 2.10. *Let X be a stochastically continuous affine process. For $i \in I$, the derivatives*

$$\frac{\partial}{\partial u_i}\Phi(t,u), \qquad \frac{\partial}{\partial u_i}\psi(t,u)$$

exist and are continuous for $(t,u) \in \mathcal{U}^\circ \cap \mathcal{Q}$.

Proof. Let $i \in I$ and let K be a compact subset of $\mathcal{U}^\circ$. It holds that

$$\left|\frac{\partial}{\partial u_i}\exp\left(\langle u, X_t\rangle\right)\right| = \left|X_t^i\right| \cdot \exp\left(\langle \operatorname{Re} u, X_t\rangle\right)\ . \tag{2.10}$$

The right hand side is uniformly bounded for all $t \in \mathbb{R}_{\geqslant 0}, u \in K$, and thus in particular uniformly integrable. We may conclude that $\frac{\partial}{\partial u_i}\mathbb{E}^x\left[e^{\langle X_t,u\rangle}\right]$ exists and is a continuous function of $(t,u) \in \mathbb{R}_{\geqslant 0}\times K$ for any $x \in D$. If in addition $(t,u)\in\mathcal{Q}$, then Lemma 2.5 states that $\mathbb{E}^x\left[e^{\langle X_t,u\rangle}\right] = \Phi(t,u)\exp\left(\langle x,\psi(t,u)\rangle\right)$. Since K was an arbitrary compact subset of $\mathcal{U}^\circ$ the claim follows. □

3. Affine Processes are Feller Processes

In this section we prove the Feller property for all affine processes. For *regular* affine processes, this has been shown in Duffie et al. (2003); here we give a proof that does not require a regularity assumption. The key to the proof are the following properties of the function $\psi(t,u)$ for a given stochastically continuous affine process X, which will also be used in the proof of our main result in Section 5:

Property A: $\psi(t,.)$ maps $\mathcal{U}^\circ$ to $\mathcal{U}^\circ$.
Property B: $\psi_J(t,u) = e^{\beta t}u_J$ for all $(t,u)\in\mathcal{Q}$, with β a real $n\times n$-matrix.

Let us give here an intuitive example illustrating the second property, which has already been observed in Dawson and Li (2006, Prop 2.1 and Cor. 2.1): Consider

an affine process with one-dimensional state space $D = \mathbb{R}$, and the property that $\Phi(t,u) = 1$. Then for any initial value $x \in \mathbb{R}$

$$\mathbb{E}^x\left[f_{iy}(X_t)\right] = e^{x\psi(t,iy)} \quad \text{and} \quad \mathbb{E}^{-x}\left[f_{iy}(X_t)\right] = e^{-x\psi(t,iy)}$$

are both characteristic functions, and moreover reciprocal to each other. But a well-known result (cf. Lukacs (1960, Thm. 2.1.4)) states that the only characteristic functions, whose reciprocals are also characteristic functions correspond to degenerate distributions, i.e. Dirac measures. Here, this implies that $\psi(t,iy) = iym(t)$, for $m(t)$ a deterministic function. Moreover, by the Markov property $m(t+s) = m(t)m(s)$, which is Cauchy's functional equation with the unique continuous solution $m(t) = e^{\lambda t}m(0)$, for some $\lambda \in \mathbb{R}$. Hence, $\psi(t,iy)$ is necessarily of the form $e^{\lambda t}\, iy$ and satisfies therefore *Property B*.

As we shall show, the argument can be extended to the case of arbitrary $\Phi(t,u)$ and to the general state space $D = \mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$. The next Lemma is the first step in this direction:

Lemma 3.1. *Let X be a stochastically continuous affine process. Let $K \subseteq \{1,\dots,d\}$, $k \in \{1,\dots,d\}$, and let $(t_n)_{n\in\mathbb{N}}$ be a sequence such that $t_n \downarrow 0$. Define $\Omega_K := \left\{y \in \mathbb{R}^d : y_i = 0 \text{ for } i \notin K\right\}$, and suppose that*

$$\operatorname{Re}\psi_k(t_n, iy) = 0 \qquad \text{for all } y \in \Omega_K \text{ and } n \in \mathbb{N}\,.$$

Then there exist $\zeta_k(t_n) \in \mathbb{R}^{|K|}$ and an increasing sequence of positive numbers R_n such that $R_n \uparrow \infty$ and

$$\psi_k(t_n, iy) = \langle \zeta_k(t_n), iy_K\rangle$$

for all $y \in \Omega_K$ with $|y| < R_n$.

For the proof we will use the following result:

Lemma 3.2. *Let Θ be a positive definite function on $\mathbb{R}^d$ with $\Theta(0) = 1$. Then*

$$|\Theta(y+z) - \Theta(y)\Theta(z)|^2 \leq \left(1 - |\Theta(y)|^2\right)\left(1 - |\Theta(z)|^2\right) \leq 1$$

for all $y, z \in \mathbb{R}^d$.

Proof of Lemma 3.2. The result follows from considering the matrix

$$M_\Theta(y,z) := \begin{pmatrix} \Theta(0) & \overline{\Theta(y)} & \Theta(z) \\ \Theta(y) & \Theta(0) & \Theta(y+z) \\ \overline{\Theta(z)} & \overline{\Theta(y+z)} & \Theta(0) \end{pmatrix}, \qquad y, z \in \mathbb{R}^d, y \neq z\,,$$

which is positive semi-definite by definition of Θ. The inequality is then derived from the fact that $\det M_\Theta(y,z) \geq 0$. See Jacob (2001, Lemma 3.5.10) for details. □

Proof of Lemma 3.1. As the characteristic function of the (possibly defective) random variable X_{t_n} under $\mathbb{P}^x$, the function $y \mapsto \mathbb{E}^x\left[f_{iy}(X_{t_n})\right]$ is positive definite for any $x \in D, n \in \mathbb{N}$. We define now for every $y \in \Omega_K$, $c > 0$, and $n \in \mathbb{N}$, the function

$$\Theta(y; n, c) := \frac{1}{\Phi(t_n, 0)}\mathbb{E}^{ce_k}\left[f_{iy}(X_{t_n})\right] = \frac{\Phi(t_n, iy)}{\Phi(t_n, 0)} \exp\Big(c \cdot \psi_k(t_n, iy)\Big)\,.$$

Clearly, as a function of $y \in \Omega_K$, also $\Theta(y; n, c)$ is positive definite. In addition it satisfies $\Theta(0; n, c) = \exp\left(c \cdot \psi_k(t_n, 0)\right) = 1$, for large enough n, say $n \geq N$, by the following argument: It should be obvious, that $\exp(c \cdot \psi_k(t_n, 0)$ is always a real quantity. By assumption, $\psi_k(t_n, 0)$ is purely imaginary, such that it must be an

integer multiple of π for all $n \in \mathbb{N}$. But $\psi_k(0,0) = 0$, and $\psi_k(t,0)$ is continuous in t by Proposition 2.9, and we conclude $\psi_k(t_n, 0) = 0$ for large enough n.

Thus, for $n \geq N$, we may apply Lemma 3.2 to Θ, and conclude that for any $y, z \in \Omega_K$, $c > 0$ and $n \geq N$

$$|\Theta(y+z;n,c) - \Theta(y;n,c) \cdot \Theta(z;n,c)|^2 \leq 1 \,. \tag{3.1}$$

For compact notation we define the abbreviations

$$\begin{aligned} r_1 &= \left|\frac{\Phi(t_n, i(y+z))}{\Phi(t_n,0)}\right|, & r_2 &= \left|\frac{\Phi(t_n,iy)\Phi(t_n,iz)}{\Phi(t_n,0)^2}\right|, \\ \alpha_1 &= \operatorname{Arg}\frac{\Phi(t_n, i(y+z))}{\Phi(t_n,0)}, & \alpha_2 &= \operatorname{Arg}\frac{\Phi(t_n,iy)\Phi(t_n,iz)}{\Phi(t_n,0)^2}, \\ \beta_1 &= \operatorname{Im}\psi_k(t_n, i(y+z)), & \beta_2 &= \operatorname{Im}\psi_k(t_n,iy) + \operatorname{Im}\psi_k(t_n,iz), \end{aligned}$$

where we suppress the dependency on y, t, z for the moment. It holds that

$$\left|r_1 e^{(\alpha_1 + c\beta_1)i} - r_2 e^{(\alpha_2 + c\beta_2)i}\right|^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\Big(\alpha_1 - \alpha_2 + (\beta_1 - \beta_2)c\Big) \,.$$

Using the elementary inequality $2r_1r_2 \leq r_1^2 + r_2^2$, we derive

$$2r_1r_2\left\{1 - \cos\Big(\alpha_1 - \alpha_2 + (\beta_1 - \beta_2)c\Big)\right\} \leq \left|r_1 e^{(\alpha_1 + c\beta_1)i} - r_2 e^{(\alpha_2 + c\beta_2)i}\right|^2 \,,$$

which combined with inequality (3.1) yields

$$r_1 r_2 \left(1 - \cos\Big(\alpha_1 - \alpha_2 + (\beta_1 - \beta_2)c\Big)\right) \leq \frac{1}{2} \,. \tag{3.2}$$

Define now $R_n = 0$ for $n < N$, and

$$R_n := \sup\left\{\rho \geq 0 : r_1(y,t_n,z)r_2(y,t_n,z) > \frac{1}{2} \text{ for } y,z \in \Omega_K \text{ with } |y| \leq \rho, |z| \leq \rho\right\}$$

for $n \geq N$. Note that $R_n \uparrow \infty$: This follows from $r_1(y,0,z) = r_2(y,0,z) = 1$ for all $y, z \in \Omega_K$, and the continuity of r_1 and r_2.

Suppose that

$$\beta_1 - \beta_2 = \operatorname{Im}\psi_k(t_n, i(y+z)) - \operatorname{Im}\psi_k(t_n,iy) - \operatorname{Im}\psi_k(t_n,iz) \neq 0$$

for any $n \in \mathbb{N}$ and $y, z \in \Omega_K$ with $|y| < R_n$, $|z| < R_n$. Then there exists an $c > 0$ such that

$$\cos\Big(\alpha_1 - \alpha_2 + (\beta_1 - \beta_2)c\Big) = -1 \,.$$

Inserting into (3.2) we obtain

$$\frac{1}{2} \cdot 2 < r_1 r_2 \left(1 - \cos\Big(\alpha_1 - \alpha_2 + (\beta_1 - \beta_2)c\Big)\right) \leq \frac{1}{2} \,,$$

a contradiction. We conclude that

$$\beta_1 - \beta_2 = \operatorname{Im}\psi_k(t_n, i(y+z)) - \operatorname{Im}\psi_k(t_n,iy) - \operatorname{Im}\psi_k(t_n,iz) = 0 \,, \tag{3.3}$$

for all $y, z \in \Omega_K$ with $|y| < R_n$, $|z| < R_n$. Equation (3.3) is nothing but Cauchy's first functional equation. Since $\psi(t,.)$ is continuous, it follows that $\operatorname{Im}\psi_k$ is a linear function of y_K. In addition, $\operatorname{Re}\psi_k(t_n, y)$ is zero, by assumption, such that there exists some real vector $\zeta_k(t_n)$ with

$$\psi_k(t_n, iy) = \langle \zeta_k(t_n), iy_K \rangle \,. \tag{3.4}$$

for all $y \in \Omega_K$ with $|y| < R_n$, and the Lemma is proved. □

We use the above Lemma to show the following Proposition, which implies Property B of ψ, that was introduced at the beginning of the section:

Proposition 3.3. *Let $(X_t)_{t\geq 0}$ be a stochastically continuous affine process on $D = \mathbb{R}^m_{\geqslant 0}\times\mathbb{R}^n$ and denote by J its real-valued components. Then there exists a real $n\times n$-matrix β such that $\psi_J(t,u) = e^{t\beta}u_J$ for all $(t,u)\in\mathcal{Q}$.*

Proof. Consider the definition of $\mathcal{U}$ in (2.2). Since $\psi(t,u)$ takes by Proposition 2.9 values in $\mathcal{U}$ it is clear that $\operatorname{Re}\psi_J(t,iy) = 0$ for *any* $(t,y)\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}^d$. Fix now some $t_* > 0$ and define $t_n := t_*/n$ for all $n\in\mathbb{N}$. We can apply Lemma 3.1 with $K = \{1,\dots,d\}$ and any choice of $k\in J$, to obtain a sequence $R_n\uparrow\infty$ (even independent of k), such that

$$\psi_J(t_n, iy) = \Xi(t_n)\cdot iy\ , \tag{3.5}$$

for all $y\in\mathbb{R}^d$ with $|y| < R_n$. Here $\Xi(t_n)$ denotes the real $n\times d$-matrix formed by concatenating the column vectors $(\zeta_k(t_n))_{k=1,\dots,d}$ obtained from Lemma 3.1.

Let $i\in I$, $n\in\mathbb{N}$, define $\Omega_n := \{\omega\in\mathbb{C} : |\omega|\leq R_n, (t_n, e_i\omega)\in\mathcal{Q}\}$, and consider the function

$$h_n : \Omega_n\to\mathbb{C}^n : \quad \omega\mapsto\psi_J(t_n,\omega e_i) - \Xi(t)\cdot\omega e_i\ .$$

This is an analytic function on Ω_n° and continuous on Ω_n. According to the Schwarz reflection principle, h_n can be extended to an analytic function on an open superset of Ω_n. But (3.5) implies that the function h_n takes the value 0 on a subset with an accumulation point in $\mathbb{C}$. We conclude that h_n is zero everywhere. In particular we have that

$$0 = \operatorname{Re}\psi_J(t_n,\omega e_i) - \Xi(t_n)\cdot\operatorname{Re}\omega e_i = \Xi(t_n)\cdot\operatorname{Re}\omega e_i\ ,$$

for all $\omega\in\Omega_n$. This can only hold true, if the i-th column of $\Xi(t_n)$ is zero. Since $i\in I$ was arbitrary we have reduced (3.5) to

$$\psi_J(t_n,u) = \Xi_0(t_n)\cdot u_J\ , \tag{3.6}$$

for all $(t_n,u)\in\mathcal{Q}$, such that $|u_J| < R_n$. Here $\Xi_0(t_n)$ denotes the $n\times n$-submatrix of $\Xi(t_n)$ that results from dropping the zero-columns.

Fix an arbitrary $u_*\in\mathcal{U}$ with $(t_*,u_*)\in\mathcal{Q}$. By Proposition 2.9 we know that also $(t,u_*)\in\mathcal{Q}$ for all $t\in[0,t_*]$, such that $R := \sup\{|\psi_K(t,u_*)| : t\in[0,t_*]\}$ is well-defined. Since $\psi(t,u)$ is continuous, R is finite. Choose N such that $R_n > R$ for all $n\geq N$. Using the semi-flow equation we can write $\psi_J(t_*,u_*)$ as

$$\begin{aligned}\psi_J(t_*,u_*) = \psi_J\left(t_n,\psi(t_*\tfrac{n-1}{n},u_*)\right) = \\ = \Xi_0(t_n)\cdot\psi_J(t_*\tfrac{n-1}{n},u_*) = \cdots = \Xi_0(t_n)^n\cdot u_*\ ;\end{aligned} \tag{3.7}$$

for any $n\geq N$. Thus, the functional equation $\psi(t,u) = \Xi_0(t)\cdot u_J$ actually holds for all $(t,u)\in\mathcal{Q}$. Another application of the semi-flow property yields then, that

$$\Xi_0(t+s) = \Xi_0(t)\Xi_0(s), \qquad \text{for all } t,s\geq 0\ .$$

Since $\Xi_0(0) = 1$, Ξ_0 is continuous and satisfies the second Cauchy functional equation, it follows that $\Xi_0(t) = e^{\beta t}$ for some real $n\times n$-matrix β, which completes the proof. □

The next proposition shows that also Property A holds true for ψ, as we have claimed at the beginning of the section.

Proposition 3.4. *Suppose that $(t,u)\in\mathcal{Q}$. If $u\in\mathcal{U}^\circ$, then $\psi(t,u)\in\mathcal{U}^\circ$.*

Proof. For a contradiction, assume there exists $(t,u) \in \mathcal{Q}$ such that $u \in \mathcal{U}^\circ$, but $\psi(t,u) \notin \mathcal{U}^\circ$. This implies that there exists $k \in I$, such that $\operatorname{Re}\psi_k(t,u) = 0$. Let $\mathcal{Q}_{t,k} = \{\omega \in \mathbb{C} : \operatorname{Re}\omega \leq 0;\, (t, \omega e_k) \in \mathcal{Q}\}$. From the inequalities of Proposition 2.9.vii we deduce that

$$0 = \operatorname{Re}\psi_k(t,u) \leq \psi_k(t, \operatorname{Re}\omega \cdot e_k) \leq 0\,, \tag{3.8}$$

and thus that $\psi_k(t, \operatorname{Re}\omega \cdot e_k) = 0$ for all $\omega \in \mathcal{Q}_{t,k}$ with $\operatorname{Re} u_k \leq \operatorname{Re}\omega$. By Proposition 2.9.(vi), $\psi_k(t, \omega e_k)$ is an analytic function of ω. Since it takes the value zero on a set with an accumulation point, it is zero everywhere, i.e. $\psi_k(t, \omega e_k) = 0$ for all $\omega \in \mathcal{Q}_{t,k}$. The same statement holds true for t replaced by $t/2$: Set $\lambda := \operatorname{Re}\psi_k(t/2, u)$. If $\lambda = 0$, we can proceed exactly as above, only with $t/2$ instead of t. If $\lambda < 0$, then we have, by another application of Proposition 2.9.vii, that

$$0 = \operatorname{Re}\psi_k(t,u) = \operatorname{Re}\psi_k(t/2, \psi(t/2,u)) \leq \psi_k(t/2, \lambda e_k) \leq \psi_k(t/2, \operatorname{Re}\omega e_k) \leq 0\,,$$

for all $\omega \in \mathcal{Q}_{t/2,k}$ such that $\lambda \leq \operatorname{Re}\omega$. Again we use that an analytic function that takes the value zero on a set with accumulation point, is zero everywhere, and obtain that $\psi_k(t/2, \omega e_k) = 0$ for all $\omega \in \mathcal{Q}_{t/2,k}$. Repeating this argument, we finally obtain a sequence $t_n \downarrow 0$, such that

$$\psi_k(t_n, \omega e_k) = 0 \qquad \text{for all } \omega \in \mathcal{Q}_{t_n,k}. \tag{3.9}$$

We can now apply Lemma 3.1 with $K = \{k\}$, which implies that ψ_k is of the linear form

$$\psi_k(t_n, \omega e_k) = \zeta_k(t_n) \cdot \omega, \quad \text{for all} \quad \omega \in \mathcal{Q}_{t_n,k} \text{ with } |\omega| \leq R_n,$$

where $\zeta_k(t_n)$ are real numbers, and $R_n \uparrow \infty$. Note that since $\zeta_k(t_n) \to 1$ as $t_n \to 0$, we have that $\zeta_k(t_n) > 0$ for n large enough. Choosing now some ω_* with $\operatorname{Re}\omega_* < 0$ it follows that $\operatorname{Re}\psi_k(t_n, \omega_* e_k) < 0$ – with strict inequality. This is a contradiction to (3.9), and the assertion is shown. □

We are now prepared to show the main result of this section:

Theorem 3.5. *Every stochastically continuous affine process X is a Feller process.*

Remark 3.6. As an immediate Corollary to this theorem, every stochastically continuous affine process has a càdlàg version, see for instance Rogers and Williams (1994).

Proof. By stochastic continuity of $(X_t)_{t\geq 0}$ and dominated convergence, it follows immediately that $P_t f(x) = \mathbb{E}^x[f(X_t)] \to f(x)$ as $t \to 0$ for all $f \in C_0(D)$ and $x \in D$. To prove the Feller property of $(X_t)_{t\geq 0}$ it remains to show that $P_t(C_0(D)) \subseteq C_0(D)$: For $u_I \in \mathbb{C}^m$ with $\operatorname{Re} u_I < 0$ and $g \in C_c^\infty(\mathbb{R}^n)$, i.e. a smooth function with bounded support, define the functions

$$h(x; u_I, g) = e^{\langle u_I, x_I\rangle} \int_{\mathbb{R}^n} f_{iy}(x_J) g(y)\, dy$$

mapping D to $\mathbb{C}$, and the set

$$P := \left\{ h(x; u_I, g) \,:\, u_I \in \mathbb{C}^d,\, \operatorname{Re} u_I < 0, g \in C_c^\infty(\mathbb{R}^n) \right\}\,.$$

Denote by $\mathcal{L}(P)$ the set of (complex) linear combinations of functions in P. From the Riemann-Lebesgue Lemma it follows that $\int_{\mathbb{R}^n} f_{iy}(x_J) g(y)\, dy$ vanishes at infinity, and thus that $\mathcal{L}(P) \subset C_0(D)$. It is easy to see that $\mathcal{L}(P)$ is a subalgebra of $C_0(D)$, that is in addition closed under complex conjugation and multiplication. (Note

that the product of two Fourier transforms of compactly supported functions g_1, g_2 is the Fourier transform of a compactly supported function, namely $g_1 * g_2$.) It is also straight-forward to check that $\mathcal{L}(P)$ is point separating and vanishes nowhere (i.e. there is no $x_0 \in D$ such that $h(x_0) = 0$ for all $h \in \mathcal{L}(P)$). Using a suitable version of the Stone-Weierstrass theorem (e.g. Semadeni (1971, Corollary 7.3.9)), it follows that $\mathcal{L}(P)$ is dense in $C_0(D)$.

Fix some $t \in \mathbb{R}_{\geqslant 0}$ and let $h(x) \in P$. By Lemma 2.5 it holds that $\mathbb{E}^x [f_u(X_t)] = \Phi(t,u) \exp(\langle x, \psi(t,u)\rangle)$ whenever $(t,u) \in \mathcal{Q}$, and $\mathbb{E}^x[f_u(X_t)] = 0$ whenever $(t,u) \notin \mathcal{Q}$. Moreover, by Proposition 3.3 we know that $\psi_J(t,u) = e^{\beta t} u_J$ for all $(t,u) \in \mathcal{Q}$. Thus, writing $u = (u_I, iy)$, we have

(3.10)

$$\begin{aligned} P_t h(x) &= \mathbb{E}^x \left[\int_{\mathbb{R}^n} f_{(u_I,iy)}(X_t) g(y)\, dy \right] = \int_{\mathbb{R}^n} \mathbb{E}^x \left[f_{(u_I,iy)}(X_t) \right] g(y)\, dy = \\ &= \int_{\{u \in \mathcal{U} : (t,u) \in \mathcal{Q}\}} \mathbb{E}^x \left[f_{(u_I,iy)}(X_t) \right] g(y)\, dy = \\ &= \int_{\{u \in \mathcal{U} : (t,u) \in \mathcal{Q}\}} \Phi(t, u_I, iy) \exp\left(\langle x_I, \psi_I(t, u_I, iy)\rangle + \langle x_J, e^{t\beta} iy \rangle \right) g(y)\, dy\,. \end{aligned}$$

Since $(u_I, iy) \in \mathcal{U}^\circ$ it follows by Proposition 3.4 that also $\operatorname{Re} \psi_I(t, u_I, iy) < 0$ for any $y \in \mathbb{R}^n$. This shows that $P_t h(x) \to 0$ as $|x_I| \to \infty$. In addition, as a function of x_J, (3.10) can be interpreted as the Fourier transformation of a compactly supported density. The Riemann-Lebesgue Lemma then implies that $P_t h(x) \to 0$ as $|x_J| \to \infty$, and we conclude that $P_t h \in C_0(D)$. The assertion extends by linearity to every $h \in \mathcal{L}(P)$, and finally by the density of $\mathcal{L}(P)$ to every $h \in C_0(D)$. This proves that the semi-group $(P_t)_{t \geq 0}$ maps $C_0(D)$ into $C_0(D)$, and hence that $(X_t)_{t \geq 0}$ is a Feller process. □

4. All semi-homogeneous affine processes are regular

Definition 4.1. We say that a stochastically continuous affine process is *semi-homogeneous*, if for all $x \in D$, $(t,u) \in \mathbb{R}_{\geqslant 0} \times i\mathbb{R}^d$

$$\mathbb{E}^x \left[e^{\langle X_t, u\rangle} \right] = e^{\langle x_J, u_J\rangle} \cdot \mathbb{E}^{(x_I, 0)} \left[e^{\langle X_t, u\rangle} \right]\,. \tag{4.1}$$

The above condition is equivalent to the statement that for any y of the form $y = (0, y_J)$, the law of $X_t + y$ under $\mathbb{P}^x$ equals the law of X_t under $\mathbb{P}^{(x+y)}$. If this held true for any $y \in D$ we would speak of a (space-)homogeneous Markov process. Since we impose the condition only for y of the form $(0, y_J)$, we call the process semi-homogeneous. Note that semi-homogeneous affine processes are frequently encountered in mathematical finance: Affine stochastic volatility models (e.g. the Heston model) are typically based on a semi-homogeneous affine process; see Keller-Ressel (2008a).

Combining Definition 4.1 with the affine property (2.1), it is easy to see that the following holds:

Lemma 4.2. *A stochastically continuous affine process is semi-homogeneous, if and only if $\psi_J(t,u) = u_J$ for all $(t,u) \in \mathbb{R}_{\geqslant 0} \times i\mathbb{R}^d$ (or equivalently for all $(t,u) \in \mathcal{Q}$).*

The main result of this section is the following:

Theorem 4.3. *Every semi-homogeneous, stochastically continuous affine process is regular.*

Our proof uses the techniques originally presented in Montgomery and Zippin (1955) for continuous transformation groups, and follows in part the proof of (Dawson and Li 2006, Theorem 4.1).

Proof. To simplify calculations we embed $\Phi(t,u)$ and $\psi(t,u)$ into the extended semi-flow $\Upsilon(t,u)$, that is we set $\widehat{\mathcal{Q}} := \mathcal{Q} \times \mathbb{C}$ and define

$$\Upsilon : \widehat{\mathcal{Q}} \to \mathbb{C}^{d+1}, \quad (t, u_1, \dots u_d, u_{d+1}) \mapsto \begin{pmatrix} \psi(t,(u_1,\dots,u_d)) \\ \Phi(t,(u_1,\dots,u_d)) \cdot u_{d+1} \end{pmatrix}. \tag{4.2}$$

Note that all vectors u have now a $(d+1)$-th component added; this component will be assigned to the non-negative components I, such that under slight abuse of notation we now write $I = \{1,\dots,m,d+1\}$. The semi-flow property is preserved by $\Upsilon(t,u)$, i.e. $\Upsilon(t+s,u) = \Upsilon(t,\Upsilon(s,u))$ for all $(t+s,u) \in \widehat{\mathcal{Q}}$. The semi-homogeneity condition on X implies that $\Upsilon_J(t,u) = u_J$ for all $(t,u) \in \widehat{\mathcal{Q}}$. Clearly, the time derivative exists and vanishes, i.e.,

$$\frac{\partial}{\partial t}\Upsilon_J(t,u)\Big|_{t=0} = 0$$

for all $u \in \mathcal{U} \times \mathbb{C}$. In the rest of the proof we thus focus on the remaining (not space-homogeneous) components. Let $u \in \widehat{\mathcal{U}}^\circ := \mathcal{U}^\circ \times \mathbb{C}$ be fixed and assume that $t,s \in \mathbb{R}_{\geqslant 0}$ are small enough such that $\Phi(t+s,u)$, $\psi(t+s,u)$ and their u-derivatives are always well-defined (cf. Lemma 2.10). Denote by $\frac{\partial \Upsilon_I}{\partial u_I}(t,u)$ the Jacobian of Υ_I with respect to u_I. Using a Taylor expansion we have that

$$\int_0^s \Upsilon_I(r,\Upsilon(t,u))\,dr - \int_0^s \Upsilon_I(t,u)\,dr = \int_0^s \frac{\partial \Upsilon_I}{\partial u_I}(r,u)\,dr \cdot (\Upsilon_I(t,u) - u_I) + \\ + o\left(\|\Upsilon_I(t,u) - u_I\|\right) . \tag{4.3}$$

On the other hand, using the semi-flow property of Υ we can write the left side of (4.3) as

$$\begin{aligned} &\int_0^s \Upsilon_I(r,\Upsilon(t,u))\,dr - \int_0^s \Upsilon_I(r,u)\,dr = \int_0^s \Upsilon_I(r+t,u)\,dr - \int_0^s \Upsilon_I(r,u)\,dr = \\ &= \int_t^{s+t} \Upsilon_I(r,u)\,dr - \int_0^s \Upsilon_I(r,u)\,dr = \int_s^{s+t} \Upsilon_I(r,u)\,dr - \int_0^t \Upsilon_I(r,u)\,dr = \\ &= \int_0^t \Upsilon_I(r+s,u)\,dr - \int_0^t \Upsilon_I(r,u)\,dr\,. \end{aligned} \tag{4.4}$$

Denoting the last expression by $I(s,t)$ and combining (4.3) with (4.4) we obtain

$$\lim_{t \downarrow 0} \frac{\left\|\frac{1}{s} I(s,t)\right\|}{\|\Upsilon_I(t,u) - u_I\|} = \left\| \frac{1}{s} \int_0^s \frac{\partial \Upsilon_I}{\partial u_I}(r,u)\,dr \right\| .$$

Define $M(s,u) := \frac{1}{s}\int_0^s \frac{\partial \Upsilon_I}{\partial u_I}(r,u)\,dr$. Note that as $s \to 0$, it holds that $M(s,u) \to \frac{\partial \Upsilon_I}{\partial u_I}(0,u) = I_I$ (the identity matrix). Thus for s small enough $\|M(s,u)\| \neq 0$, and

we conclude that

$$\lim_{t\downarrow 0}\frac{1}{t}\left\|\Upsilon_I(t,u)-u_I\right\| = \tag{4.5}$$
$$= \left\|\lim_{t\downarrow}\frac{I(s,t)}{st}\right\|\cdot\|M(s,u)\|^{-1} = \left\|\frac{\Upsilon_I(s,u)-u_I}{s}\right\|\cdot\|M(s,u)\|^{-1}\ .$$

The right hand side of (4.5) is well-defined and finite, implying that also the limit on the left hand side is. Thus, combining (4.3) and (4.4), dividing by st and taking the limit $t\downarrow 0$ we obtain

$$\lim_{t\downarrow 0}\frac{\Upsilon_I(t,u)-u_I}{t} = \frac{\Upsilon_I(s,u)-u_I}{s}\cdot M(s,u)^{-1}\ .$$

Again we may choose s small enough, such that $M(s,u)$ is invertible, and the right hand side of the above expression is well-defined. The existence and finiteness of the right hand side then implies the existence of the limit on the left. In addition the right hand side is a continuous function of $u\in\widehat{\mathcal{U}}^\circ$, such that also the left hand side is. Adding back the components J, for which a time derivative trivially exists (recall that $\psi_J(t,u)=u_J$ for all $t\geq 0$), we obtain that

$$\mathcal{R}(u) := \lim_{t\downarrow 0}\frac{\Upsilon(t,u)-u}{t} = \left.\frac{\partial}{\partial t}\Upsilon(t,u)\right|_{t=0} \tag{4.6}$$

exists and is a continuous function of $u\in\widehat{\mathcal{U}}^\circ$. Denoting the first d components of $\mathcal{R}(u)$ by $R(u)$ and the $d+1$-th component by $F(u)$ we can 'disentangle' the extended semi-flow Υ, drop the $(d+1)$-th component of u, and see that

$$F(u) := \left.\frac{\partial}{\partial t}\Phi(t,u)\right|_{t=0} \quad\text{and}\quad R(u) := \left.\frac{\partial}{\partial t}\psi(t,u)\right|_{t=0} \tag{4.7}$$

are likewise well-defined and continuous on $\mathcal{U}^\circ$.

To show that $(X_t)_{t\geq 0}$ is regular affine it remains to show that (4.7) extends continuously to $\mathcal{U}$: To this end let $t_n\downarrow 0$, $x\in D$, $u\in\mathcal{U}^\circ$, and rewrite (4.7) as

$$F(u)+\langle x,R(u)\rangle = \lim_{n\to\infty}\frac{\Phi(t_n,u)\exp\left(\langle x,\psi(t_n,u)-u\rangle\right)-1}{t_n} = \tag{4.8}$$
$$= \lim_{n\to\infty}\frac{f_{-u}(x)\,\mathbb{E}^x\left[f_u(X_{t_n})\right]-1}{t_n} = \lim_{n\to\infty}\frac{1}{t_n}\left\{\int_D e^{\langle\xi-x,u\rangle}\,p_{t_n}(x,d\xi)-1\right\} =$$
$$= \lim_{n\to\infty}\frac{1}{t_n}\left\{\int_{D-x}\left(e^{\langle\xi,u\rangle}-1\right)\,\widetilde{p}_{t_n}(x,d\xi)+\frac{p_{t_n}(x,D)-1}{t_n}\right\}\ ,$$

where $p_t(x,d\xi)$ is the transition kernel of the Markov process $(X_t)_{t\geq 0}$, and $\widetilde{p}_t(x,d\xi)$ is its 'shifted transition kernel' $\widetilde{p}_t(x,d\xi) := p_t(x,d\xi+x)$. The right hand side of (4.8) can be regarded as a limit of log-characteristic functions of (infinitely divisible) sub-stochastic measures[2]. That is, there exist infinitely divisible sub-stochastic measures $\mu_n(x,d\xi)$, such that

$$\exp\left(F(u)+\langle x,R(u)\rangle\right) = \lim_{n\to\infty}\int_{\mathbb{R}^d}e^{\langle u,\xi\rangle}\,\mu_n(x,d\xi),\quad\text{for all } u\in\mathcal{U}^\circ.$$

[2]Note that $\exp\left(\frac{1}{t_n}\left\{\int_{D-x}\left(e^{\langle\xi,u\rangle}-1\right)\,\widetilde{p}_t(x,d\xi)\right\}\right)$ is the characteristic function of a compound Poisson distribution with intensity $\frac{1}{t_n}$ and jump measure $\widetilde{p}_t(x,d\xi)$.

Let now $\theta \in \mathbb{R}^d$ with $\theta_I < 0$ and $\theta_J = 0$ (note that $\theta \in \mathcal{U}^\circ$) and consider the exponentially tilted measures $e^{\langle \theta, \xi \rangle}\, \mu_n(x, d\xi)$. Their characteristic functions converge to $\exp\left(F(u+\theta) + \langle x, R(u+\theta)\rangle\right)$. Thus, by Lévy's continuity theorem, there exists $\mu_*(x, d\xi)$ such that $e^{\langle \theta, \xi \rangle}\, \mu_n(x, d\xi) \to \mu_*(x, d\xi)$ weakly. On the other hand, by Helly's selection theorem, $\mu_n(x, d\xi)$ has a vaguely convergent subsequence, which converges to some measure $\mu(x, d\xi)$. By uniqueness of the weak limit we conclude that $\mu(x, d\xi) = e^{\langle -\theta, \xi \rangle} \mu_*(x, d\xi)$. Thus we have that for all $x \in D$ and $u \in \mathcal{U}^\circ$ with $\operatorname{Re} u$ in a neighborhood of θ,

$$\exp\left(F(u) + \langle x, R(u)\rangle\right) = \lim_{n\to\infty} \int_{\mathbb{R}^d} e^{\langle u, \xi\rangle}\, \mu_n(d, d\xi) = \tag{4.9}$$

$$= \lim_{n\to\infty} \int_{\mathbb{R}^d} e^{\langle u-\theta, \xi\rangle}\, e^{\langle \theta, \xi\rangle} \mu_n(x, d\xi) = \int_{\mathbb{R}^d} e^{\langle u-\theta, \xi\rangle}\, \mu_*(x, d\xi) = \int_{\mathbb{R}^d} e^{\langle u, \xi\rangle}\, \mu(x, d\xi)\,.$$

But the choice of θ was arbitrary, such that (4.9) extends to all $u \in \mathcal{U}^\circ$. Applying dominated convergence to the last term of (4.9) shows that both F and R have a continuous extension to all of $\mathcal{U}$, which we also denote by F and R respectively.

It remains to show that (4.7) remains valid on $\mathcal{U}$: Let $u \in \mathcal{U}$ and $(u_n)_{n\in\mathbb{N}} \in \mathcal{U}^\circ$ such that $u_n \to u$. Remember that by Proposition 3.4 $u_n \in \mathcal{U}^\circ$ implies that also $\psi(t, u_n) \in \mathcal{U}^\circ$ for any $t \geq 0$. Thus we have

$$\int_0^t R(\psi(s,u))\, ds = \int_0^t \lim_{u_n \to u} R(\psi(s,u_n))\, ds = \lim_{u_n\to u} \int_0^t R(\psi(s,u_n))\, ds = \tag{4.10}$$

$$= \lim_{u_n \to u} \int_0^t \frac{\partial}{\partial t} \psi(s, u_n)\, ds = \lim_{u_n \to u} \psi(t, u_n) - u_n = \psi(t,u) - u\,.$$

Since the left hand side of (4.10) is t-differentiable, also the right hand side is, and we obtain $R(u) = \frac{\partial}{\partial t}\psi(t,u)\big|_{t=0}$ for all $u \in \mathcal{U}$. A similar calculation as above can be made upon replacing R with F, resulting in $F(u) = \frac{\partial}{\partial t}\Phi(t,u)\big|_{t=0}$ for all $u \in \mathcal{U}$, and thus showing that the semi-homogeneous affine process $(X_t)_{t\geq 0}$ is regular. □

5. All affine processes are regular

In this final section we reduce the question of regularity of general stochastically continuous affine processes to stochastically continuous, *semi-homogeneous* affine processes. Recall that for those processes we have shown regularity in the preceding section. The transformation of general affine processes to semi-homogeneous processes is based on the method of the moving frame, which has been successfully applied in the context of SPDEs several times; see for instance Filipović, Tappe, and Teichmann (2008) and Filipović, Tappe, and Teichmann (2009).

Theorem 5.1. *Every stochastically continuous affine process X is regular.*

Proof. By Theorem 3.5 X is a Feller process, and thus has a càdlàg version. Clearly, choosing a càdlàg version will not alter the functions $\Phi(t,u)$ and $\psi(t,u)$ defined by (2.1). Furthermore, we know by Proposition 3.3 that

$$\psi_J(t,u) = \exp(t\beta) u_J \tag{5.1}$$

for $(t,u) \in \mathcal{Q}$ and a real $n \times n$ matrix β. We define the $d \times d$ matrix

$$K = \begin{pmatrix} \mathrm{id}_m & 0 \\ 0 & \beta \end{pmatrix}, \tag{5.2}$$

and the transformation $\mathcal{T}$

$$Z_t = \mathcal{T}[X]_t := X_t - K^\top \int_0^t X_s \, ds, \tag{5.3}$$

transforming the process X path-by-path into a process Z. Note that the transformation is well-defined due to the càdlàg property of the trajectories, and preserves the stochastic continuity of X. Moreover, the transformation can be inverted by

$$\mathcal{T}^{-1}[Z]_t = Z_t + K^\top \int_0^t \exp\big((t-s)K^\top\big) Z_s \, ds, \tag{5.4}$$

which is seen directly by inserting (5.3) and integrating by parts.

We claim that the transformed process $Z = \mathcal{T}[X]$ is a semi-homogeneous affine process. For this purpose we calculate the conditional characteristic function: Let $u \in i\mathbb{R}^d$, and for each $N \in \mathbb{N}$ and $k \in \{0, \dots, N\}$, define $t_k = kt/N$, such that $t_0, \dots, t_N$, is an equidistant partition of $[0,t]$ into intervals of mesh t/N. By writing the time-integral as a limit of Riemann sums, and using dominated convergence we have that

$$\begin{aligned}\mathbb{E}^x\left[\exp\left(\langle u, Z_{t+s}\rangle\right)\middle|\, \mathcal{F}_s\right] = \exp\left(-\left\langle u, K^\top \int_0^s X_r \, dr\right\rangle\right) \cdot \\ \lim_{N\to\infty} \mathbb{E}^x\left[\exp\left(\langle u, X_{t+s}\rangle - \frac{t}{N}\left\langle Ku, \sum_{k=0}^{N-1} X_{s+t_k}\right\rangle\right)\middle|\, \mathcal{F}_s\right].\end{aligned} \tag{5.5}$$

With the shorthands $h := t/N$ and $\Sigma_n := \sum_{k=0}^n X_{s+t_k}$, and using the tower law as well as the affine property of $(X_t)_{t\geq 0}$, the expectation on the right side can be written as

$$\begin{aligned}&\mathbb{E}^x\left[\exp\Big(\langle u, X_{t+s}\rangle - h\,\langle Ku, \Sigma_{N-1}\rangle\Big)\middle|\, \mathcal{F}_s\right] = \\ &\mathbb{E}^x\left[\exp\Big(-h\,\langle Ku, \Sigma_{N-2}\rangle\Big)\cdot \mathbb{E}^x\left[\exp\Big(\langle (\mathrm{id}_d - hK)u, X_{t+s}\rangle\Big)\middle|\, \mathcal{F}_{s+t_{N-1}}\right]\middle|\, \mathcal{F}_s\right] = \\ &\Phi(h, (\mathrm{id}_d - hK)u)\cdot \\ &\qquad \cdot \mathbb{E}^x\left[\exp\Big(\big\langle \psi(h, (\mathrm{id}_d - hK)u), X_{s+t_{N-1}}\big\rangle - h\,\langle Ku, \Sigma_{N-2}\rangle\Big)\middle|\, \mathcal{F}_s\right].\end{aligned}$$

Applying the tower law $(N-1)$-times in the same way (conditioning on $\mathcal{F}_{s+t_{N-1}}, \mathcal{F}_{s+t_{N-2}}, \dots, \mathcal{F}_{s+t_1}$, respectively) we arrive at the equation

$$\begin{aligned}&\mathbb{E}^x\left[\exp\left(\langle u, X_{t+s}\rangle - \frac{t}{N}\left\langle Ku, \sum_{k=0}^{N-1} X_{s+t_k}\right\rangle\right)\middle|\, \mathcal{F}_s\right] \\ &= p(N-1;t,u)\exp\Big(\langle X_s, q(N-1;t,u)\rangle\Big),\end{aligned}$$

where the quantities $p(N-1;t,u)$ and $q(N-1;t,u)$ are defined through the following recursion:

$$p(0;t,u) = 1, \qquad p(k+1;t,u) = \Phi\Big(h, (\mathrm{id}_d - hK)q(k;t,u)\Big)\cdot p(k;t,u)\,, \tag{5.6a}$$

$$q(0;t,u) = u, \qquad q(k+1;t,u) = \psi\Big(h, (\mathrm{id}_d - hK)q(k;t,u)\Big)\,, \tag{5.6b}$$

Since the Riemannian sums in (5.5) converge point by point, we conclude that the quantities $p(N-1;t,u)$ and $q(N-1;t,u)$ converge to some functions $p(t,u), q(t,u)$

as $N \to \infty$, and thus

$$\mathbb{E}^x \left[\exp\left(\langle u, Z_{t+s} \rangle \right) | \, \mathcal{F}_s \right] = p(t,u) \exp\left(\langle q(t,u), X_s \rangle - \left\langle u, K^\top \int_0^s X_r \, dr \right\rangle \right) \tag{5.7}$$

for all $t, s \geq 0$ and $u \in i\mathbb{R}^d$. Let now $q_J(t,u)$ denote the J-components of $q(t,u)$. Based on the recursion (5.6) and the fact that $\psi_J(t,u) = e^{\beta t} u_J$ it holds that

$$q_J(t,u) = \lim_{N \to \infty} e^{t\beta} \Big(\mathrm{id}_n - \frac{t\beta}{N} \Big)^{N-1} u_J = e^{t\beta} e^{-t\beta} u_J = u_J \, .$$

Thus we can rewrite (5.7) as

$$\mathbb{E}^x \left[\exp\left(\langle u, Z_{t+s} \rangle \right) | \, \mathcal{F}_s \right] = p(t,u) \exp\left(\left\langle q_I(t,u), Z^I_s \right\rangle + \left\langle u_J, Z^J_s \right\rangle \right)$$

which shows that Z is indeed a stochastically continuous, semi-homogeneous affine process. By Theorem 4.3 such a process is regular. Hence, the functions $\widetilde{F}(u) = \frac{\partial}{\partial t} p(t,u)\big|_{t=0}$ and $\widetilde{R}(u) = \frac{\partial}{\partial t} q(t,u)\big|_{t=0}$ exist and satisfy the admissibility conditions in Duffie et al. (2003, Def. 2.6). By Duffie et al. (2003, Thm. 2.7), the functions $\widetilde{F}(u)$ and $\widetilde{R}(u) + Ku$ are also admissible, and thus define a regular affine process $\widetilde{X}$. Using now the Feynman-Kac formula in Duffie et al. (2003, Prop. 11.2), it is seen that the transformation $\mathcal{T}$ transforms the regular affine process $\widetilde{X}$ into the regular affine process $\mathcal{T}[\widetilde{X}]$ characterized by $\widetilde{F}(u)$ and $\widetilde{R}(u)$ – that is into a process equal in law to Z. We have shown that

$$\mathcal{T}[X] = Z \qquad \text{and} \qquad \mathcal{T}[\widetilde{X}] = Z \, ,$$

where equality is understood in law. Since the transformation $\mathcal{T}$ can be inverted path-by-path, we conclude that $X = \widetilde{X}$ in law, and thus that X is regular. □

Remark 5.2. The intuition behind the 'moving frame' transformation used above is the following: given the process X we first construct a a time-dependent coordinate transformation

$$Y_t = \exp\big(-K^\top t\big) X_t,$$

the 'moving frame'. In the moving frame the process X becomes time-dependent, but can be re-scaled (in order to arrive at a time-homogeneous process) by the stochastic integral

$$dZ_t = \exp\big(K^\top t\big) dY_t.$$

The stochastic integral can be defined by integration by parts, i.e.,

$$Z_t = \exp\big(K^\top t\big) Y_t - K^\top \int_0^t \exp\big(K^\top t\big) Y_r dr,$$

which yields the transformation formula (5.3). The method of the moving frame is therefore an operation which allows to remove (or change) the linear drift of an affine process.

Remark 5.3. Now that we have shown that every stochastically continuous affine process X is regular, all the results of Duffie et al. (2003) on regular affine processes apply to X. It follows in particular that the set $\mathcal{Q}$, introduced in (2.4) is actually equal to $\mathcal{U}$, and hence simply connected. Thus, the logarithm $\phi(t,u) = \log \Phi(t,u)$ is uniquely defined by choosing the main branch of the complex logarithm, and we can write

$$\mathbb{E}^x \left[e^{\langle X_t, u \rangle} \right] = \exp\left(\phi(t,u) + \langle x, \psi(t,u) \rangle \right) \, ,$$

for all $(t,u) \in \mathcal{U}$, as in Duffie et al. (2003).

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