The space of outcomes of semi-static trading strategies need not be closed[*]

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Abstract

Semi-static trading strategies make frequent appearances in mathematical finance, where dynamic trading in a liquid asset is combined with static buy-and-hold positions in options on that asset. We show that the space of outcomes of such strategies can have very poor closure properties when all European options for a fixed date T are available for static trading. This causes problems for optimal investment, and stands in sharp contrast to the purely dynamic case classically considered in mathematical finance.

1   Introduction and main results

Given a local martingale S and a finite stopping time T defined on a stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ in discrete or continuous time, we consider outcomes at time T of semi-static trading in S. More specifically, we consider self-financing dynamic trading in S and a risk-free asset with zero interest rate, combined with static (buy-and-hold) positions in arbitrary European options written on the final value S_T. Such outcomes are of the form $(H \cdot S)_T + h(S_T)$, where $H \cdot S$ denotes the stochastic integral of the S-integrable process H with respect to S, and h is a measurable function satisfying some integrability conditions. The semi-static strategy consists of the pair (H, h), chosen by the investor. This type of

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semi-static trading strategies has been used extensively in the literature; see e.g. [Hob11, HK12, BHLP13, GHLT$^+$14, DS14] and the references therein. A key reason is that the collection of time zero prices of all such static claims pins down the law of S_T under $\mathbb{P}$, if $\mathbb{P}$ is the pricing measure.

One could also restrict the static component $h(S_T)$ to lie in a given finite-dimensional set of available options, for instance by requiring it to be of the form $h(S_T) = a_0 + a_1 C_1(S_T) + \cdots + a_n C_n(S_T)$, where $C_i(S_T) = (S_T - K_i)_+$ is a vanilla call payoff with given strike K_i, and $a_0, \ldots, a_n \in \mathbb{R}$ are chosen by the investor. Such a setup is also common in the literature; see e.g. [DH07, ABPS16, BN15]. It is however different from our setting, where h is chosen from an infinite-dimensional space of measurable functions. One of the main purposes of this paper is to clarify the sharply different properties that the two situations may exhibit.

The largest reasonable space of outcomes of semi-static trading strategies is arguably the sum $U + V = \{u + v \colon u \in U,\ v \in V\}$, where

$$U = \{(H \cdot S)_T \colon H \text{ is } S\text{-integrable and } H \cdot S \text{ is a supermartingale on } [0, T]\},$$
$$V = L^1(\Omega, \sigma(S_T), \mathbb{P}).$$

The supermartingale property of the gains processes $H \cdot S$ is a weak restriction which is implied by any reasonable admissibility or integrability condition that excludes doubling strategies (recall that S from the outset is assumed to be a local martingale). Requiring the static component to be integrable, rather than just measurable, corresponds to a finite initial capital requirement: If an outcome $f = (H \cdot S)_T + h(S_T)$ is integrable—which we interpret as requiring finite initial capital—and if $(H \cdot S)_T \in U$, then $h(S_T)$ is necessarily integrable as well.

On the other hand, the smallest reasonable space of outcomes (at least in our setting without trading constraints) is arguably the sum $U_\infty + V_\infty$, where

$$U_\infty = \{(H \cdot S)_T \colon H \text{ is } S\text{-integrable and } H \cdot S \text{ is a bounded martingale}\},$$
$$V_\infty = L^\infty(\Omega, \sigma(S_T), \mathbb{P}).$$

In particular, the dynamic components of such semi-static trading strategies clearly satisfy all admissibility and integrability conditions that have been considered in the literature to date.

The spaces U and V enjoy very strong closure properties. For V this is obvious; for U much less so. Kunita and Watanabe [KW67] proved early on that if $H^n \cdot S$ is a sequence of $\mathscr{H}^2$ martingales such that $(H^n \cdot S)_T \to f$ in L^2 for some limit f, then the limit is again of the form $f = (H \cdot S)_T$, where $H \cdot S$ is an $\mathscr{H}^2$ martingale; see e.g. [Pro05, Theorem IV.41]. The same result holds in the $\mathscr{H}^p$ and L^p case for any $p \in (1, \infty]$, and Yor proved that the statement also remains true under uniform (rather than $\mathscr{H}^2$) integrability and L^1 (rather than L^2) convergence; see [Yor78] and [DS99] for further discussion. In a similar vein, the following result is crucial for the development of arbitrage theory in mathematical

finance: if $u_n \in U$, $u_n \geq -1$, and $u_n \to f$ in probability for some random variable f, then $f \in U - L^0_+$. That is, f is dominated by some element of U. Further discussion and generalizations can be found e.g. in [DS94] and [DS98]. Note that these results imply in particular that U_∞ is closed in L^∞, that its closure in any L^p space $(p \geq 1)$ is contained in U, and that its closure in L^0 is contained in $U - L^0_+$.

A natural question is to what extent these closure properties carry over to the spaces $U + V$ and $U_\infty + V_\infty$ of outcomes of semi-static trading strategies. The answer is that they do not. The goal of the present paper is to demonstrate this by way of example. This is done in our two main results, Theorems 1.1 and 1.3, which cover the discrete and continuous time cases, respectively.

Theorem 1.1. *There is a discrete time stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \in \{0,1,2\}}, \mathbb{P})$ with countable sample space Ω, on which is defined a bounded martingale $S = (S_t)_{t \in \{0,1,2\}}$ such that the following holds: There exist random variables g and g_m, $m \geq 1$, such that*

(i) $g_m \in U_\infty + V_\infty$ *and* $g_m \geq 0$ *for each* m,

(ii) $g_m \to g$ *almost surely and in* L^p *for every* $p \in [1, \infty)$,

(iii) $g \notin U + V - L^0_+$.

Thus, the nonnegative random variables g_m are final outcomes of semi-static trading strategies of the most well-behaved kind: their dynamic and static components are both bounded. In particular, the dynamic trading strategies are admissible in the classical sense. Furthermore, the random variables g_m converge to a limit g in a rather strong sense, but this limit cannot be represented as, and not even dominated by, the final outcome of any semi-static trading strategy satisfying minimal regularity conditions. As will become clear from the construction, each g_m can be viewed as a portfolio of digital options, hedged by a position in the underlying stock; see Remark 2.3 in Section 2.

To prove Theorem 1.1 we construct final outcomes g_m converging to an integrable limit g which, if it were to have a representation $g \leq u + v$ with $u \in U$ and $v \in V$, would violate the simple bound $\|u\|_1 + \|v\|_1 < \infty$. To achieve this, we construct a sequence of simpler models, each of which admits an element of $U_\infty + V_\infty$ whose L^p norms are small, but whose components in U_∞ and V_∞ are nonetheless large in L^1. These models are then pasted together to form a new model, which admits the required sequence of elements g_m. The individual models are described in Section 2, and the pasting procedure is described in Section 3.

Remark 1.2. *Let us mention a conceivable extension of Theorem 1.1: Is it possible to strengthen part* (ii) *of Theorem 1.1 so that* $g_m \to g$ *in* L^∞? *We do not know the answer.*

We emphasize that there is nothing special about discrete time that makes Theorem 1.1 work. An analogous example may be constructed in a basic continuous-time setting, as the following result shows.

Theorem 1.3. *There is a stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ equipped with a Brownian motion W and a stopping time T such that the following holds for the price process $S = W^T$: There exist random variables g and g_m, $m \geq 1$, such that*

(i) *$g_m \in U_\infty + V_\infty$ and $g_m \geq 0$ for each m,*

(ii) *$g_m \to g$ almost surely and in L^p for every $p \in [1, \infty)$,*

(iii) *$g \notin U + V - L_+^0$.*

Furthermore, S is uniformly bounded.

The proof follows the pattern of Theorem 1.1. The only difference lies in the construction of the individual models, which is presented in Section 4. The pasting procedure then works exactly as described in Section 3, and we refrain from repeating it.

Remark 1.4. *In the context of the above theorems, two referees both asked the same question: Is it possible to obtain the same conclusion in a model where the filtration $\mathbb{F}$ is generated by S? It turns out that the answer is affirmative. In fact, slight modifications of the constructions in Sections 2, 3 and 4 yield this additional feature. We now sketch these modifications.*

Let us first discuss the continuous case. In the construction of Lemma 4.1, one changes the Brownian motion W slightly: On the event $\{X = 1\}$, W is run "twice as fast" after time σ, that is, we replace W_t by $W_{\sigma + 2(t-\sigma)}$ on this event for $t \geq \sigma$. Before σ and on $\{X = 0\}$, W is unchanged. Defining S as before, it follows that the outcome of the "coin flip" X becomes known at time σ if one observes the right-continuous filtration generated by S. This filtration therefore coincides with $(\mathscr{F}_{t \wedge T})_{t \geq 0}$. Next, note that the construction still works if W, and hence S, is started from any deterministic value in $(-1, 1)$. This now becomes useful for the pasting procedure in Section 3. Indeed, letting s_n, $n \geq 1$, be countably many distinct numbers in $(-1, 1)$, we may take $S_0^n = s_n$, so that $S_0 = s_n$ on Ω_n. This implies $\Omega_n \in \sigma(S_0)$ for each n. In summary, the right-continuous filtration generated by S now coincides with $\mathbb{F}$ up the time horizon T. Moreover, the conclusion of Theorem 1.3 is unaffected by these changes.

Let us now consider the discrete case. In the construction of Lemma 2.1, introduce an additional time point between $t = 1$ and $t = 2$, say $t = 1.5$. The definition of S_t for $t = 0, 1, 2$, as well as of X, A, $\widetilde{A}$, and f remain the same. At time $t = 1.5$, we take $S_{1.5} = S_1$ on $\{X = 1\}$, and $S_{1.5} = S_2$ on $\{X = 0\}$. We let $(\mathscr{F}_t)_{t \in \{0,1,1.5,2\}}$ be the filtration generated by $S = (S_t)_{t \in \{0,1,1.5,2\}}$, and choose the probabilities so that S is a martingale. In this way X, and hence f, is $\mathscr{F}_{1.5}$-measurable. With this construction, Lemma 2.1 remains true for the new time set $\{0, 1, 1.5, 2\}$. Indeed, properties (i)–(ii) are verified as before. Property (iii) is a consequence of the estimates in Lemma 2.2, which are still valid. The proof is almost identical: just note that $\mathbb{P}(S_2 = \pm a) \geq 1/8$ still holds, and that the first equality in (2.3) follows from the fact that $S_{1.5} = S_1$ on $\{X = 1\}$. Finally, by modifying

the pasting procedure as in the continuous case, one deduces that Theorem 1.1 holds for the new time set. This concludes the affirmative answer to the question posed above.

A simple corollary of the above theorems is that the spaces

$$\{(H \cdot S)_T + h(S_T) \colon H \cdot S \text{ is an } \mathscr{H}^p \text{ martingale, } h(S_T) \in L^p(\sigma(S_T))\}$$

need not be closed in L^p ($p \geq 1$). The closure of the corresponding space in the case $p = 2$ but with finitely many static claims was crucial for the semi-static Jacod-Yor theorem in [AL17]. Thus, we do not expect that result to carry over to the case of infinitely many static claims. The non-closedness of the above spaces is an example of the well-known fact that sums of closed subspaces of Banach spaces need not be closed; see e.g. Section 41 in [Hal74].

Another immediate corollary is that the space

$$\{(H \cdot S)_T + h(S_T) - f \colon H \text{ is 1-admissible, } h(S_T) \in L^1(\sigma(S_T)), \ f \in L^0_+\}$$

need not be closed in L^1. Here H is called 1-admissible if it is S-integrable and $H \cdot S \geq -1$. This space is a natural space of outcomes in the context of portfolio optimization with semi-static trading opportunities. Thus, existence of optimal strategies is a delicate issue in such a setting.

Finally, we provide a result demonstrating that the non-closedness in Theorems 1.1 and 1.3 is caused by the infinite-dimensionality of the space V of static claims. If V is replaced by a finite-dimensional space, then closedness is retained.

Theorem 1.5. *Let $C_1, \ldots, C_n$ be linearly independent elements of L^1. The closure in L^0 of the space*

$$W = \left\{ (H \cdot S)_T + \sum_{i=1}^{n} a_i C_i \colon H \text{ is 1-admissible, } a_1, \ldots, a_n \in \mathbb{R} \right\}$$

is contained in $W - L^0_+$. Here H is called 1-admissible if it is S-integrable and $H \cdot S \geq -1$.

Proof. Let $\{(H^m \cdot S)_T + h^m : m \geq 1\}$ be an L^0-convergent sequence in W. In particular, it is bounded in L^0; recall that a subset $\mathscr{X} \subset L^0$ is bounded if $\lim_{x \to \infty} \sup_{X \in \mathscr{X}} \mathbb{P}(|X| > x) = 0$. By 1-admissibility, $H^m \cdot S$ is a supermartingale, whence $\mathbb{E}[|(H^m \cdot S)_T|] \leq 1 + \mathbb{E}[1 + (H^m \cdot S)_T|] \leq 2$, so that the sequence $\{(H^m \cdot S)_T : m \geq 1\}$ is bounded in L^1 and hence in L^0. Thus the sequence $\{h^m : m \geq 1\}$ is bounded in L^0. Now write $h^m = r^m \sum_{i=1}^{n} a_i^m C_i$, where $r^m \geq 0$ and the vector $a^m = (a_1^m, \ldots, a_n^m)$ has unit norm, and take a subsequence to obtain $a^m \to a$ for some unit vector a. Thus $\sum_{i=1}^{n} a_i^m C_i$ converges to a random variable, which is nonzero by linear independence of $C_1, \ldots, C_n$. Boundedness in L^0 of $\{h^m : m \geq 1\}$ then implies that $\{r^m : m \geq 1\}$ is bounded, hence convergent after passing to a subsequence. To summarize, we have shown that by passing to a subsequence, we may suppose that h^m is convergent in L^0. Thus $(H^m \cdot S)_T$ also converges in L^0, say to a limit f. By Corollary 4.11 in [DS99], this limit is of the form $f = (H \cdot S)_T - g$ for some 1-admissible H and some $g \in L^0_+$. This proves the result. □

2 The discrete case

The following lemma describes the individual models used in the proof of the discrete time Theorem 1.1. These individual models are later pasted together according to the procedure described in Section 3.

Lemma 2.1. *Fix $\varepsilon \in (0, 1/2]$, $M > 0$, and $a, b \in [2, 3]$. There exists a discrete time stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \in \{0,1,2\}}, \mathbb{P})$ with finite sample space Ω, equipped with a martingale $S = (S_t)_{t \in \{0,1,2\}}$ with S_2 taking values in $\{\pm a, \pm b\}$, as well as a random variable f such that:*

(i) *$f \in U_\infty + V_\infty$ and $f \geq 0$,*

(ii) *$\|f\|_p = M(\varepsilon/2)^{1/p}$ for all $p \in [1, \infty)$,*

(iii) *any representation $f \leq u + v$ with $u \in U$ and $v \in V$ satisfies $\|u\|_1 + \|v\|_1 \geq M/16$.*

Proof. The price process $S = (S_t)_{t=0,1,2}$ and filtration $(\mathscr{F}_t)_{t=0,1,2}$ are constructed as follows. Define $S_0 = 0$ and let $\mathscr{F}_0 = \{\emptyset, \Omega\}$. Let $S_1 = \pm 1$ with probability $1/2$ each. Next, let X be a Bernoulli random variable with $\mathbb{P}(X = 1) = \varepsilon = 1 - \mathbb{P}(X = 0)$, independent of S_1. Set $\mathscr{F}_1 = \sigma(S_1, X)$. Define the event $A = \{S_1 = 1\}$, and consider the slightly larger event

$$\widetilde{A} = A \cup \{X = 1\}.$$

Now set $S_2 = \pm a$ on $\widetilde{A}$ and $S_2 = \pm b$ on $\widetilde{A}^c$. Using the martingale condition $\mathbb{E}[S_2 \mid \mathscr{F}_1] = S_1$, the conditional probabilities are pinned down,

$$
\begin{aligned}
\mathbb{P}(S_2 = a \mid \mathscr{F}_1) &= \frac{a + S_1}{2a} \qquad \text{on } \widetilde{A}, \\
\mathbb{P}(S_2 = b \mid \mathscr{F}_1) &= \frac{b + S_1}{2b} \qquad \text{on } \widetilde{A}^c.
\end{aligned}
\tag{2.1}
$$

Note that these indeed lie in $(0, 1)$ since $a, b \geq 2$ and $S_1 = \pm 1$. Finally, set $\mathscr{F} = \mathscr{F}_2 = \sigma(S_1, X, S_2)$. This completes the description of the stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ and the prices process S. In particular, observe that the above construction only involves three independent "coin flips" and can thus be accommodated on the eight-point sample space $\Omega = \{0, 1\}^3$.

The random variable f is defined to be

$$f = M(\mathbf{1}_{\widetilde{A}} - \mathbf{1}_A) = MX\mathbf{1}_{A^c}.$$

We now prove that f satisfies the properties (i)–(iii).

(i): Clearly $f \geq 0$. Observe that

$$f = -\frac{M}{2}S_1 + \frac{M}{2}\left(\mathbf{1}_{\widetilde{A}} - \mathbf{1}_{\widetilde{A}^c}\right). \tag{2.2}$$

Since $\widetilde{A} = \{|S_2| = a\}$, it is clear that $f \in U_\infty + V_\infty$.

(ii): Simply note that $\mathbb{E}[|f|^p] = M^p \varepsilon / 2$.

(iii): Suppose $f \leq u + v$ for some $u \in U$ and $v \in V$. By nonnegativity of f we have $f^2 \leq fu + fv$. Applying part (ii) and Lemma 2.2 below yields

$$\frac{1}{2} \varepsilon M^2 = \mathbb{E}[f^2] \leq \mathbb{E}[fu] + \mathbb{E}[fv] \leq 8\varepsilon M \left(\|u\|_1 + \|v\|_1 \right).$$

This completes the proof of Lemma 2.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following key property of f was used, which intuitively states that while f is an element of $U + V$, it is almost orthogonal to both U and V. This forces the components of f in U and V to be large, despite f itself being rather small. We put ourselves in the setting of the proof of Lemma 2.1.

Lemma 2.2. *The random variable f satisfies $\mathbb{E}[fu] \leq \varepsilon M \|u\|_1$ for any $u \in U$, and $\mathbb{E}[fv] \leq 8\varepsilon M \|v\|_1$ for any $v \in V$.*

Proof. Pick any $u = (H \cdot S)_2 \in U$. In the present discrete setting, $H \cdot S$ is a martingale. Thus, using also the independence of X and S_1,

$$\begin{aligned} \mathbb{E}[fu] &= M\mathbb{E}[X \mathbf{1}_{A^c} (H \cdot S)_1] \\ &= M\varepsilon \mathbb{E}[\mathbf{1}_{A^c} (H \cdot S)_1] = M\varepsilon \mathbb{E}[\mathbf{1}_{A^c} u] \leq M\varepsilon \|u\|_1. \end{aligned} \tag{2.3}$$

Next, for any $v \in V$,
$$\mathbb{E}[fv] \leq M\mathbb{E}[|v|\mathbb{E}[X \mid S_2]].$$

We claim that $\mathbb{E}[X \mid S_2] \leq 8\varepsilon$, which then completes the proof of the lemma. Since $X = 0$ on $\widetilde{A}^c$, the bound clearly holds on that event. Furthermore, in view of (2.1) and the fact that $a \in [2, 3]$ and $\mathbb{P}(\widetilde{A}) \geq 1/2$, it follows that $\mathbb{P}(S_2 = a) = \mathbb{E}[\mathbf{1}_{\widetilde{A}} \mathbb{P}(S_2 = a \mid \mathscr{F}_1)] \geq \frac{1}{2} \times \frac{a-1}{2a} \geq 1/8$. Thus
$$\mathbb{E}[X \mid S_2 = a] \leq \frac{\mathbb{E}[X]}{\mathbb{P}(S_2 = a)} \leq 8\varepsilon,$$

showing that the claimed bound holds on the event $\{S_2 = a\}$. The event $\{S_2 = -a\}$ is treated similarly. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.3. *The second part of the representation (2.2) of the payoff f can be interpreted as a digital option written on the final value S_2 of the price process. Indeed, it pays either $+M/2$ if $S_2 = \pm a$, or $-M/2$ if $S_2 = \pm b$. Thus f can be viewed as a portfolio consisting of a digital option together with the partial hedge $-(M/2)S_1$.*

3 Pasting together the individual models

We now describe the pasting procedure that produces a proof of Theorem 1.1 from the building blocks in Lemma 2.1.

Define

$$\varepsilon_n = 2^{-n^2}, \qquad M_n = 2^n,$$

and select countably many distinct numbers a_n, b_n in the interval $[2,3]$. Apply Lemma 2.1 for each n to obtain stochastic bases $(\Omega_n, \mathscr{F}^n, \mathbb{F}^n, \mathbb{P}_n)$ and corresponding price processes $S^n = (S^n_t)_{t \in \{0,1,2\}}$ and random variables f_n satisfying the properties of Lemma 2.1, with (ε, M, a, b) replaced by $(\varepsilon_n, M_n, a_n, b_n)$.

We now paste these models together. Specifically, define

$$\Omega = \bigcup_{n \geq 1} \Omega_n, \qquad \mathscr{F}_t = \sigma(A : A \in \mathscr{F}^n_t, \, n \geq 1),$$

$$\mathbb{P}(\, \cdot \mid \Omega_n) = \mathbb{P}_n, \qquad \mathbb{P}(\Omega_n) = 2^{-n},$$

where Ω is understood as a disjoint union. In particular, the collection of events $\{\Omega_n : n \geq 1\}$ constitutes an $\mathscr{F}_0$-measurable partition of Ω. Next, define the price process by

$$S_t = \sum_{n \geq 1} S^n_t \mathbf{1}_{\Omega_n},$$

and let the random variables g_m and g be given by

$$g_m = \sum_{n=1}^{m} f_n \mathbf{1}_{\Omega_n}, \qquad g = \sum_{n \geq 1} f_n \mathbf{1}_{\Omega_n}.$$

Clearly g_m converges almost surely to g. In fact, the convergence actually takes place in L^p for any $p \in [1, \infty)$. Indeed, writing $\mathbb{E}_n$ for the expectation under $\mathbb{P}_n$, we have by Lemma 2.1(ii),

$$\mathbb{E}[|g - g_m|^p] = \sum_{n=m+1}^{\infty} 2^{-n} \, \mathbb{E}_n[|f_n|^p] = \frac{1}{2} \sum_{n=m+1}^{\infty} 2^{-n} \varepsilon_n M^p_n.$$

Since $2^{-n} \varepsilon_n M^p_n = 2^{-n(n+1-p)}$, the right-hand side tends to zero as m tends to infinity.

Moreover, each g_m lies in $U_\infty + V_\infty$. Indeed, for each n, Lemma 2.1(iii) yields $f_n = (H^n \cdot S^n)_2 + h^n(S^n_2)$ for some H^n and h^n such that the two components are bounded. Thus

$$f_n \mathbf{1}_{\Omega_n} = (H^n \mathbf{1}_{\Omega_n} \cdot S)_2 + h^n(S_2) \mathbf{1}_{\Omega_n},$$

and since $\Omega_n = \{|S_2| \in \{a_n, b_n\}\}$, the second term on the right-hand side is a (bounded) function of S_2. Thus $f_n \mathbf{1}_{\Omega_n} \in U_\infty + V_\infty$. Since g_m is a finite sum of such terms, it follows that g_m lies in $U_\infty + V_\infty$ as well. Also, g_m is nonnegative since each f_n is nonnegative.

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Finally, assume for contradiction that g lies in $U+V-L_+^0$, say $g \leq u+v$ with $u = (H{\cdot}S)_2$ and $v = h(S_2)$. Then u and v lie in L^1. On the other hand, by considering the restrictions to Ω_n we deduce

$$f_n = g|_{\Omega_n} \leq u_n + v_n,$$

where $u_n = (H|_{\Omega_n} \cdot S^n)_2$ and $v_n = h(S_2^n)$. In view of Lemma 2.1(iii), therefore,

$$\|u\|_1 + \|v\|_1 = \sum_{n \geq 1} 2^{-n}\, \mathbb{E}_n[|u_n| + |v_n|] \geq \frac{1}{16} \sum_{n \geq 1} 2^{-n} M_n = \infty.$$

This contradiction shows that $g \notin U + V - L_+^0$, and completes the proof of Theorem 1.1.

4   The continuous case

The proof of Theorem 1.3 works exactly as the proof of Theorem 1.1, except that Lemma 2.1 needs to be replaced by Lemma 4.1 below when pasting together the individual models.

Lemma 4.1. *Fix $\varepsilon \in (0, 1/2]$, $M > 0$, and $a, b \in [2, 3]$. There exists a stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ equipped with a Brownian motion W, a stopping time T, and a random variable f such that the price process $S = W^T$ is bounded with $S_T \in \{\pm a, \pm b\}$, and the random variable f satisfies*

(i) *$f \in U_\infty + V_\infty$ and $f \geq 0$,*

(ii) *$\|f\|_p = M(\varepsilon/2)^{1/p}$ for all $p \in [1, \infty)$,*

(iii) *whenever $f \leq u + v$ with $u \in U$ and $v \in V$, we have $\|u\|_1 + \|v\|_1 \geq M/16$.*

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a Brownian motion W and an independent Bernoulli random variable X with $\mathbb{P}(X = 1) = \varepsilon = 1 - \mathbb{P}(X = 0)$. Let

$$\sigma = \inf\{t \geq 0 : |W_t| = 1\}$$

be the first time the Brownian motion hits level one. Now let $\mathbb{F}$ be the right-continuous filtration generated by the processes W and $X\mathbf{1}_{[\sigma,\infty)}$. Thus, prior to time σ, only the Brownian motion is observed. Then, at time σ, the realization X is observed as well. With respect to this filtration, σ is a stopping time, W is a Brownian motion, and X is $\mathscr{F}_\sigma$-measurable but independent of $\mathscr{F}_{\sigma-}$.

Next, similarly to the discrete time case, we define the events

$$A = \{W_\sigma = 1\}, \qquad \widetilde{A} = A \cup \{X = 1\},$$

and we set

$$S = W^T, \qquad T = \inf\{t \geq \sigma : |W_t| = a\mathbf{1}_{\widetilde{A}} + b\mathbf{1}_{\widetilde{A}^c}\}.$$

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Thus depending on whether $\widetilde{A}$ or $\widetilde{A}^c$ occurs, T is the first time the absolute value of the Brownian motion reaches a or b, respectively. In particular, T is a stopping time with $T > \sigma$. The price process S is a bounded martingale with $S_T \in \{\pm a, \pm b\}$.

As in the discrete time case, the random variable f is defined to be

$$f = M(\mathbf{1}_{\widetilde{A}} - \mathbf{1}_A) = MX\mathbf{1}_{A^c}.$$

The three properties of Lemma 4.1 are proved exactly as in the discrete time case, where we use that Lemma 2.2 remains valid in the present continuous time setting:

Lemma 4.2. *The random variable f satisfies $\mathbb{E}[fu] \leq \varepsilon M\|u\|_1$ for any $u \in U$, and $\mathbb{E}[fv] \leq 8\varepsilon M\|v\|_1$ for any $v \in V$.*

Proof. Pick any $u = (H \cdot S)_T \in U$, and write $Y = H \cdot S$ for simplicity. Nonnegativity of f and the supermartingale property of Y yield

$$\mathbb{E}[fu] = \mathbb{E}\left[f\,\mathbb{E}\left[Y_T - Y_\sigma \mid \mathscr{F}_\sigma\right]\right] + \mathbb{E}[f\,Y_\sigma] \leq \mathbb{E}[f\,Y_\sigma].$$

Since X is independent of $Y_\sigma \in \mathscr{F}_{\sigma-}$, we have $\mathbb{E}[f\,Y_\sigma] = M\varepsilon\mathbb{E}[\mathbf{1}_{A^c}Y_\sigma]$. The supermartingale property of Y finally yields

$$\mathbb{E}[\mathbf{1}_{A^c}Y_\sigma] = \mathbb{E}[Y_\sigma] + \mathbb{E}[\mathbf{1}_A(Y_T - Y_\sigma)] - \mathbb{E}[\mathbf{1}_AY_T] \leq -\mathbb{E}[\mathbf{1}_AY_T] \leq \|Y_T\|_1 = \|u\|_1,$$

whence $\mathbb{E}[fu] \leq M\varepsilon\|u\|_1$ as claimed.

The statement regarding $v \in V$ follows exactly as in the proof of Lemma 2.2, where instead of (2.1) one relies on the identities

$$\mathbb{P}(S_T = a \mid \mathscr{F}_\sigma) = \frac{a + S_\sigma}{2a} \qquad \text{on } \widetilde{A},$$
$$\mathbb{P}(S_T = b \mid \mathscr{F}_\sigma) = \frac{b + S_\sigma}{2b} \qquad \text{on } \widetilde{A}^c,$$

which are direct consequences of the martingale property of S and the definition of T. $\square$

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