A TRAJECTORIAL INTERPRETATION OF DOOB'S MARTINGALE INEQUALITIES

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ABSTRACT. We present a unified approach to Doob's L^p maximal inequalities for $1 \leq p < \infty$. The novelty of our method is that these martingale inequalities are obtained as consequences of elementary *deterministic* counterparts. The latter have a natural interpretation in terms of robust hedging. Moreover our deterministic inequalities lead to new versions of Doob's maximal inequalities. These are best possible in the sense that equality is attained by properly chosen martingales.

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1. Introduction

In this paper we derive estimates for the running maximum of a martingale or non-negative submartingale in terms of its terminal value. Given a function f we write $\bar{f}(t) = \sup_{u \leq t} f(u)$. Among other results, we establish the following martingale inequalities:

Theorem 1.1. *Let $(S_n)_{n=0}^T$ be a non-negative submartingale. Then*

$$\mathbb{E}\left[\bar{S}_T^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[S_T^p], \quad 1 < p < \infty, \tag{Doob-L^p}$$

$$\mathbb{E}[\bar{S}_T] \leq \frac{e}{e-1}\Big[\, \mathbb{E}[S_T \log(S_T)] + \mathbb{E}\left[S_0 \left(1 - \log(S_0)\right)\right] \Big]. \tag{Doob-L^1}$$

Here (Doob-L^p) is the classical Doob L^p-inequality, $p \in (1, \infty)$, [8, Theorem 3.4]. The second result (Doob-L^1) represents the Doob L^1-inequality in the sharp form derived by Gilat [10] from the $L \log L$ Hardy-Littlewood inequality.

Trajectorial inequalities. The novelty of this note is that the above martingale inequalities are established as consequences of *deterministic* counterparts. We postpone the general statements (Proposition 2.1) and illustrate the spirit of our approach by a simple result that may be seen as the *trajectorial version* of Doob's L^2-inequality:

Let $s_0, \dots, s_T$ be real numbers. Then

$$\bar{s}_T^2 + 4\Big[\, \textstyle\sum_{n=0}^{T-1} \bar{s}_n (s_{n+1} - s_n)\Big] \leq 4 s_T^2. \tag{Path-L^2}$$

Inequality (Path-L^2) is completely elementary and the proof is straightforward: it suffices to rearrange terms and to complete squares . The significance of (Path-L^2) rather lies in the fact that it implies (Doob-L^2). Indeed, if $S = (S_n)_{n=1}^T$ is a non-negative submartingale, we may apply (Path-L^2) to each trajectory of S. The decisive observation is that, by the submartingale property,

$$\mathbb{E}\Big[\, \textstyle\sum_{n=0}^{T-1} \bar{S}_n (S_{n+1} - S_n)\Big] \geq 0, \tag{1.1}$$

hence (Doob-L^2) follows from (Path-L^2) by taking expectations.

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Inequalities in continuous time – sharpness. Passing to the continuous time setting, it is clear that (Doob-L^p) and (Doob-L^1) carry over verbatim to the case where $S = (S_t)_{t\in[0,T]}$ is a non-negative càdlàg submartingale, by the usual limiting argument. It is not surprising that also in continuous time one has trajectorial counterparts of those inequalities, the sum in (Path-L^2) being replaced by a – carefully defined – integral. Moreover, in the case $p = 1$ the inequality can be *attained* by a martingale in continuous time (cf. [10] and [11]). Notably, this does not hold for $1 < p < \infty$. We discuss this for the case $p = 2$ in the L^2-norm formulation: Given a non-negative càdlàg submartingale $S = (S_t)_{t\in[0,T]}$ we have

$$\|\bar{S}_T\|_2 \leq 2\|S_T\|_2. \qquad \text{(Doob-}L^2\text{)}$$

Dubins and Gilat [9] showed that the constant 2 in (Doob-L^2) is optimal, i.e. can not be replaced by a strictly smaller constant. It is also natural to ask whether equality can be attained in (Doob-L^2). It turns out that this happens only in the trivial case $S \equiv 0$; otherwise the inequality is strict. Keeping in mind that equality in (Doob-L^1) is attained, one may try to improve also on (Doob-L^2) by incorporating the starting value of the martingale. Indeed, we obtain the following result:

Theorem 1.2. *For every non-negative càdlàg submartingale $S = (S_t)_{t\in[0,T]}$*

$$\|\bar{S}_T\|_2 \leq \|S_T\|_2 + \|S_T - S_0\|_2. \tag{1.2}$$

Inequality (1.2) *is sharp. More precisely, given $x_0, x_1 \in \mathbb{R}$, $0 < x_0 \leq x_1$, there exists a positive, continuous martingale $S = (S_t)_{t\in[0,T]}$ such that $\|S_0\|_2 = x_0, \|S_T\|_2 = x_1$ and equality holds in* (1.2)*.*

In Theorem 3.1 we formulate the result of Theorem 1.2 for $1 < p < \infty$, thus establishing an optimal a priori estimate on $\|\bar{S}_T\|_p$.

We emphasize that the idea that (Doob-L^p) can be improved by incorporating the starting value S_0 into the inequality is not new. Cox [7], Burkholder [5] and Peskir [18] show that

$$\mathbb{E}[\bar{S}_T^2] \leq 4\mathbb{E}[S_T^2] - 2\mathbb{E}[S_0^2]. \tag{1.3}$$

Here the constants 4 resp. 2 are sharp (cf. [18]) with equality in (1.3) holding iff $S \equiv 0$. [1]

Financial interpretation. We want to stress that (Path-L^2) has a natural interpretation in terms of mathematical finance.

Financial intuition suggests to consider the positive martingale $S = (S_n)_{n=0}^T$ as the process describing the price evolution of an asset under the so-called "risk-neutral measure", so that $\Phi(S_0, \ldots, S_T) = (\bar{S}_T)^2$, resp. $\varphi(S_T) = S_T^2$, have the natural interpretation of a so-called *exotic* option, resp. a *European* option, written on S. In finance, a European option φ, resp. exotic option Φ, is a function that depends on the final value S_T of S, resp. on its whole path $S_0, \ldots, S_T$. The seller of the option Φ pays the buyer the random amount $\Phi(S_0, \ldots, S_T)$ after its expiration at time T. Following [2] we may interpret $\mathbb{E}[\Phi]$ as the price that the buyer pays for this option at time 0 (Cf. [19, Ch. 5] for an introductory survey on risk-neutral pricing).

Here we take the point of view of an economic agent who sells the option Φ and wants to protect herself in all possible scenarios $\omega \in \Omega$, i.e., against all possible values $\Phi(S_0(\omega), \ldots, S_T(\omega))$, which she has to pay to the buyer of Φ. This means that she will trade in the market in order to arrive at time T with a portfolio value which is at least as

[1] That (1.2) implies (1.3) follows from a simple calculation. Alternatively the sharpness of (1.3) is a consequence of the fact that equality in (1.2) can be attained for all possible values of $\|S_0\|_2, \|S_T\|_2$.

high as the value of Φ. By buying a European option $\varphi(S_T) = S_T^2$, she can clearly protect herself in case the asset reaches its maximal value at maturity T. However, she still faces the risk of S having its highest value at some time n before T. To protect against that possibility, one way for her is to "go short" in the underlying (i.e., to hold negative positions in S). By scaling, her protecting strategy should be proportional to the running maximum $\bar{S}_n$. At this point our educated guess is to follow the strategy $H_n = -4\bar{S}_n$, meaning that from time n to time $n+1$ we keep an amount H_n of units of the asset S in our portfolio. The portfolio strategy produces the following value at time T:

$$\sum_{n=0}^{T-1} H_n(S_{n+1} - S_n) = -4\sum_{n=0}^{T-1} \bar{S}_n(S_{n+1} - S_n). \tag{1.4}$$

The reason why we have chosen the special form $H_n = -4\bar{S}_n$ now becomes apparent when considering (Path-L^2) and (1.1). In our "financial mind experiment" this may be interpreted as follows: by buying 4 European options S_T^2 and following the self-financing trading strategy $H = (H_n)_{n=0}^{T-1}$, the seller of the option $\Phi = (\bar{S}_T)^2$ covers her position at maturity T, whatever the outcome $(S_0(\omega), \dots, S_T(\omega))$ of the price evolution is. Thus an upper bound for the price of the exotic option Φ in terms of the European option φ is given by

$$\mathbb{E}[(\bar{S}_T)^2] \leq 4\mathbb{E}[S_T^2].$$

We note that Henry-Labordère [12] derived (Doob-L^p) in a related fashion.

The idea of robust pricing and pathwise hedging of exotic options seemingly goes back to Hobson [13], see also [4, 6, 15]. We refer the reader to [14] for a thorough introduction to the topic.

Organisation of the paper. In Section 2 we prove Doob's inequalities (Doob-L^p) and (Doob-L^1) after establishing the trajectorial counterparts (Path-L^p) and (Path-L^1). We prove Theorem 1.2 and its L^p version in Section 3.

2. Proof of Theorem 1.1

The aim of this section is to prove Doob's maximal inequalities in Theorem 1.1 by means of deterministic inequalities, which are established in Proposition 2.1 below. The proof of Theorem 1.1 is given at the end of this section. As regards (Doob-L^p), we prove the stronger result

$$\mathbb{E}\left[\bar{S}_T^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[S_T^p] - \frac{p}{p-1}\mathbb{E}[S_0^p], \quad 1 < p < \infty, \tag{2.1}$$

which was obtained in [7, 18].

Proposition 2.1. *Let $s_0, \dots, s_T$ be non-negative numbers.*

(I) For $1 < p < \infty$ and $h(x) := -\frac{p^2}{p-1}x^{p-1}$, we have

$$\bar{s}_T^p \leq \sum_{i=0}^{T-1} h(\bar{s}_i)(s_{i+1} - s_i) - \frac{p}{p-1}s_0^p + \left(\frac{p}{p-1}\right)^p s_T^p. \tag{Path-L^p}$$

(II) For $h(x) := -\log(x)$, we have

$$\bar{s}_T \leq \frac{e}{e-1}\Big(\sum_{i=0}^{T-1} h(\bar{s}_i)(s_{i+1} - s_i) + s_T\log(s_T) + s_0(1 - \log(s_0))\Big). \tag{Path-L^1}$$

We note that for $p = 2$ inequality (Path-L^p) is valid also in the case where $s_0, \dots, s_T$ are real (possibly negative) numbers. A continuous time counterpart of (Path-L^p) is given in Remark 3.5 below.

In the proof of Proposition 2.1, we need the following identity:

Lemma 2.2. *Let $s_0, \dots, s_T$ be real numbers and $h : \mathbb{R} \to \mathbb{R}$ any function. Then*

$$\sum_{i=0}^{T-1} h(\bar{s}_i)(s_{i+1} - s_i) = \sum_{i=0}^{T-1} h(\bar{s}_i)(\bar{s}_{i+1} - \bar{s}_i) + h(\bar{s}_T)(s_T - \bar{s}_T). \tag{2.2}$$

Proof. This follows by properly rearranging the summands. Indeed, observe that for a term on the right-hand side there are two possibilities: if $\bar{s}_{i+1} = \bar{s}_i$ resp. $s_T = \bar{s}_T$, it simply vanishes. Otherwise it equals a sum $h(\bar{s}_k)(s_{k+1} - s_k) + \ldots + h(\bar{s}_m)(s_{m+1} - s_m)$ where $\bar{s}_k = \ldots = \bar{s}_m$. In total, every summand on the left-hand side of (2.2) is accounted exactly once on the right. □

We note that Lemma 2.2 is a special case of [17, Lemma 3.1].

Proof of Proposition 2.1. (I) By convexity, $x^p + px^{p-1}(y - x) \le y^p, x, y \ge 0$. Hence Lemma 2.2 yields

$$\begin{aligned}
\sum_{i=0}^{T-1} h(\bar{s}_i)(s_{i+1} - s_i) &= -\frac{p^2}{p-1} \sum_{i=0}^{T-1} \bar{s}_i^{p-1}(\bar{s}_{i+1} - \bar{s}_i) - \frac{p^2}{p-1} \bar{s}_T^{p-1}(s_T - \bar{s}_T) \\
&\ge -\frac{p}{p-1} \sum_{i=0}^{T-1} \bar{s}_{i+1}^p - \bar{s}_i^p - \frac{p^2}{p-1} \bar{s}_T^{p-1}(s_T - \bar{s}_T) \qquad (2.3)\\
&= p\bar{s}_T^p - \frac{p^2}{p-1} \bar{s}_T^{p-1} s_T + \frac{p}{p-1} \bar{s}_0^p.
\end{aligned}$$

We therefore have

$$\sum_{i=0}^{T-1} h(\bar{s}_i)(s_{i+1} - s_i) + \left(\frac{p}{p-1}\right)^p s_T^p - \frac{p}{p-1} \bar{s}_0^p - \bar{s}_T^p \ge (p-1)\bar{s}_T^p - \frac{p^2}{p-1} \bar{s}_T^{p-1} s_T + \left(\frac{p}{p-1}\right)^p s_T^p. \tag{2.4}$$

To establish (Path-L^p) it is thus sufficient to show that the right-hand side of (2.4) is non-negative. Defining c such that $S_n = c\bar{S}_n$, this amounts to showing that

$$g(c) = (p-1) - \frac{p^2}{p-1} c + \left(\frac{p}{p-1}\right)^p c^p \ge 0. \tag{2.5}$$

Using standard calculus we obtain that g reaches its minimum at $\hat{c} = \frac{p-1}{p}$ where $g(\hat{c}) = 0$.

(II) By Lemma 2.2 we have

$$\begin{aligned}
\sum_{i=0}^{T-1} h(\bar{s}_i)(s_{i+1} - s_i) &= -\sum_{i=0}^{T-1} \log(\bar{s}_i)(\bar{s}_{i+1} - \bar{s}_i) - \log(\bar{s}_T)(s_T - \bar{s}_T) \\
&\ge \sum_{i=0}^{T-1} \left(\bar{s}_{i+1} - \bar{s}_i - \bar{s}_{i+1} \log(\bar{s}_{i+1}) + \bar{s}_i \log(\bar{s}_i)\right) - \log(\bar{s}_T)(s_T - \bar{s}_T) \\
&= \bar{s}_T - s_0 + s_0 \log(s_0) - s_T \log(\bar{s}_T),
\end{aligned}$$

where the inequality follows from the convexity of $x \mapsto -x + x\log(x)$, $x > 0$. If $s_T = 0$ then the above inequality shows that (Path-L^1) holds true. Otherwise, we have

$$\bar{s}_T \le \sum_{i=0}^{T-1} h(\bar{s}_i)(s_{i+1} - s_i) + s_0 - s_0\log(s_0) + s_T\log(s_T) + s_T\log\left(\frac{\bar{s}_T}{s_T}\right).$$

Note that the function $x \mapsto x\log(y/x)$ on $(0,\infty)$, for any fixed $y > 0$, has a maximum in $\hat{x} = y/e$, where it takes the value y/e. This means that $s_T\log(\bar{s}_T/s_T) \le \bar{s}_T/e$ which concludes the proof. □

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 2.1 (I), for $h(x) := -\frac{p^2}{p-1}x^{p-1}$ we have

$$\bar{S}_T^p \le \sum_{i=0}^{T-1} h(\bar{S}_i)(S_{i+1} - S_i) - \frac{p}{p-1}S_0^p + \left(\frac{p}{p-1}\right)^p S_T^p. \tag{2.6}$$

Since S is a submartingale and h is negative, $\mathbb{E}[\sum_{i=0}^{T-1} h(\bar{S}_i)(S_{i+1} - S_i)] \le 0$ and thus (2.1) – and consequently (Doob-L^p) – follow from (2.6) by taking expectations.

Inequality (Doob-L^1) follows from Proposition 2.1 (II) in the same fashion. □

Remark 2.3. *Given the terminal law μ of a martingale S, Hobson* [14, Section 3.7] *also provides pathwise hedging strategies for lookback options on S. As opposed to the strategies given in Proposition 2.1, we emphasize that the strategies in* [14] *depend on μ.*

3. Qualitative Doob L^p Inequality – Proof of Theorem 1.2

In this section we prove Theorem 1.2 as well as the following result which pertains to $p \in (1,\infty)$.

Theorem 3.1. *Let $(S_t)_{t\in[0,T]}$ be a non-negative submartingale, $S \ne 0$, and $1 < p < \infty$. Then*

$$\|\bar{S}_T\|_p \le \frac{p}{p-1}\|S_T\|_p - \frac{1}{p-1}\frac{\|S_0\|_p^p}{\|\bar{S}_T\|_p^{p-1}}. \tag{3.1}$$

Given the values $\|S_0\|_p$ and $\|S_T\|_p$, inequality (3.1) *is best possible. More precisely, given $x_0, x_1 \in \mathbb{R}$, $0 < x_0 \le x_1$, there exists a positive, continuous martingale $S = (S_t)_{t\in[0,T]}$ such that $\|S_0\|_p = x_0$, $\|S_T\|_p = x_1$ and equality holds in* (3.1)*.*

Moreover, equality in (3.1) *holds if and only if S is a non-negative martingale such that $\bar{S}$ is continuous and $\bar{S}_T = \alpha S_T$, where $\alpha \in [1, \frac{p}{p-1})$.*

Remark 3.2. *We prove Theorem 3.1 by introducing a pathwise integral in continuous time. Note that inequality* (3.1) *can also be obtained without defining such an integral. However, the definition of the pathwise integral will allow us to characterize all submartingales for which equality in* (3.1) *holds.*

Connection between Theorem 1.2 and Theorem 3.1. We now discuss under which conditions Theorem 1.2 and Theorem 3.1 are equivalent for $p = 2$. Recall that Theorem 1.2 asserts that

$$\|\bar{S}_T\|_2 \le \|S_T\|_2 + \|S_T - S_0\|_2 \tag{3.2}$$

and Theorem 1.2 reads in the case of $p = 2$ as

$$\|\bar{S}_T\|_2 \le 2\|S_T\|_2 - \frac{\|S_0\|_2^2}{\|\bar{S}_T\|_2}. \tag{3.3}$$

• If S is a *martingale*, then (3.2) and (3.3) are equivalent. Indeed, rearranging (3.3) yields

$$\psi(\|\bar{S}_T\|_2) := \frac{1}{2}\|\bar{S}_T\|_2 + \frac{\|S_0\|_2^2}{2\|\bar{S}_T\|_2} \leq \|S_T\|_2, \tag{3.4}$$

and by inverting the strictly monotone function ψ on $[\|S_0\|_2, \infty)$ we obtain

$$\|\bar{S}_T\|_2 \leq \psi^{-1}(\|S_T\|_2) = \|S_T\|_2 + \sqrt{\|S_T\|_2^2 - \|S_0\|_2^2}.$$

Since S is a martingale, $\sqrt{\|S_T\|_2^2 - \|S_0\|_2^2} = \|S_T - S_0\|_2$, which gives (3.2).
• If S is a true submartingale, then the estimate in (3.2) is in fact stronger than (3.3). This follows from the above reasoning and the fact that for a true submartingale we have $\sqrt{\|S_T\|_2^2 - \|S_0\|_2^2} > \|S_T - S_0\|_2$.
• Clearly, it would be desirable to obtain also for general p an inequality of the type (3.2), which is in the case of a martingale S equivalent to (3.1), and where $\bar{S}_T$ only appears on the left-hand side. By similar reasoning as for $p = 2$, finding such an inequality is tantamount to inverting the function

$$\psi(x) = \frac{p-1}{p}x + \frac{\|S_0\|_p^p}{px^{p-1}},$$

which is strictly monotone on $[\|S_0\|_p, \infty)$. Since finding ψ^{-1} amounts to solving an algebraic equation, there is in general no closed form representation of ψ^{-1} unless $p \in \{2, 3, 4\}$.

Definition of the continuous-time integral. For a general account on the theory of pathwise stochastic integration we refer to Bichteler [3] and Karandikar [16]. Here we are interested in the particular case where the integrand is of the form $h(\bar{S})$ and h is monotone and continuous. In this setup a rather naive and ad hoc approach is sufficient (see Lemma 3.3 below).

Fix càdlàg functions $f, g : [0, T] \to [0, \infty)$ and assume that g is monotone. We set

$$\int_0^T g_{t-}\, df_t := \lim_{n\to\infty} \sum_{t_i \in \pi_n} g_{t_i-}\,(f_{t_{i+1}} - f_{t_i}) \tag{3.5}$$

if the limit exists for *every* sequence of finite partitions π_n with mesh converging to 0. The standard argument of mixing sequences then implies uniqueness. We stress that (3.5) exists if and only if the "non predictable version" $\int_0^T g_t\, df_t = \lim_{n\to\infty} \sum_{t_i\in\pi_n} g_{t_i}\,(f_{t_{i+1}} - f_{t_i})$ exists; in this case the two values coincide.

By rearranging terms one obtains the identity

$$\sum_{t_i\in\pi} g_{t_i}(f_{t_{i+1}} - f_{t_i}) = -\sum_{t_i\in\pi} f_{t_i}(g_{t_{i+1}} - g_{t_i}) + g_T f_T - g_0 f_0 - \overbrace{\sum_{t_i\in\pi}(g_{t_{i+1}} - g_{t_i})(f_{t_{i+1}} - f_{t_i})}^{(*)}. \tag{3.6}$$

If it is possible to pass to a limit on either of the two sides, one can do so on the other. Hence, $\int_0^T g_t\, df_t$ is defined whenever $\int_0^T f_t\, dg_t$ is defined and vice versa, since the monotonicity of g implies that $(*)$ converges. In this case we obtain the integration by parts formula

$$\int_0^T g_t\, df_t = -\int_0^T f_t\, dg_t + g_T f_T - g_0 f_0 - \sum_{0\leq t\leq T}(g_t - g_{t-})(f_t - f_{t-}). \tag{3.7}$$

Below we will need that the integrals $\int_0^T h(\bar{f}_t)\, df_t$ and $\int_0^T f_t\, dh(\bar{f}_t)$ are well defined whenever h is continuous, monotone and f is càdlàg. In the case of $\int_0^T f_t\, dh(\bar{f}_t)$ this can be seen by splitting f in its continuous and its jump part. Existence of $\int_0^T h(\bar{f}_t)\, df_t$ is then a consequence of (3.7).

The following lemma establishes the connection of the just defined pathwise integral with the standard Ito-intgral.

Lemma 3.3. *Let S be a martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and h be a monotone and continuous function. Then*

$$(h(\bar{S}) \bullet S)_T(\omega) = \int_0^T h(\bar{S}_{t\text{-}}(\omega))\, dS_t(\omega) \quad \mathbb{P}\text{-}a.s., \tag{3.8}$$

where the left hand side refers the Ito-integral while the right hand side appeals to the pathwise integral defined in (3.5)*.*

Proof. Karandikar ([16, Theorem 2]) proves that

$$(h(\bar{S}) \bullet S)_T(\omega) = \lim_{n\to\infty} \sum_{t_i \in \pi_n} h(\bar{S}_{t_i\text{-}}(\omega))\, (S_{t_{i+1}}(\omega) - S_{t_i}(\omega))$$

for a suitably chosen sequence of random partions $\pi_n, n \geq 1$. According to the above discussion, $\int_0^T h(\bar{S}_{t\text{-}}(\omega))\, dS_t(\omega) = \lim_{n\to\infty} \sum_{t_i\in\pi_n} h(\bar{S}_{t_i\text{-}}(\omega))\, (S_{t_{i+1}}(\omega) - S_{t_i}(\omega))$ for any choice of partions $\pi_n(\omega), n \geq 1$ with mesh converging to 0. $\square$

We are now able to establish a continuous-time version of Proposition 2.1

Proposition 3.4. *Let $f : [0, T] \to [0, \infty)$ be càdlàg. Then for $h(x) := -\frac{p^2}{p-1} x^{p-1}$ we have*

$$\bar{f}_T^p \leq \int_0^T p^{-1} h(\bar{f}_t)\, df_t + \frac{p}{p-1} \bar{f}_T^{p-1} f_T - \frac{1}{p-1} f_0^p. \tag{3.9}$$

Equality in (3.9) *holds true if and only if $\bar{f}$ is continuous. Similarly, a continuous-time version of* (Path-L^1) *also holds true.*

Proof. Inequality (3.9) follows from (2.3) by passing to limits. We now show that equality in (3.9) holds iff $\bar{f}$ is continuous. To simplify notation, we consider the case $p = 2$. (3.7) implies

$$\begin{aligned} \int_0^T h(\bar{f}_t)\, df_t &= 4 \int_0^T f_t\, d\bar{f}_t - 4\bar{f}_T f_T + 4 f_0^2 + 4 \sum_{0\leq t\leq T} (\bar{f}_t - \bar{f}_{t\text{-}})(f_t - f_{t\text{-}}), \\ &\geq 2\bar{f}_T^2 - 4\bar{f}_T f_T + 2\bar{f}_0^2, \end{aligned} \tag{3.10}$$

where equality in (3.10) holds iff $\bar{f}$ is continuous. Hence, equality in (3.9) holds true iff $\bar{f}$ is continuous. $\square$

If we choose f to be the path of a continuous martingale, the integral in (3.9) is a pathwise version of an Azéma-Yor process, cf. [17, Theorem 3].

Remark 3.5. *Passing to limits in* (Path-L^p) *in Section 2 we obtain that for every càdlàg function $f : [0, T] \to [0, \infty)$*

$$\bar{f}_T^p \leq -\int_0^T \frac{p^2}{p-1} \bar{f}_t^{p-1}\, df_t + \left(\frac{p}{p-1}\right)^p f_T^p - \frac{p}{p-1} f_0^p, \quad 1 < p < \infty.$$

Alternatively this can be seen as a consequence of (3.9)*.*

Lemma 3.6. *Let $(S_t)_{t\in[0,T]}$ be a non-negative submartingale and $1 < p < \infty$. Set $S = M + A$, where M is a martingale and A is an increasing, predictable process with $A_0 = 0$. Then*

$$\mathbb{E}[\bar{S}_T^p] \leq -\frac{p}{p-1}\mathbb{E}[S_0^{p-1}A_T] + \frac{p}{p-1}\mathbb{E}[\bar{S}_T^{p-1}S_T] - \frac{1}{p-1}\mathbb{E}[S_0^p]. \tag{3.11}$$

Equality holds in (3.11) *if and only if S is a martingale such that $\bar{S}$ is a.s. continuous.*

Proof. By Proposition 3.4 we find for $h(x) = -\frac{p^2}{p-1}x^{p-1}$

$$\bar{S}_T^p \leq \int_0^T p^{-1}h(\bar{S}_t)\,dS_t + \frac{p}{p-1}\bar{S}_T^{p-1}S_T - \frac{1}{p-1}S_0^p, \tag{3.12}$$

where equality holds iff $\bar{S}$ is continuous. Since

$$\mathbb{E}\left[\int_0^T p^{-1}h(\bar{S}_t)\,dA_t\right] \leq -\frac{p}{p-1}\mathbb{E}[S_0^{p-1}A_T], \tag{3.13}$$

(3.11) follows by taking expectations in (3.12). As the estimate in (3.13) is an equality iff $A = 0$, we conclude that equality in (3.11) holds iff S is a martingale such that $\bar{S}$ is continuous. □

We note that in the case of $p = 2$ also [1, Corollary 2.2.2'] implies that equality in (3.11) holds for every continuous martingale S.

Proof of Theorem 3.1 and Theorem 1.2. By Lemma 3.6 and Hölder's inequality we have

$$\|\bar{S}_T\|_p^p \leq -\frac{p}{p-1}\mathbb{E}[S_0^{p-1}A_T] + \frac{p}{p-1}\|\bar{S}_T^{p-1}S_T\|_1 - \frac{1}{p-1}\|S_0\|_p^p, \tag{3.14}$$

$$\leq -\frac{p}{p-1}\mathbb{E}[S_0^{p-1}A_T] + \frac{p}{p-1}\|\bar{S}_T\|_p^{p-1}\|S_T\|_p - \frac{1}{p-1}\|S_0\|_p^p, \tag{3.15}$$

where equality in (3.14) holds for every martingale S such that $\bar{S}$ is continuous, and equality in (3.15) holds whenever S_T is a constant multiple of $\bar{S}_T$. Since $\mathbb{E}[S_0^{p-1}A_T] \geq 0$ we obtain (3.1) after dividing by $\|\bar{S}_T\|_p^{p-1}$.

In order to establish (1.2) in Theorem 1.2 for $p = 2$, we rearrange terms in (3.15) to obtain

$$\psi(\|\bar{S}_T\|_2) := \frac{1}{2}\|\bar{S}_T\|_2 + \frac{2\mathbb{E}[S_0A_T] + \|S_0\|_2^2}{2\|\bar{S}_T\|_2} \leq \|S_T\|_2.$$

Similarly as in the discussion after Remark 3.2 above, inverting ψ on $[\|S_0\|_2, \infty)$ implies

$$\|\bar{S}_T\|_2 \leq \|S_T\|_2 + \sqrt{\|S_T\|_2^2 - 2\mathbb{E}[S_0A_T] - \|S_0\|_2^2}.$$

Since for every submartingale S we have $\sqrt{\|S_T\|_2^2 - 2\mathbb{E}[S_0A_T] - \|S_0\|_2^2} = \|S_T - S_0\|_2$, this proves (1.2).

In order to prove that (3.1), resp. (1.2), is attained, we have to ensure the existence of a p-integrable martingale S such that $\bar{S}$ is continuous and S_T is a constant multiple of $\bar{S}_T$. To this end we may clearly assume that $x_0 = 1$. Fix $\alpha \in (1, \frac{p}{p-1})$ and let $B = (B_t)_{t\geq 0}$ be a Brownian Motion starting at $B_0 = 1$. Consider the process $B^{\tau_\alpha} = (B_{t\wedge\tau_\alpha})_{t\geq 0}$ obtained by stopping B at the stopping time

$$\tau_\alpha := \inf\left\{t > 0 : B_t \leq \bar{B}_t/\alpha\right\}.$$

This stopping rule corresponds to the Azéma-Yor solution of the Skorokhod embedding problem (B, μ), cf. [1], where the probability measure μ is given by

$$\frac{d\mu}{dx} = \frac{\alpha^{-\frac{1}{\alpha-1}}}{(\alpha-1)} x^{-\frac{2\alpha-1}{\alpha-1}} 1_{[\alpha^{-1},\infty)}(x).$$

Clearly B^{τ_α} is a uniformly integrable martingale. Therefore the process $(S_t)_{t\in[0,T]}$ defined as $S_t := B_{\frac{t}{T-t}\wedge\tau_\alpha}$ is a non-negative martingale satisfying $\underline{S}_T = \overline{S}_T/\alpha$. S_T is p-integrable for $\alpha \in (1, \frac{p}{p-1})$ and $\|S_T\|_p$ runs through the interval $(1,\infty)$ while α runs in $(1, \frac{p}{p-1})$. This concludes the proof.

Note that the proof in fact shows that equality in (3.1) holds if and only if S is a non-negative martingale such that $\overline{S}$ is continuous and $\overline{S}_T = \alpha S_T$, where $\alpha \in [1, \frac{p}{p-1})$. □

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