

COVER’S UNIVERSAL PORTFOLIO, STOCHASTIC PORTFOLIO THEORY AND THE NUMÉRAIRE PORTFOLIO

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ABSTRACT. Cover’s celebrated theorem states that the long run yield of a properly chosen “universal” portfolio is almost as good as that of the best *retrospectively chosen* constant rebalanced portfolio. The “universality” refers to the fact that this result is *model-free*, i.e., not dependent on an underlying stochastic process. We extend Cover’s theorem to the setting of stochastic portfolio theory: the market portfolio is taken as the numéraire, and the rebalancing rule need not be constant anymore but may depend on the current state of the stock market. By fixing a stochastic model of the stock market this model-free result is complemented by a comparison with the numéraire portfolio. Roughly speaking, under appropriate assumptions the asymptotic growth rate coincides for the three approaches mentioned in the title of this paper. We present results in both discrete and continuous time.

1. INTRODUCTION

In [19] the question was raised whether there is a relation between T. Cover’s theory of universal portfolio (which appeared as the very first paper of the present journal, see [9]) and stochastic portfolio theory (SPT henceforth) as initiated by R. Fernholz (see [17] and the references therein). After all, both theories ask for general recipes for choosing in a *preference-free* way good (at least in the long run) portfolios among d assets, whose prices over time are given by

$$S = (S_t^1, \dots, S_t^d).$$

Here the time t varies in \mathbb{T} , where \mathbb{T} stands either for $\mathbb{N} = \{0, 1, \dots\}$ (discrete time) or $\mathbb{R}_+ = [0, \infty)$ (continuous time). In many cases S is modeled by a stochastic process defined on some probability space. We note, however, that one may also consider a *model-free* approach where $S = (s_t^1, \dots, s_t^d)_{t \in \mathbb{T}}$ is just a deterministic trajectory with values in $(0, \infty)^d$. Indeed, Cover and Ordentlich’s discrete time results in [9, 10] are formulated in this model-free sense. The situation is more subtle in continuous time due to stochastic integration. In [25], F. Jamshidian extended Cover’s universal portfolio to continuous time under a setting of Itô processes satisfying some asymptotic stability conditions.

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In stochastic portfolio theory one also seeks robust investment strategies. More precisely, the strategies should be constructed using only observable quantities (such as market weights and their quadratic variations) and should not depend on quantities that are non-observable or difficult to estimate. In particular, no drift estimation is involved which is usually required in expected utility maximization. These are exactly the principles behind the concept of *functionally generated portfolios* (see [17, Chapter 3]). While in most of the literature an Itô process setting is assumed, much of SPT can be developed in a model-free setting as done by S. Pal and L. Wong [30] in discrete time and by A. Schied et al. [31] in continuous time. The reason why it works in continuous time is that the value processes of functionally generated portfolios can be defined without stochastic integration.

In this paper we connect the two theories and provide additionally a comparison with the numéraire portfolio, which corresponds to the classical log-optimal portfolio.¹ Relationships between the two theories were studied in the recent papers by T. Ichiba and M. Brod [23, 4] as well as L. Wong [33]. In particular, Wong [33] extends Cover’s approach to the family of functionally generated portfolios in discrete time and shows that the distribution of wealth in this family satisfies a pathwise large deviation principle.

1.1. Summary and discussion of the main results. In this article we work under the setting of SPT. Namely, the market portfolio is taken as the benchmark, or “numéraire”, so that the primary assets are the market weights which take values in the open d -simplex defined by $\Delta^d = \{x \in (0, 1)^d \mid \sum_{i=1}^d x^i = 1\}$. Its closure is denoted by $\bar{\Delta}^d = \{x \in [0, 1]^d \mid \sum_{i=1}^d x^i = 1\}$. This enables us to analyze strategies which depend on the market weights, and the performance of *relative* wealth with respect to the market portfolio.

1.1.1. Discrete time. We start by summarizing our results in discrete time. We extend Cover’s universal portfolio to a class of M -Lipschitz portfolio maps denoted by \mathcal{L}^M . Each element of \mathcal{L}^M maps the market weights to *long-only* portfolio weights in $\bar{\Delta}^d$ (see Definition 3.1).

Denoting by $(V_t^\pi)_{t=0}^\infty$ the *relative wealth process* corresponding to a portfolio strategy² $(\pi_t)_{t=1}^\infty$, we are interested in comparing the asymptotic growth rates

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log(V_T^\pi),$$

for certain “optimal” portfolio choices π . More precisely, under suitable conditions we establish asymptotic equality of the growth rates of the following portfolios:

- the *best retrospectively chosen portfolio* at time T in the class $\mathcal{L} := \bigcup_{M=1}^\infty \mathcal{L}^M$ (in this context $V_T^{*,M}$ will denote the relative wealth at time T achieved by investing according to the best strategy in \mathcal{L}^M over the time interval $[0, T]$);
- the *analogue of Cover’s universal portfolio* whose relative wealth process $(V_t(\nu))_{t=0}^\infty$ is defined in (3.2) (here ν is a probability measure on \mathcal{L} with *full support* on each \mathcal{L}^M);
- the *log-optimal portfolio* among the class of *long-only* strategies, whose relative wealth process is denoted by $(\widehat{V}_t)_{t=0}^\infty$.

¹Henceforth, we only use the terminology log-optimal portfolio.

²Here, the portfolio weight π_t is chosen at time $t - 1$ and is used over the time interval $[t - 1, t]$.

The first two portfolios can be compared in a model-free way (see Theorem 3.9). To compare them with the log-optimal portfolio we have to introduce a probabilistic setting. Our main result can then be roughly stated as follows:

Theorem 1.1. *Let $(\mu_t)_{t=0}^\infty$ be a time-homogenous ergodic Markov process in discrete time describing the dynamics of the market weights. Then*

$$(1.1) \quad \lim_{M \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{1}{T} \log(V_T^{*,M}) = \lim_{T \rightarrow \infty} \frac{1}{T} \log(V_T(\nu)) = \lim_{T \rightarrow \infty} \frac{1}{T} \log(\widehat{V}_T)$$

holds almost surely.

Intuitively, this theorem says that a suitable *full support* mixture of strategies (given by the universal portfolio) is asymptotically as good as the best one chosen with hindsight, and the log-optimal portfolio constructed with full knowledge of the underlying process.

1.1.2. *Continuous time.* Theorem 1.1, which involves Lipschitz portfolio maps, cannot be extended directly to continuous time because of stochastic integrals. Instead, we consider *functionally generated portfolios* (see Section 4) whose relative wealth processes can be defined in a pathwise manner (see e.g. [31]). This choice not only allows model-free considerations but also perfectly connects Cover’s theory with SPT in continuous time. By replacing the set \mathcal{L}^M by certain spaces of functionally generated portfolios and assuming that the log-optimal portfolio is functionally generated, we get essentially the same theorem as above.

Apart from the work by F. Jamshidian [25], universal portfolio theory has only been studied sparingly in continuous time; see for example the paper [24] which studied the performance of the universal portfolio under the “Hybrid Atlas” model. To the best of our knowledge generalizations to nonparametric families of portfolio maps (in continuous time) have not been considered so far. In this sense, our results significantly extend the continuous time literature.

While our approach focuses on the mathematical aspects, universal portfolio strategies have also been studied extensively in an algorithmic framework. See [29] for a recent survey and in particular [21].

1.1.3. *Discussion of the results.* Our model-free approach has clear advantages over classical ones which heavily rely on a particular model choice. Even in the case when the model class (e.g. the Heston model or Lévy models) is correctly specified, model parameters cannot be estimated precisely and always come with a confidence interval. So, in practice the *estimated optimal* portfolio is always different from the *true optimal* one. Our results support the idea that a Bayesian average in the spirit of Cover’s universal portfolio is, in the long run, better than a suboptimal estimate.

As for the original theorems of Cover and Jamshidian, a valid criticism is of course that we only establish asymptotic equality on a first-order log-return basis. As such, a lot of important information is lost in the limit. However, one cannot expect to obtain any information on higher-order terms unless further quantitative assumptions are made on the considered models. Cover’s aim and also the goal of the present article is to be as model-free as possible.³ Nevertheless, it is of great theoretical and practical interest to strengthen the asymptotic results to quantitative ones under suitable additional conditions. We hope to address this important question in future research.

³We are grateful to one of anonymous referees for pointing this out.

The remainder of the paper is organized as follows. In Section 2 we provide a brief overview (in discrete time for convenience) of the main topics of this paper, i.e., Cover’s theorem, the setting of SPT and the log-optimal portfolio. In Section 3 we establish Theorem 1.1 in discrete time (see Theorem 3.10 and Corollary 3.11), while Section 4 is dedicated to proving the corresponding statements in continuous time in the setting of functionally generated portfolios and – for the comparison with the log-optimal portfolio – under the assumption that the market weights follow an ergodic Itô diffusion (see Theorem 4.11 and Corollary 4.13). Some auxiliary and technical proofs are gathered in the appendix.

2. OVERVIEW OF THE THREE PORTFOLIOS

For expositional simplicity time is discrete in this section.

2.1. Cover’s universal portfolio. Cover’s insight reveals that the “wisdom of hindsight” does not give significant advantages over a properly chosen “universal” portfolio constructed using only historical and current prices of the assets. The relevant optimality criterion here is the asymptotic growth rate of the portfolio.

Let us sketch this – at first glance surprising – result in a particularly easy setting (compare [9, 10]): Fix $T \in \mathbb{N}$ and think of an investor who at time T looks back which stock she should have bought at time $t = 0$ (by investing her initial endowment and subsequently holding the stock). There is an obvious solution: pick $i \in \{1, \dots, d\}$ which maximizes the normalized logarithmic return

$$(2.1) \quad \frac{1}{T}(\log(S_T^i) - \log(S_0^i)).$$

The problem with this trading strategy is, of course, that we have to make our choice at time $t = 0$ instead of $t = T$. Here is the remedy (compare e.g., [3]): at time $t = 0$ simply divide the initial endowment, say 1€ , into d portions of $\frac{1}{d}\text{€}$, invest each portion in each of the stocks and then hold the resulting portfolio. At time T the normalized logarithmic return satisfies⁴

$$(2.2) \quad \frac{1}{T} \log(V_T) \geq \frac{1}{T} \log \left(\frac{1}{d} \sum_{j=1}^d \frac{S_T^j}{S_0^j} \right) \geq \frac{1}{T} (\log(S_T^i) - \log(S_0^i) - \log d),$$

where again i denotes the stock which performed best during the time interval $[0, T]$. Hence the difference between (2.1) and (2.2) can be bounded by $\frac{\log(d)}{T}$ which tends to zero as $T \rightarrow \infty$. Hence this buy-and-hold portfolio, which corresponds to a universal portfolio in the sense of Cover, has asymptotically the same normalized logarithmic return as the – only retrospectively known – best performing stock.

Instead of these “pure” investments Cover considered a more ambitious setting, namely all *constant rebalanced portfolio strategies*: let $b = (b^1, \dots, b^d) \in \bar{\Delta}^d$, i.e., $b^j \geq 0$ and $\sum_{j=1}^d b^j = 1$. The value of the corresponding constant rebalanced portfolio $(V_t(b))_{t=0}^\infty$ starting at $V_0(b) = 1$ is defined by holding throughout the proportion b^j of the current wealth in stock j , so that $V_0(b) = 1$ and

$$(2.3) \quad \frac{V_{t+1}(b)}{V_t(b)}(s) = \sum_{j=1}^d b^j \frac{s_{t+1}^j}{s_t^j},$$

⁴Later we will use V to denote instead the *relative* wealth of the portfolio.

for each trajectory $s = ((s_t^j)_{j=1}^d)_{t=0}^\infty \subset (0, \infty)^d$ of the stocks.

Fix again T and define the quantity V_T^* by

$$(2.4) \quad V_T^*(s) = \max_{b \in \bar{\Delta}^d} V_T(b)(s),$$

which is a function of the trajectory $s = (s_t^1, \dots, s_t^d)_{t=0}^T$. Again, the idea is that, with hindsight, i.e., knowing $(s_t^1, \dots, s_t^d)_{t=0}^T$, one considers the best weight $b \in \bar{\Delta}^d$ which attains the maximum (2.4). Cover's goal is to construct a portfolio which generates wealth that performs asymptotically as well as the process $(V_T^*)_{T=0}^\infty$ as $T \rightarrow \infty$, uniformly over all price paths. For this reason the portfolio is said to be *universal*. In order to do so, let ν be a probability measure on $\bar{\Delta}^d$ which replaces the previous uniform distribution over the d stocks. The universal portfolio is built by investing at time 0 the portion $d\nu(b)$ of initial capital in the constant rebalanced portfolio $V(b)$ and by subsequently following the constant rebalanced portfolio process $(V_t(b))_{t=0}^T$. The explicit formula for the wealth is

$$(2.5) \quad V_t(\nu)(s) = \int_{\bar{\Delta}^d} V_t(b)(s) d\nu(b),$$

where $V_t(b)$ is defined by (2.3). The portfolio weight of the corresponding *universal portfolio* is given by the wealth-weighted average

$$(2.6) \quad b_t^\nu(s) = \frac{\int_{\bar{\Delta}^d} b V_t(b)(s) d\nu(b)}{\int_{\bar{\Delta}^d} V_t(b)(s) d\nu(b)}.$$

Let us now recall Cover's celebrated result:

Theorem 2.1. (Cover [9]): *Let ν be a probability measure on $\bar{\Delta}^d$ with full support. Then*

$$(2.7) \quad \lim_{T \rightarrow \infty} \frac{1}{T} (\log(V_T(\nu)(s)) - \log(V_T^*(s))) = 0,$$

for all trajectories $s = (s_t^1, \dots, s_t^d)_{t=0}^\infty$ for which there are constants $0 < c \leq C < \infty$ such that

$$(2.8) \quad c \leq \frac{s_{t+1}^j}{s_t^j} \leq C, \quad \text{for all } j = 1, \dots, d \quad \text{and all } t \in \mathbb{N}.$$

The proof is given in the Appendix.

Remark 2.2. As shown by T. Cover and E. Ordentlich [10], the condition (2.8) can be dropped at least when ν is the uniform or Dirichlet($\frac{1}{2}, \dots, \frac{1}{2}$) distribution on Δ^d (see also A. Blum and A. Kalai [3] for an elegant proof in case of the uniform distribution).

Remark 2.3. Let $\mathcal{M}_1(\bar{\Delta}^d)$ be the set of probability measures on $\bar{\Delta}^d$. For each $\mu \in \mathcal{M}_1(\bar{\Delta}^d)$, consider the value $\int_{\bar{\Delta}^d} V_T(b)(s) d\mu(b)$ of the mixture portfolio with initial measure μ . Note that the constant rebalanced portfolio $V_T(b)$ corresponds to the case where μ is the point mass at b . It is easy to see that

$$\sup_{\mu \in \mathcal{M}_1(\bar{\Delta}^d)} \int_{\bar{\Delta}^d} V_T(b)(s) d\mu(b) = V_T^*(s),$$

where $V_T^*(s)$ is defined by (2.4). It follows that the universal portfolio (2.5) (with initial measure ν) is still asymptotically optimal in the larger class

$$(2.9) \quad \left\{ \left(\frac{\int_{\bar{\Delta}^d} bV_t(b)d\mu(b)}{\int_{\bar{\Delta}^d} V_t(b)(s)d\mu(b)} \right)_{t \geq 0} \mid \mu \in \mathcal{M}_1(\bar{\Delta}^d) \right\}.$$

2.2. Stochastic portfolio theory, portfolio maps and the corresponding universal portfolio. In SPT we let (s^1, \dots, s^d) denote the market capitalizations of the stocks rather than their prices. Then we define the vector of *market weights* $(\mu^1, \dots, \mu^d) \in \Delta^d$ by

$$(\mu^1, \dots, \mu^d) = \left(\frac{s^1}{s^1 + \dots + s^d}, \dots, \frac{s^d}{s^1 + \dots + s^d} \right).$$

This amounts to taking the *market portfolio* (whose value at time t is $\sum_{j=1}^d s_t^j$) as the *numéraire* (compare [11] and [15]).

The *relative wealth process* $(V_t^\pi)_{t=0}^\infty$, expressed in units of the market portfolio and starting at $V_0 = 1$, is obtained by the following recursive relation⁵

$$(2.10) \quad \frac{V_{t+1}^\pi}{V_t^\pi} = \sum_{j=1}^d \pi_{t+1}^j \frac{\mu_{t+1}^j}{\mu_t^j}.$$

In general, we allow all predictable, admissible trading strategies $(\pi_t)_{t=1}^\infty$, where the portfolio weight π_t is used over the time interval $[t-1, t]$. In this paper all trading strategies are fully invested in the equity market, i.e., the portfolio weights sum to 1 for all t . In particular, the strategies do not lend or borrow money. Henceforth all wealth processes are measured in units of the market portfolio.

We will focus on trading strategies defined by (deterministic) *portfolio maps*. These are (Borel) measurable functions

$$(2.11) \quad \pi : \Delta^d \rightarrow \bar{\Delta}^d$$

which associate to the current market capitalization $\mu_t = (\mu_t^1, \dots, \mu_t^d)$ the weights $(\pi(\mu_t) = (\pi^1(\mu_t), \dots, \pi^d(\mu_t)))$ according to which an agent distributes current wealth among the d stocks at time t . The *constant rebalanced portfolio strategies* considered by Cover correspond to the constant functions $\pi : \Delta^d \rightarrow \bar{\Delta}^d$.

In this paper we extend Cover's theory of *constant rebalanced portfolios* to certain families of *portfolio maps*. First we note that Cover's and Jamshidian's definition of a universal portfolio as in (2.6) and (2.5) can be easily extended to a general setting. Let \mathcal{G} denote some appropriate space of portfolio maps, $\mathcal{B}(\mathcal{G})$ its Borel σ -algebra and ν some probability measure on \mathcal{G} .

Definition 2.4. *Let ν be a probability measure on $(\mathcal{G}, \mathcal{B}(\mathcal{G}))$. Then the corresponding universal portfolio at time t is given by the wealth-weighted average*

$$(2.12) \quad \pi_t^\nu = \frac{\int_{\mathcal{G}} \pi V_t^\pi d\nu(\pi)}{\int_{\mathcal{G}} V_t^\pi d\nu(\pi)}.$$

From (2.10) it is easily seen that the wealth generated by π^ν is given by

$$(2.13) \quad V_T(\nu) = \int_{\mathcal{G}} V_T^\pi d\nu(\pi).$$

⁵Here it is assumed implicitly that the stocks do not pay dividends. This assumption is common in universal and stochastic portfolio theory and allows us to focus on the main ideas.

2.3. The log-optimal portfolio. To define the log-optimal portfolio we consider a probabilistic setting. The stock price process $S = (S_t^1, \dots, S_t^d)_{t=0}^\infty$ and the corresponding relative market capitalizations $\mu = (\mu_t^1, \dots, \mu_t^d)_{t=0}^\infty$ are now assumed to be stochastic processes defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^\infty, \mathbb{P})$.

There is a large literature on the log-optimal portfolio (see e.g., [2], [26] and the references given there). For a fixed horizon T , this portfolio is by definition the maximizer of the expected logarithmic growth rate

$$(2.14) \quad \mathbb{E}[\log(V_T^\pi)] = \mathbb{E} \left[\sum_{t=0}^{T-1} \log \left(\sum_{j=1}^d \pi_{t+1}^j \frac{\mu_{t+1}^j}{\mu_t^j} \right) \right]$$

over all predictable, admissible trading strategies $(\pi_t)_{t=1}^T$. Under mild assumptions on the process a unique optimizer exists; see e.g., [2, 28].

To connect the log-optimal portfolio with universal portfolios in the sense of Definition 2.4 we need appropriate assumptions. We will assume that μ is a time-homogenous Markov process, and we will restrict to *long-only* portfolios in the optimization of (2.14). These imply that the optimal portfolio in (2.14) (over the set of predictable processes taking values in $\bar{\Delta}^d$) has the form $\pi_t = \pi(\mu_{t-1})$, where $\pi : \Delta^d \mapsto \bar{\Delta}^d$ as in (2.11). We denote the corresponding optimizer by $\hat{\pi}$.

The Markovian assumption can be motivated by the stability of capital distributions of equity markets (see [17, Chapter 5]). In SPT, this led to systems of interacting Brownian particles whose dynamics depend on their relative rankings. Under suitable conditions, these systems show behaviors observed in large equity markets. See, for example [1, 24] for “Atlas”-type models and the references therein⁶. We also refer to [27] which studies the growth optimal portfolio in a Markovian setting with uncertainties.

3. A COMPARISON OF THE THREE APPROACHES - THE DISCRETE TIME CASE

Throughout this section we work in discrete time and assume that the market weights are described by a d -dimensional path $\mu = (\mu_t)_{t=0}^\infty$ with values in Δ^d . We consider as far as possible a model-free approach, but will introduce a probabilistic setting when the log-optimal portfolio is involved.

3.1. Definitions of the portfolios. We start by defining rigorously, in the present setting, the three portfolios introduced in Section 1 and Section 2.

3.1.1. The best retrospectively chosen portfolio. Consider Cover’s theme of choosing retrospectively at time T a strategy which is optimal within a certain class of strategies, in our case portfolio maps $\pi : \Delta^d \rightarrow \bar{\Delta}^d$. A moment’s reflection reveals that it does not make sense to allow to choose among *all measurable* functions $\pi : \Delta^d \rightarrow \bar{\Delta}^d$. Indeed, there is no restriction to choose π such that $\pi(\mu_t) = e_{j(t)}$, where $j(t) \in \{1, \dots, d\}$ maximizes μ_{t+1}^j / μ_t^j . This is asking for too much clairvoyance and does not allow for meaningful results (compare [10] and [3, Section 5]).

However, it does make sense (economically as well as mathematically) to restrict to more regular trading strategies. In particular, we work with the following set of M -Lipschitz portfolio maps. For $\epsilon > 0$ we let $\bar{\Delta}_\epsilon^d$ denote the set of $x \in \Delta^d$ satisfying $x^j \geq \frac{\epsilon}{d}$, for $j = 1, \dots, d$. Also we let $\|\cdot\|_1$ be the usual 1-norm.

⁶A comparison between the log-optimal portfolio and Cover’s universal portfolio is studied in [24].

Definition 3.1. For $M > 0$ we denote by \mathcal{L}^M the set of all M -Lipschitz functions $\Delta^d \rightarrow \bar{\Delta}_{M-1}^d$, i.e., $\|\pi(x) - \pi(y)\|_1 \leq L\|x - y\|_1$, $x, y \in \Delta^d$.

Remark 3.2. The set \mathcal{L}^M of M -Lipschitz functions $\pi : \Delta^d \rightarrow \bar{\Delta}_{M-1}^d$ is a compact metric space with respect to the topology of uniform convergence induced by the norm $\|\pi\|_\infty = \sup\{\|\pi(x)\|_1 : x \in \Delta^d\}$.

Remark 3.3. Instead of Lipschitz functions we could just as well consider other compact function spaces, e.g., Hölder spaces equipped with a proper norm. This is done in the context of functionally generated portfolios in Section 4.

The retrospectively chosen best performing portfolio among the above Lipschitz maps is defined as follows:

Definition 3.4. For a given trajectory $(\mu_t)_{t=0}^T \in (\Delta^d)^{T+1}$ we define

$$(3.1) \quad V_T^{*,M} = \sup_{\pi \in \mathcal{L}^M} V_T^\pi = \sup_{\pi \in \mathcal{L}^M} \prod_{t=0}^{T-1} \left(\sum_{j=1}^d \pi^j(\mu_t) \frac{\mu_{t+1}^j}{\mu_t^j} \right).$$

By compactness (see Remark 3.2) and continuity of the map $\pi \mapsto V_T^\pi$ there exists an optimizer $\pi^{*,M} \in \mathcal{L}^M$ (not necessarily unique) such that $V_T^{*,M} = V_T^{\pi^{*,M}}$, thus the sup above can be replaced by max.

3.1.2. The universal portfolio. Our aim is to find a *predictable* process $\pi^M = (\pi_t^M)_{t=1}^\infty$, i.e., one which depends only on the history of the market weights, such that the performance of $(V_t^{\pi^M})_{t=0}^\infty$ is asymptotically as good as that of $(V_t^{*,M})_{t=0}^\infty$. This can be achieved by the universal portfolio introduced in Definition 2.4, where the \mathcal{G} is now \mathcal{L}^M as in Definition 3.1. As \mathcal{L}^M is a compact metric space, we may find a (Borel) probability measure ν on $(\mathcal{L}^M, \|\cdot\|_\infty)$ with full support; this will be essential for establishing an analog to Theorem 2.1. The (relative) wealth of the universal portfolio is given, as in (2.13), by

$$(3.2) \quad V_T^M(\nu) = \int_{\mathcal{L}^M} V_T^\pi d\nu(\pi).$$

3.1.3. The log-optimal portfolio. In order to relate the universal portfolio to the (long-only) log-optimal portfolio, we assume that $\mu = (\mu_t)_{t=0}^\infty$ is a time-homogeneous Markov process (see Section 2.3). Here is a precise statement.

Assumption 3.5. The process μ is a time homogeneous, ergodic Markov process with a unique invariant measure ϱ on the open simplex Δ^d .

We denote the transition kernel of the chain by $(\varrho(x, \cdot))_{x \in \Delta^d}$, i.e., for all Borel sets $A \subseteq \bar{\Delta}^d$ we have $\mathbb{P}[\mu_{t+1} \in A | \mathcal{F}_t] = \varrho(\mu_t, A)$. For further notions concerning ergodic Markov processes we refer to [14].

The *long-only log-optimal* trading strategy $\hat{\pi}$, as noted above, is given in terms of a portfolio map. Given that $\mu_t = x \in \Delta^d$, we know the conditional law $\varrho(x, \cdot)$ of μ_{t+1} . We therefore choose $\hat{\pi}(x) \in \bar{\Delta}^d$ as the maximizer

$$(3.3) \quad \hat{\pi}(x) = \arg \max_{p \in \bar{\Delta}^d} \left(\int_{\Delta^d} \log(\langle p, \frac{y}{x} \rangle) \varrho(x, dy) \right)$$

and assume that $\widehat{\pi}(\cdot)$ can be chosen to be measurable (here $\langle \cdot, \cdot \rangle$ denotes the Euclidean dot product). For $x \in \Delta^d$ define the number $L(x)$ as the value of the optimization problem (3.3), i.e.,

$$(3.4) \quad L(x) = \max_{p \in \bar{\Delta}^d} \left(\int_{\Delta^d} \log(\langle p, \frac{y}{x} \rangle) \varrho(x, dy) \right) = \int_{\Delta^d} \log(\langle \widehat{\pi}(x), \frac{y}{x} \rangle) \varrho(x, dy).$$

Considering $\pi(x) = x$ (which corresponds to the market portfolio) we clearly have $L(x) \geq 0$ for each $x \in \Delta^d$. We obtain the a.s. relation

$$L(x) = \mathbb{E} \left[\log \left(\frac{\widehat{V}_{t+1}}{\widehat{V}_t} \right) \middle| \mu_t = x \right],$$

where $\widehat{V} = (\widehat{V}_t)_{t=0}^\infty$ denotes the long-only log-optimal wealth process $V^{\widehat{\pi}}$ defined by the portfolio map $\widehat{\pi}$ via (2.10).

Assumption 3.6. *Using the above notation we assume that*

$$(3.5) \quad L := \int_{\Delta^d} L(x) d\varrho(x) < \infty.$$

Applying Birkhoff's ergodic theorem for discrete time Markov processes (see [14, Theorem 2.2, Section 2.1.4]) we have the following result.

Theorem 3.7. *Under Assumptions 3.5 and 3.6, we have that, for ϱ -a.e. starting value $\mu_0 \in \Delta^d$,*

$$(3.6) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \log(\widehat{V}_T) = L,$$

the limit holding true a.s. as well as in L^1 .

More generally, let $\pi : \Delta^d \rightarrow \bar{\Delta}^d$ be any measurable portfolio map such that

$$(3.7) \quad L^\pi := \int_{\Delta^d} \left(\int_{\Delta^d} \log \left(\langle \pi(x), \frac{y}{x} \rangle \right) \varrho(x, dy) \right) d\varrho(x) > -\infty.$$

We then have, for ϱ -a.s. starting value μ_0 , that

$$(3.8) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \log(V_T^\pi) = L^\pi$$

a.s. as well as in L^1 .

In general there is little reason why the function $\widehat{\pi}$ should have better regularity properties than being just measurable. On the other hand, we may approximate $\widehat{\pi}$ by more regular functions, in particular by functions in \mathcal{L}^M . This will be crucial for comparing the asymptotic growth rates. The following result is intuitively obvious, but the proof turns out to be quite technical and will be given in the appendix.

Lemma 3.8. *Under Assumptions 3.5 and 3.6, for any $\epsilon > 0$ there exist $M > 0$ and an M -Lipschitz function $\pi_{Lip} \in \mathcal{L}^M$ such that*

$$L^{\pi_{Lip}} > L - \epsilon,$$

where L and L^π are given in (3.5) and (3.7) respectively. In particular, we have $L = \sup_M \sup_{\pi \in \mathcal{L}^M} L^\pi$.

3.2. Asymptotically equivalent growth rates. We are now ready to compare the asymptotic performance of the three approaches. We first establish an analogue of Theorem 2.1.

Theorem 3.9. *Fix $M > 0$ and a Borel probability measure ν with full support on \mathcal{L}^M . For every trajectory $(\mu_t)_{t=0}^\infty$ in Δ^d we have*

$$(3.9) \quad \lim_{T \rightarrow \infty} \frac{1}{T} (\log(V_T^{*,M}) - \log(V_T^M(\nu))) = 0.$$

Proof. The inequality “ \geq ” is obvious. For the reverse inequality we follow the argument of [3]. As \mathcal{L}^M is compact and ν has full support, it is not difficult to see that for any $\eta > 0$, there exists $\delta > 0$ such that every η -neighbourhood of a point $\pi \in \mathcal{L}^M$ has ν -measure bigger than δ .

Let a trajectory $(\mu_t)_{t=0}^\infty$ in Δ^d be given. For a fixed time T let $\pi^{*,M} \in \mathcal{L}^M$ be an optimizer of (3.1). Consider a portfolio map $\pi^M \in \mathcal{L}^M$ with $\|\pi^M - \pi^{*,M}\|_\infty < \eta$, i.e., such that, for every $x \in \Delta^d$ we have $\|\pi^M(x) - \pi^{*,M}(x)\|_1 = \sum_{j=1}^d |\pi^M(x)^j - \pi^{*,M}(x)^j| < \eta$.

Choose $\eta > 0$ small enough so that $\alpha = \eta M d < 1$ and define, for $x \in \Delta^d$,

$$(3.10) \quad \tilde{\pi}(x) = \frac{1}{\alpha} \pi^M(x) - \frac{1-\alpha}{\alpha} \pi^{*,M}(x).$$

Rearranging, we have

$$(3.11) \quad \pi^M(x) = (1-\alpha)\pi^{*,M}(x) + \alpha\tilde{\pi}(x).$$

It is easy to see that that $\tilde{\pi}$ maps Δ^d into $\bar{\Delta}^d$.

Using (3.11), we have the estimate

$$(3.12) \quad \begin{aligned} \frac{1}{T} \log V_T^{\pi^M} &= \frac{1}{T} \sum_{t=0}^{T-1} \log(\langle \pi^M(\mu_t), \frac{\mu_{t+1}}{\mu_t} \rangle) \\ &\geq \frac{1}{T} \sum_{t=0}^{T-1} \log(\langle (1-\alpha)\pi^{*,M}(\mu_t), \frac{\mu_{t+1}}{\mu_t} \rangle) \\ &= \frac{1}{T} \log(V_T^{*,M}) + \log(1-\alpha). \end{aligned}$$

Fix $\epsilon > 0$. Choosing $\eta > 0$ sufficiently small we can make $\alpha = \eta M d$ small enough such that the final term is bigger than $-\epsilon$. Summing up, we have

$$(3.13) \quad \frac{1}{T} [\log(V_T^{*,M}) - \log(V_T^{\pi^M})] < \epsilon$$

whenever $\|\pi^M - \pi^{*,M}\|_\infty < \eta$.

Denote by $B = B_\eta(\pi^{*,M})$ the $\|\cdot\|_\infty$ -ball with radius η in \mathcal{L}^M which has ν -measure at least $\delta > 0$, where δ only depends on η . As each element π^M of B satisfies (3.13) we have

$$(3.14) \quad \frac{1}{T} \log(V_T^M(\nu)) \geq \frac{\log(\delta)}{T} + \frac{1}{T} \log(V_T^{*,M}) - \epsilon.$$

Now (3.9) is proved by sending in (3.14) T to infinity and letting ϵ to zero. \square

Note that in Theorem 3.9 we do not need the uniform boundedness condition (2.8) (compare this result with [33, Lemma 3.3]). We now combine Lemma 3.8 (which is probabilistic) with Theorem 3.9 (which is pathwise) to obtain – under suitable assumptions – equality of the asymptotic performance among the three portfolios. We first consider the space \mathcal{L}^M for a fixed M . In Corollary 3.11 below we then formulate a result for $\mathcal{L} = \bigcup_M \mathcal{L}^M$.

Theorem 3.10. *Let $\Omega = (\Delta^d)^\mathbb{N}$ be the canonical path space equipped with its natural filtration and a probability measure \mathbb{P} . Define $\mu = (\mu_t)_{t=0}^\infty$ to be the canonical process, i.e., $\mu_t(\omega) = \omega_t$, which takes values in Δ^d and satisfies Assumptions 3.5 and 3.6. Moreover, let $M > 0$ be a fixed Lipschitz constant for the space \mathcal{L}^M . Consider the following objects that are defined for each trajectory $(\mu_t)_{t=0}^\infty$:⁷*

- (i) *Define for each $T \in \mathbb{N}$ the portfolio map $\pi^{*,M} \in \mathcal{L}^M$ as well as the corresponding wealth $V_T^{*,M} := V_T^{\pi^{*,M}}$ as in (3.1).*
- (ii) *Fix a probability measure ν on \mathcal{L}^M with full support and consider the wealth process of the universal portfolio $(V_t^M(\nu))_{t=0}^\infty$ as of (3.2).*
- (iii) *Define the log-optimal portfolio among the portfolio maps $\pi \in \mathcal{L}^M$ by*

$$(3.15) \quad \hat{\pi}^M = \arg \max_{\pi \in \mathcal{L}^M} \int_{\Delta^d} \left[\int_{\Delta^d} \log(\langle \pi(x), \frac{y}{x} \rangle) \varrho(x, dy) \right] d\varrho(x)$$

and the corresponding wealth process $(\hat{V}_t^M)_{t=0}^\infty = (V_t^{\hat{\pi}^M})_{t=0}^\infty$ via (2.10).

Then, we have \mathbb{P} -a.s.

$$(3.16) \quad \liminf_{T \rightarrow \infty} \frac{1}{T} \log(V_T^{*,M}) = \liminf_{T \rightarrow \infty} \frac{1}{T} \log(V_T^M(\nu)) = \lim_{T \rightarrow \infty} \frac{1}{T} \log(\hat{V}_T^M) = \sup_{\pi \in \mathcal{L}^M} L^\pi,$$

where L^π is given in (3.7). In addition, the first equality holds for all trajectories $(\mu_t)_{t=0}^\infty$ in Δ^d .

Proof. We first note that $\hat{\pi}^M$ is well-defined; simply use the compactness of \mathcal{L}^M with respect to $\|\cdot\|_\infty$ (compare the proof of Lemma 3.8). Note also that by the ergodic theorem (Theorem 3.7), we have for each $\pi \in \mathcal{L}^M$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log V_T^\pi = L^\pi \quad \mathbb{P}\text{-a.s.},$$

where L^π is defined by (3.7). In particular, as $\hat{\pi}^M \in \mathcal{L}^M$ by definition, we have

$$(3.17) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \log \hat{V}_T^M = \sup_{\pi \in \mathcal{L}^M} L^\pi \quad \mathbb{P}\text{-a.s.}$$

That the first equality in (3.16) holds for *all* trajectories $(\mu_t)_{t=0}^\infty$ in Δ^d was shown in Theorem 3.9.

For each fixed $T \in \mathbb{N}$ we obviously have

$$(3.18) \quad \frac{1}{T} \log(\hat{V}_T^M) \leq \frac{1}{T} \log(V_T^{*,M}) \quad \mathbb{P}\text{-a.s.}$$

Using (3.17), (3.18) and Theorem 3.9 we thus have \mathbb{P} -a.s.

$$(3.19) \quad \sup_{\pi \in \mathcal{L}^M} L^\pi = \lim_{T \rightarrow \infty} \frac{1}{T} \log(\hat{V}_T^M) \leq \liminf_{T \rightarrow \infty} \frac{1}{T} \log(V_T^{*,M}) = \liminf_{T \rightarrow \infty} \frac{1}{T} \log(V_T^M(\nu)).$$

⁷To simplify the notations we will suppress ω .

On the other hand, by the definition of $(\widehat{V}_t^M)_{t=0}^\infty$ as the log-optimizer within the class \mathcal{L}^M , we have

$$(3.20) \quad \mathbb{E}[\log(V_T^M(\nu))] \leq \sup_{\pi \in \mathcal{L}^M} \mathbb{E}[\log(V_T^\pi)] = \mathbb{E}[\log(\widehat{V}_T^M)].$$

To see this, note that the universal portfolio is given by (2.12). By the time-homogenous Markovianity it is thus sufficient to dominate the left hand side of (3.20) by taking the supremum over elements in \mathcal{L}^M .

Combining now (3.20), Theorem 3.7 and (3.19) yields that

$$\begin{aligned} \mathbb{E} \left[\liminf_{T \rightarrow \infty} \frac{1}{T} \log(V_T^M(\nu)) \right] &\leq \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}[\log(V_T^M(\nu))] \\ &\leq \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}[\log(\widehat{V}_T^M)] \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \log(\widehat{V}_T^M) \\ &\leq \liminf_{T \rightarrow \infty} \frac{1}{T} \log V_T^{*,M} \\ &= \liminf_{T \rightarrow \infty} \frac{1}{T} \log(V_T^M(\nu)), \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Here, the first inequality follows from Fatou's lemma (note here that $\frac{1}{T} \log(V_T^M(\nu))$ is bounded from below, see e.g., (3.14)). From this we see that the quantity $\liminf_{T \rightarrow \infty} \frac{1}{T} \log(V_T^M(\nu))$ is \mathbb{P} -a.s. constant and equal to $\lim_{T \rightarrow \infty} \frac{1}{T} \log(\widehat{V}_T^M)$. This completes the proof of the theorem. \square

Next we will send M to infinity in the following way. For $M = 1, 2, 3, \dots$ choose a measure ν^M on \mathcal{L}^M with full support. Define $\nu = \sum_{M=1}^\infty 2^{-M} \nu^M$ and the wealth of the universal portfolio $V(\nu)$ as in (3.2) by

$$(3.21) \quad V_t(\nu) = \int_{\mathcal{L}} V_t^\pi d\nu(\pi), \quad t \in \mathbb{N}.$$

where $\mathcal{L} = \bigcup_{M=1}^\infty \mathcal{L}^M$. Recall that $(\widehat{V}_t)_{t=0}^\infty$ is the wealth process of the (long-only) log-optimal portfolio (3.3).

Corollary 3.11. *Under the assumptions of Theorem 3.10 we have \mathbb{P} -a.s.*

$$(3.22) \quad \lim_{M \rightarrow \infty} \liminf_{T \rightarrow \infty} \frac{1}{T} \log V_T^{*,M} = \lim_{T \rightarrow \infty} \frac{1}{T} \log V_T(\nu) = \lim_{T \rightarrow \infty} \frac{1}{T} \log \widehat{V}_T = L,$$

where L is defined in (3.5).

Proof. Letting $M \rightarrow \infty$ in (3.16), we have

$$\lim_{M \rightarrow \infty} \liminf_{T \rightarrow \infty} \frac{1}{T} \log V_T^{*,M} = \lim_{M \rightarrow \infty} \sup_{\pi \in \mathcal{L}^M} L^\pi = L = \lim_{T \rightarrow \infty} \frac{1}{T} \log \widehat{V}_T,$$

where the last equality follows from Theorem 3.7 and the second last follows from Lemma 3.8. By construction $V_T(\nu) \geq 2^{-M} V_T^M(\nu^M)$ for every M , so we have by Theorem 3.9 for every M

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log V_T(\nu) \geq \liminf_{T \rightarrow \infty} \frac{1}{T} (-M \log 2 + \log V_T^M(\nu^M)) = \liminf_{T \rightarrow \infty} \frac{1}{T} \log V_T^{*,M},$$

and hence also

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log V_T(\nu) \geq \lim_{M \rightarrow \infty} \liminf_{T \rightarrow \infty} \frac{1}{T} \log V_T^{*,M}.$$

Using the same argument as in the last part of the proof of Theorem 3.10, we get

$$(3.23) \quad \lim_{M \rightarrow \infty} \liminf_{T \rightarrow \infty} \frac{1}{T} \log V_T^{*,M} = \liminf_{T \rightarrow \infty} \frac{1}{T} \log V_T(\nu) = \lim_{T \rightarrow \infty} \frac{1}{T} \log \widehat{V}_T = L.$$

Now the corollary is proved if

$$(3.24) \quad \limsup_{T \rightarrow \infty} \frac{1}{T} (\log V_T(\nu) - \log \widehat{V}_T) = \limsup_{T \rightarrow \infty} \frac{1}{T} \log \left(\frac{V_T(\nu)}{\widehat{V}_T} \right) = 0,$$

holds \mathbb{P} -a.s. As by Lemma 3.12, $(\frac{V_t(\nu)}{\widehat{V}_t})_{t=0}^\infty$ is a non-negative supermartingale, it converges \mathbb{P} -a.s. to a finite limit as $t \rightarrow \infty$. This in turn implies (3.24) and proves the assertion. \square

Lemma 3.12. *The process $(\frac{V_t(\nu)}{\widehat{V}_t})_{t=0}^\infty$ is a non-negative supermartingale.*

Proof. First note that for any $\pi : \Delta^d \rightarrow \bar{\Delta}^d$, $(\frac{V_t^\pi}{\widehat{V}_t})_{t=0}^\infty$ is a non-negative supermartingale. Indeed, by Lemma 3.13 below we have

$$E \left[\frac{V_{t+1}^\pi}{\widehat{V}_{t+1}} \middle| \mathcal{F}_t \right] = \frac{V_t^\pi}{\widehat{V}_t} \int_{\Delta^d} \frac{\langle \pi(\mu_t), \frac{y}{\mu_t} \rangle}{\langle \widehat{\pi}(\mu_t), \frac{y}{\mu_t} \rangle} \varrho(\mu_t, dy) \leq \frac{V_t^\pi}{\widehat{V}_t}.$$

By Fubini's theorem we get the supermartingale property of $(\frac{V_t(\nu)}{\widehat{V}_t})_{t=0}^\infty$,

$$\begin{aligned} E \left[\frac{V_{t+1}(\nu)}{\widehat{V}_{t+1}} \middle| \mathcal{F}_t \right] &= E \left[\int_{\mathcal{L}} \frac{V_{t+1}^\pi}{\widehat{V}_{t+1}} d\nu(\pi) \middle| \mathcal{F}_t \right] \\ &= \int_{\mathcal{L}} E \left[\frac{V_{t+1}^\pi}{\widehat{V}_{t+1}} \middle| \mathcal{F}_t \right] d\nu(\pi) \\ &\leq \int_{\mathcal{L}} \frac{V_t^\pi}{\widehat{V}_t} d\nu(\pi) = \frac{V_t(\nu)}{\widehat{V}_t}. \end{aligned}$$

\square

Here we establish the supermartingale property used in the previous proof.

Lemma 3.13. *Let $\widehat{\pi}$ be given by (3.3). Then for every $\pi : \Delta^d \rightarrow \bar{\Delta}^d$ and every $x \in \Delta^d$,*

$$\int_{\Delta^d} \frac{\langle \pi(x), \frac{y}{x} \rangle}{\langle \widehat{\pi}(x), \frac{y}{x} \rangle} \varrho(x, dy) \leq 1.$$

Proof. We proceed as in the proof of [2, Proposition 4.3]. Fix π and $\alpha \in (0, 1)$ and define $\pi^\alpha = \alpha\pi + (1-\alpha)\widehat{\pi}$. Then by the (long only) log-optimality of $\widehat{\pi}$ we have for every $x \in \Delta^d$

$$\begin{aligned} 0 &\leq \int_{\Delta^d} \left(\log \langle \widehat{\pi}(x), \frac{y}{x} \rangle - \log \langle \pi^\alpha(x), \frac{y}{x} \rangle \right) \varrho(x, dy) = \int_{\Delta^d} \left(\int_{\langle \pi^\alpha(x), \frac{y}{x} \rangle}^{\langle \widehat{\pi}(x), \frac{y}{x} \rangle} \frac{1}{z} dz \right) \varrho(x, dy) \\ &\leq \int_{\Delta^d} \frac{\langle \widehat{\pi}(x), \frac{y}{x} \rangle - \langle \pi^\alpha(x), \frac{y}{x} \rangle}{\langle \pi^\alpha(x), \frac{y}{x} \rangle} \varrho(x, dy) = \int_{\Delta^d} \frac{\langle \alpha(\widehat{\pi}(x) - \pi(x)), \frac{y}{x} \rangle}{\langle \pi^\alpha(x), \frac{y}{x} \rangle} \varrho(x, dy). \end{aligned}$$

Hence,

$$\int_{\Delta^d} \frac{\langle \pi(x), \frac{y}{x} \rangle}{\langle \pi^\alpha(x), \frac{y}{x} \rangle} \varrho(x, dy) \leq \int_{\Delta^d} \frac{\langle \widehat{\pi}(x), \frac{y}{x} \rangle}{\langle \pi^\alpha(x), \frac{y}{x} \rangle} \varrho(x, dy) \leq \int_{\Delta^d} \frac{\langle \widehat{\pi}(x), \frac{y}{x} \rangle}{\langle (1-\alpha)\widehat{\pi}(x), \frac{y}{x} \rangle} \varrho(x, dy),$$

where the last equality follows from $\pi^\alpha \geq (1-\alpha)\widehat{\pi}$. By Fatou's lemma we therefore have

$$\begin{aligned} \int_{\Delta^d} \frac{\langle \pi(x), \frac{y}{x} \rangle}{\langle \widehat{\pi}(x), \frac{y}{x} \rangle} \varrho(x, dy) &= \int_{\Delta^d} \lim_{\alpha \rightarrow 0} \frac{\langle \pi(x), \frac{y}{x} \rangle}{\langle \pi^\alpha(x), \frac{y}{x} \rangle} \varrho(x, dy) \leq \lim_{\alpha \rightarrow 0} \int_{\Delta^d} \frac{\langle \pi(x), \frac{y}{x} \rangle}{\langle \pi^\alpha(x), \frac{y}{x} \rangle} \varrho(x, dy) \\ &\leq \lim_{\alpha \rightarrow 0} \frac{1}{1-\alpha} \int_{\Delta^d} \frac{\langle \widehat{\pi}(x), \frac{y}{x} \rangle}{\langle \widehat{\pi}(x), \frac{y}{x} \rangle} \varrho(x, dy) = 1. \end{aligned}$$

□

4. THE CONTINUOUS TIME CASE WITH FUNCTIONALLY GENERATED PORTFOLIOS

This section is dedicated to a similar analysis in continuous time and with *functionally generated* portfolio maps [17, Chapter 3]. Using the pathwise Itô calculus developed by H. Föllmer [20], we can define the corresponding wealth processes in a pathwise manner for any continuous market path admitting a quadratic variation process. This allows us to define the best retrospectively chosen portfolio which is not well-defined in general (and in particular for the Lipschitz portfolio maps).

4.1. Functionally generated portfolios. We consider the following set of concave functions. For some fixed $M > 0$ and $0 \leq \alpha \leq 1$, we define

$$\mathcal{G}^{M,\alpha} = \left\{ G \in C^{2,\alpha}(\bar{\Delta}^d), \text{ concave such that } \|G\|_{C^{2,\alpha}} \leq M \text{ and } G \geq \frac{1}{M} \right\},$$

where $C^{2,\alpha}(\bar{\Delta}^d)$ denotes the Hölder space of 2-times continuously differentiable functions from $\bar{\Delta}^d \rightarrow \mathbb{R}$ whose derivatives are α -Hölder continuous. That is,

$$C^{2,\alpha}(\bar{\Delta}^d) = \{G \in C^2(\bar{\Delta}^d) \mid \|G\|_{C^{2,\alpha}} < \infty\},$$

where

$$\|G\|_{C^{2,\alpha}} = \max_{|\mathbf{k}| \leq 2} \|D^{\mathbf{k}}G\|_\infty + \max_{|\mathbf{k}|=2} \sup_{x \neq y} \frac{|D^{\mathbf{k}}G(x) - D^{\mathbf{k}}G(y)|}{\|x - y\|^\alpha}$$

with \mathbf{k} denoting a multi-index in \mathbb{N}^2 . For $\alpha = 0$ the second term in this norm is left away. Note that G is only defined on the simplex Δ^d . In order that the partial derivatives are well defined, we assume that each G is extended to an open neighborhood of Δ^d such that $G(x) = G(x')$, where x' is the orthogonal projection of x onto Δ^d . The choice of the extension is irrelevant.

Here is an analytical lemma whose proof is given in the appendix.

Lemma 4.1. *For any $M, \alpha > 0$ the set $\mathcal{G}^{M,\alpha}$ is compact with respect to $\|\cdot\|_{C^{2,0}}$.*

To the set of generating functions $\mathcal{G}^{M,\alpha}$ we associate now the set of functionally generated portfolios $\mathcal{FG}^{M,\alpha}$ in the spirit of [17] defined by

(4.1)

$$\mathcal{FG}^{M,\alpha} = \left\{ \pi^G : \Delta^d \rightarrow \bar{\Delta}^d, \right. \\ \left. x \mapsto (\pi^G(x))^i = x^i \left(\frac{D^i G(x)}{G(x)} + 1 - \sum_{j=1}^d \frac{D^j G(x)}{G(x)} x^j \right), i = 1, \dots, d, | G \in \mathcal{G}^{M,\alpha} \right\}.$$

By the concavity of G , π^G takes values in $\bar{\Delta}^d$, i.e., it is long-only (see e.g. [19, Remark 11.1]). The corresponding wealth processes are denoted by V^{π^G} or V^G .

For these portfolios it is possible to obtain a pathwise expression for V^{π^G} . We refer the reader to [31] for extensions of this pathwise approach to time-dependent and path-dependent generating functions. There this is achieved by applying the functional Itô calculus developed by B. Dupire [13] and R. Cont and D. Fournié [7, 8], which generalizes Föllmer's Itô calculus to path-dependent functionals. In this paper we only consider functionally generated portfolio maps as defined in (4.1).

We adopt the notation of [31] and fix a refining sequence of partitions $(\mathbb{T}_n)_{n=1}^\infty$ of $[0, \infty)$, i.e., $\mathbb{T}_n = \{t_0, t_1, \dots\}$ is such that $0 = t_0^n < t_1^n < \dots$ and $t_k^n \rightarrow \infty$ as $k \rightarrow \infty$, and $\mathbb{T}_1 \subset \mathbb{T}_2 \subset \dots$. Moreover, the mesh of \mathbb{T}_n tends to zero on each compact interval as $n \rightarrow \infty$. Furthermore, we denote the successor of $t \in \mathbb{T}_n$ by t' . That is, $t' = \min\{u \in \mathbb{T}_n \mid u > t\}$. Throughout this section the market weights are described by a d -dimensional continuous path $\mu = (\mu_t)_{t \geq 0}$ with values in Δ^d . Here and henceforth we let S_d^+ be the set of $d \times d$ positive definite matrices.

Assumption 4.2. *The path $(\mu_t)_{t \geq 0}$ admits a continuous S_d^+ -valued quadratic variation $[\mu]$ along (\mathbb{T}_n) in the sense of [20], i.e., for any $1 \leq i, j \leq d$ and all $t \geq 0$ the sequence*

$$\sum_{s \in \mathbb{T}_n, s \leq t} (\mu_{s'}^i - \mu_s^i)(\mu_{s'}^j - \mu_s^j)$$

converges to a finite limit, as $n \rightarrow \infty$, denoted $[\mu^i, \mu^j]_t$, such that $t \mapsto [\mu^i, \mu^j]_t$ is continuous.

The dynamics of the relative wealth process V^{π^G} built by investing according to $\pi^G \in \mathcal{FG}^{M, \alpha}$ are given in this continuous time case by

$$(4.2) \quad \frac{dV_t^{\pi^G}}{V_t^{\pi^G}} = \sum_{i=1}^d (\pi^G(\mu_t))^i \frac{d\mu_t^i}{\mu_t^i} = \sum_{i=1}^d \frac{D^i G(\mu_t)}{G(\mu_t)} d\mu_t^i, \quad V_0^\pi = 1,$$

(compare (2.10) in the discrete time case), where the right hand side has to be understood as Föllmer's pathwise integral (c.f. Equation (2.5) in [31]). Note that the second equality holds by the definition of π^G and the fact that $\sum_{i=1}^d d\mu_t^i = 0$.

Using (4.2) and Föllmer's Itô calculus, we have the following pathwise version of Fernholz's [17] master equation (also see [31, Theorem 2.9]).

Corollary 4.3. *Let $G \in C^2(\bar{\Delta}^d)$ and π^G be defined as in (4.1). Let $(\mu_t)_{t \geq 0}$ be a continuous path satisfying Assumption 4.2. Then V^{π^G} satisfies*

$$(4.3) \quad V_T^{\pi^G} \equiv V_T^G = \frac{G(\mu_T)}{G(\mu_0)} e^{\mathfrak{g}([0, T])}, \quad 0 \leq T < \infty,$$

where $\mathfrak{g}(dt) = -\frac{1}{2G(\mu_t)} \sum_{i,j} D^{ij} G(\mu_t) d[\mu^i, \mu^j]_t$.

4.2. Definitions of the portfolios. We again consider (i) the best retrospectively chosen portfolio, (ii) the universal portfolio and (iii) the log-optimal portfolio. To define the log-optimal portfolio we will restrict to a specific stochastic model introduced in Section 4.2.3. In Section 4.2.4 we derive the asymptotic growth rate for this model class under an additional ergodicity assumption.

4.2.1. *The best retrospectively chosen portfolio.* We consider the set of functionally generated portfolios $\mathcal{FG}^{M,\alpha}$ and a given continuous path $(\mu_t)_{t \geq 0}$ satisfying Assumption 4.2. For $M, \alpha > 0$ fixed, we define

$$(4.4) \quad V_T^{*,M,\alpha} = \sup_{\pi^G \in \mathcal{FG}^{M,\alpha}} V_T^{\pi^G} = \sup_{G \in \mathcal{G}^{M,\alpha}} V_T^G.$$

We first prove that an optimizer exists by establishing the following continuity property whose proof can be found in the appendix.

Lemma 4.4. *Let $T, M, \alpha > 0$ be fixed and $(\mu_t)_{t \geq 0}$ be a continuous path satisfying Assumption 4.2. Consider the function $G \mapsto V_T^G$ where V_T^G is given by (4.3). Then $G \mapsto V_T^G$ is continuous from $(\mathcal{G}^{M,\alpha}, \|\cdot\|_{C^{2,0}})$ to \mathbb{R} .*

Proposition 4.5. *Let T be fixed and $(\mu_t)_{t \geq 0}$ be a continuous path satisfying Assumption 4.2. Let $V_T^{*,M,\alpha}$ be defined by (4.4). Then there exists an optimizer $G_T^* \in \mathcal{G}^{M,\alpha}$ and in turn a portfolio π_T^* generated by G_T^* such that*

$$V_T^{*,M,\alpha} = V_T^{\pi_T^*} = V_T^{G_T^*}.$$

Proof. This is simply a consequence of continuity as proved in Lemma 4.4 and compactness of $(\mathcal{G}^{M,\alpha}, \|\cdot\|_{C^{2,0}})$ as shown in Lemma 4.1. \square

4.2.2. *Universal portfolio.* To define the analogue of Cover's/Jamshidian's portfolio in the present setting, let m be a Borel probability measure on $(\mathcal{G}^{M,\alpha}, \|\cdot\|_{C^{2,0}})$. Consider the map

$$(4.5) \quad F : \mathcal{G}^{M,\alpha} \rightarrow \mathcal{FG}^{M,\alpha}; \quad G \mapsto F(G) = \pi^G,$$

where π^G is given by (4.1). Define now on $(\mathcal{FG}^{M,\alpha}, \|\cdot\|_\infty)$ a Borel probability measure ν via the pushforward $\nu = F_*m$. As in Definition 2.4, we then define the corresponding universal portfolio via

$$(4.6) \quad \pi_T^\nu = \frac{\int_{\mathcal{FG}^{M,\alpha}} \pi^G(\mu_T) V_T^{\pi^G} d\nu(\pi^G)}{\int_{\mathcal{FG}^{M,\alpha}} V_T^{\pi^G} d\nu(\pi^G)}.$$

Analogous to (2.13), the value of the universal portfolio is given by

$$(4.7) \quad V_T^{M,\alpha}(\nu) := V_T^{\pi^\nu} = \int_{\mathcal{FG}^{M,\alpha}} V_T^{\pi^G} d\nu(\pi^G) = \int_{\mathcal{G}^{M,\alpha}} V_T^G dm(G).$$

Remark 4.6. More precisely, we need to verify that the universal portfolio still allows for pathwise integration and that the value of the portfolio (as a pathwise integral) is given by the right hand side of (4.7). These claims can be easily checked using the definitions and results in [31], so we omit the details.

4.2.3. *Functionally generated log-optimal portfolios.* By definition, the log-optimal portfolios requires a stochastic model for the market weights. We suppose that $\mu = (\mu_t^1, \dots, \mu_t^d)_{t \geq 0}$ follows a *time-homogeneous Markovian Itô diffusion*, defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with values in Δ^d , given by

$$(4.8) \quad \mu_t = \mu_0 + \int_0^t c(\mu_s) \lambda(\mu_s) ds + \int_0^t \sqrt{c(\mu_s)} dW_s, \quad \mu_0 \in \Delta^d,$$

where $\sqrt{\cdot}$ denotes the matrix square root, W is a d -dimensional Brownian motion, λ is a Borel measurable function from $\Delta^d \rightarrow \mathbb{R}^d$ and c is a Borel measurable function from $\Delta^d \rightarrow S_+^d$, satisfying

$$(4.9) \quad \int_0^T \lambda^\top(\mu_t) c(\mu_t) \lambda(\mu_t) dt < \infty, \quad \forall T \in [0, \infty),$$

$$(4.10) \quad c(x) \mathbf{1} = 0, \quad \sum_{i,j} c^{ij}(x) \lambda(x)^j = 0, \quad \forall x \in \Delta^d.$$

The requirements in (4.10) are necessary to guarantee that the process μ lies in Δ^d . Note that $(\mu_t)_{t \geq 0}$ given by (4.8) satisfies the so called *structure condition* (see [32]) (because of (4.9) and the fact that the drift part is of form $\int_0^t c(\mu_s) \lambda(\mu_s) ds$). This structural condition characterizes the condition of “no unbounded profit with bounded risk” (NUPBR) in the case of continuous semimartingales (see e.g., [22]).

In this setting the proportions of current (relative) wealth invested in each of the assets are described by processes π in the following set

$$(4.11) \quad \Pi = \{\pi \mid H^d\text{-valued, predictable, } R\text{-integrable}\},$$

where the process R is defined componentwise by $R_t^i = \int_0^t \frac{d\mu_s^i}{\mu_s^i}$. Here, H^d denotes the hyperplane corresponding to portfolio weights that are *not* necessarily long-only, i.e., $H^d = \{x \in \mathbb{R}^d \mid \sum_{j=1}^d x^j = 1\}$. Note that the set $\mathcal{FG}^{M,\alpha}$ is clearly a subset of long-only strategies in Π . The relative wealth process V^π satisfies

$$(4.12) \quad \frac{dV_t^\pi}{V_t^\pi} = \sum_{i=1}^d \pi_t^i \frac{d\mu_t^i}{\mu_t^i}, \quad V_0^\pi = 1.$$

In contrast to Section 4.1, this is a usual stochastic integral because we are dealing with general integrands π . Note that we can also write

$$(4.13) \quad \begin{aligned} V_T^\pi &= \mathcal{E}((\pi \bullet R))_T = \exp \left(\int_0^T \left(\frac{\pi}{\mu_t} \right)^\top d\mu_t - \frac{1}{2} \int_0^T \left(\frac{\pi}{\mu_t} \right)^\top c(\mu_t) \frac{\pi}{\mu_t} dt \right) \\ &= \exp \left(\int_0^T \sum_{i=1}^d \pi_t^i \frac{d\mu_t^i}{\mu_t^i} - \frac{1}{2} \int_0^T \sum_{i,j} \frac{\pi_t^i \pi_t^j}{\mu_t^i \mu_t^j} c^{ij}(\mu_t) dt \right), \end{aligned}$$

where, for two vectors $p, q \in \mathbb{R}^d$, p/q always denotes the componentwise quotient $(\frac{p^1}{q^1}, \dots, \frac{p^d}{q^d})$.

Next we consider the log-optimal portfolio defined by (2.14) (but in continuous time now). As in [16, Section 3.1], we derive the ratio of two wealth processes V^π and V^θ for $\pi, \theta \in \Pi$. Using (4.12) (for the processes π and θ) and Itô's lemma, this ratio is given by

$$\begin{aligned} d \left(\frac{V_t^\pi}{V_t^\theta} \right) &= \frac{V_t^\pi}{V_t^\theta} \left(\frac{\pi_t}{\mu_t} - \frac{\theta_t}{\mu_t} \right)^\top \left(d\mu_t - c(\mu_t) \frac{\theta_t}{\mu_t} dt \right) \\ &= \frac{V_t^\pi}{V_t^\theta} \left(\frac{\pi_t}{\mu_t} - \frac{\theta_t}{\mu_t} \right)^\top \left(\sqrt{c(\mu_t)} dW_t + c(\mu_t) \left(\lambda(\mu_t) - \frac{\theta_t}{\mu_t} \right) dt \right). \end{aligned}$$

The finite variation part of the expression vanishes *for every* $\pi \in \Pi$ if we choose $\theta \in \Pi$ such that

$$(4.14) \quad c(\mu_t) \left(\frac{\theta_t}{\mu_t} - \lambda(\mu_t) \right) = 0, \quad \mathbb{P}\text{-a.s. for all } t \geq 0.$$

By passing from the scaled relative weights θ/μ to ordinary portfolio weights via [16, Equation (5)], the generic solution of (4.14), which we denote by $\hat{\pi}^8$, is given by

$$(4.15) \quad \hat{\pi}_t^i = \mu_t^i \left(\lambda^i(\mu_t) + 1 - \sum_{j=1}^d \mu_t^j \lambda^j(\mu_t) \right).$$

Let \hat{V} be the associated wealth process. From (4.14), the ratio V_t^π/\hat{V}_t is, for any $\pi \in \Pi$, a non-negative local martingale and therefore a supermartingale. Hence \hat{V}_t yields the relative wealth process corresponding to the log-optimal portfolio (see e.g., [26, 16]). Indeed, by the supermartingale property and Jensen's inequality

$$\mathbb{E} \left[\log(V_T^\pi) - \log(\hat{V}_T) \right] = \mathbb{E} \left[\log \left(\frac{V_T^\pi}{\hat{V}_T} \right) \right] \leq \log \left(\mathbb{E} \left[\frac{V_T^\pi}{\hat{V}_T} \right] \right) \leq 0.$$

Thus $\mathbb{E}[\log(V_T^\pi)] \leq \mathbb{E}[\log(\hat{V}_T)]$ for all $\pi \in \Pi$.

By (4.13), the expected value of the log-optimal portfolio is given by

$$\sup_{\pi \in \Pi} \mathbb{E}[\log V_T^\pi] = \frac{1}{2} \mathbb{E} \left[\int_0^T \lambda^\top(\mu_t) c(\mu_t) \lambda(\mu_t) dt \right].$$

So far we have optimized over *all* strategies in Π . In the sequel we shall mainly consider suprema taken over smaller sets, in particular over $\mathcal{FG}^{M,\alpha}$. Note that in this case the optimizer will still be a function of the market weights due to the Markov property of $(\mu_t)_{t \geq 0}$.

In this context let us also answer the question of when the log-optimal portfolio is functionally generated. This is needed to relate its asymptotic growth rate to the one of the best retrospectively chosen portfolio and the universal portfolio.

Proposition 4.7. *Let $(\mu_t)_{t \geq 0}$ be of the form (4.8). Then the log-optimal portfolio is generated by a differentiable function G , i.e.,*

$$\hat{\pi}_t^i = \mu_t^i \left(\frac{D^i G(\mu_t)}{G(\mu_t)} + 1 - \sum_{j=1}^d \mu_t^j \frac{D^j G(\mu_t)}{G(\mu_t)} \right), \quad i = 1, \dots, d,$$

if the drift characteristic λ satisfies

$$\lambda(x) = \nabla \log G(x) = \frac{\nabla G(x)}{G(x)}, \quad x \in \Delta^d.$$

Proof. The assertion follows from expression (4.15). □

⁸By a slight abuse of notation, we here write $\hat{\pi}$ although we do not restrict to long-only portfolios as in Section 2.3.

4.2.4. *Asymptotic growth rates for an ergodic market weights process.*

Assumption 4.8. *The process μ as given in (4.8) is an ergodic process with stationary measure ϱ on Δ^d .*

With this assumption we derive an expression of the asymptotic growth rate $\lim_{T \rightarrow \infty} \frac{1}{T} \log V_T^\pi$. For the precise notion of ergodicity in continuous time we refer to [14, Section 2.2., Theorem 2.4 and Section 2.2.3]. Assumption 4.8 is essentially satisfied under a mean reversion condition. Examples include polynomial models for the market weights staying in the interior of the simplex (see [5, Theorem 5.1]) with the subclass of volatility stabilized models [18].

In the following theorem we consider portfolio maps which are not necessarily long-only, but can take values in the hyperplane H^d .

Theorem 4.9. *Under Assumption 4.8 the following statements hold true:*

- (i) *Let $\pi : \Delta^d \rightarrow H^d$ be any (ϱ -measurable) portfolio map such that*

$$(4.16) \quad \int_{\Delta^d} \left| \left(\frac{\pi(x)}{x} \right)^\top c(x) \lambda(x) \right| \varrho(dx) < \infty,$$

$$Q^\pi := \int_{\Delta^d} \left(\frac{\pi(x)}{x} \right)^\top c(x) \left(\frac{\pi(x)}{x} \right) \varrho(dx) < \infty.$$

We then have, for ϱ -a.e. starting value μ_0 , that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log(V_T^\pi) = L^\pi := \int_{\Delta^d} \left(\frac{\pi(x)}{x} \right)^\top c(x) \lambda(x) \varrho(dx) - \frac{1}{2} \int_{\Delta^d} \left(\frac{\pi(x)}{x} \right)^\top c(x) \left(\frac{\pi(x)}{x} \right) \varrho(dx), \quad \mathbb{P}\text{-a.s.}$$

- (ii) *Assume that $L := \frac{1}{2} \int_{\Delta^d} \lambda^\top(x) c(x) \lambda(x) \varrho(dx) < \infty$. Then, for ϱ -a.e. starting value μ_0 , it holds that*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log \widehat{V}_T = L, \quad \mathbb{P}\text{-a.s.}$$

The proof of Theorem 4.9 relies on the following lemma which is stated and proved in [17, Lemma 1.3.2].

Lemma 4.10. *Let M be a continuous local martingale such that*

$$(4.17) \quad \lim_{T \rightarrow \infty} \frac{1}{T^2} \langle M, M \rangle_T \log \log T = 0, \quad \mathbb{P}\text{-a.s.}$$

Then $\lim_{T \rightarrow \infty} \frac{1}{T} M_T = 0$, \mathbb{P} -a.s.

Proof of Theorem 4.9. Let us start by proving statement (i). By (4.12), $\log V_T^\pi$ reads as

$$(4.18) \quad \log V_T^\pi = \int_0^T \left(\frac{\pi(\mu_t)}{\mu_t} \right)^\top c(\mu_t) \lambda(\mu_t) dt - \frac{1}{2} \int_0^T \left(\frac{\pi(\mu_t)}{\mu_t} \right)^\top c(\mu_t) \frac{\pi(\mu_t)}{\mu_t} dt + \int_0^T \left(\frac{\pi(\mu_t)}{\mu_t} \right)^\top \sqrt{c(\mu_t)} dW_t.$$

The local martingale part

$$M_T^\pi := \int_0^T \left(\frac{\pi(\mu_t)}{\mu_t} \right)^\top \sqrt{c(\mu_t)} dW_t$$

satisfies Condition (4.17) of Lemma 4.10 below. Indeed, by the ergodic theorem in continuous time (see e.g., [14, Theorem 2.4 and Section 2.2.3]) and (4.16) we have

$$\frac{1}{T} \langle M^\pi, M^\pi \rangle_T = \frac{1}{T} \int_0^T \left(\frac{\pi(\mu_t)}{\mu_t} \right)^\top c(\mu_t) \frac{\pi(\mu_t)}{\mu_t} dt \xrightarrow{T \rightarrow \infty} Q^\pi < \infty, \quad \mathbb{P}\text{-a.s.}$$

Multiplying the left hand side with $(\log \log T)/T$, therefore yields Condition (4.17) and

$$\frac{1}{T} M_T^\pi = \frac{1}{T} \int_0^T \left(\frac{\pi(\mu_t)}{\mu_t} \right)^\top \sqrt{c(\mu_t)} dW_t \rightarrow 0, \quad \mathbb{P}\text{-a.s.}$$

Hence, evoking again the ergodic theorem yields

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \log V_T^\pi &= \lim_{T \rightarrow \infty} \frac{1}{T} \left(\int_0^T \left(\frac{\pi(\mu_t)}{\mu_t} \right)^\top c(\mu_t) \lambda(\mu_t) dt - \frac{1}{2} \int_0^T \left(\frac{\pi(\mu_t)}{\mu_t} \right)^\top c(\mu_t) \frac{\pi(\mu_t)}{\mu_t} dt \right) \\ &= \int_{\Delta^d} \left(\frac{\pi(x)}{x} \right)^\top c(x) \lambda(x) \varrho(dx) - \frac{1}{2} \int_{\Delta^d} \left(\frac{\pi(x)}{x} \right)^\top c(x) \left(\frac{\pi(x)}{x} \right) \varrho(dx), \end{aligned}$$

\mathbb{P} -a.s. (and also in $L^1(\Omega, \mathcal{F}, P)$) and thus assertion (i).

Concerning statement (ii), note from (4.14) that the scaled relative weights corresponding to the log-optimal portfolio satisfy

$$c(x) \left(\frac{\widehat{\pi}(x)}{x} - \lambda(x) \right) = 0.$$

Thus, by (4.18) and (4.12), $\log \widehat{V}_T$ simplifies to

$$\log \widehat{V}_T = \frac{1}{2} \int_0^T \lambda^\top(\mu_t) c(\mu_t) \lambda(\mu_t) dt + \int_0^T \lambda^\top(\mu_t) \sqrt{c(\mu_t)} dW_t.$$

In this case we have

$$\frac{1}{T} \langle M^{\widehat{\pi}}, M^{\widehat{\pi}} \rangle_T = \frac{1}{T} \int_0^T \lambda^\top(\mu_t) c(\mu_t) \lambda(\mu_t) dt \xrightarrow{T \rightarrow \infty} 2L, \quad \mathbb{P}\text{-a.s.},$$

which yields by the same argument as above

$$\frac{1}{T} M_T^{\widehat{\pi}} = \frac{1}{T} \int_0^T \lambda^\top(\mu_t) \sqrt{c(\mu_t)} dW_t \rightarrow 0, \quad \mathbb{P}\text{-a.s.}$$

and in turn

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log \widehat{V}_T = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_0^T \lambda^\top(\mu_t) c(\mu_t) \lambda(\mu_t) dt = L, \quad \mathbb{P}\text{-a.s.}$$

□

4.3. Asymptotically equivalent growth rates. As in discrete time we will establish asymptotic equality of the growth rates of all three portfolio types introduced in Section 4.2. First we compare the best retrospectively chosen portfolio with the universal one. For an analogous result in the context of optimal arbitrage see Theorem 4.5 of Kardaras and Robertson [27].

Theorem 4.11. *Let $M, \alpha > 0$ be fixed and let $(\mu_t)_{t \geq 0}$ be a continuous path satisfying Assumption 4.2 such that for all $i \in \{1, \dots, d\}$*

$$(4.19) \quad \lim_{T \rightarrow \infty} \frac{1}{T} [\mu^i, \mu^i]_T < \infty.$$

Consider a probability measure m on $\mathcal{G}^{M,\alpha}$ with full support and set $\nu = F_*m$ with F defined in (4.5). Then

$$\lim_{T \rightarrow \infty} \frac{1}{T} (\log V_T^{*,M,\alpha} - \log V_T^{M,\alpha}(\nu)) = 0,$$

where $V^{*,M,\alpha}$ and $V^{M,\alpha}(\nu)$ are defined in (4.4) and (4.7) respectively.

Proof. The inequality “ \geq ” is obvious. For the converse inequality we proceed similarly as in the previous section (using only generating functions). As m has full support and $\mathcal{G}^{M,\alpha}$ is compact, we have that, for $\eta > 0$ there exists some $\delta > 0$, such that every η -neighborhood of a point $G \in \mathcal{G}^{M,\alpha}$ has m -measure bigger than δ .

Let $T \geq 1$ and denote by G_T^* the optimizer as of Proposition 4.5. Consider now a generating function G such $\|G - G_T^*\|_{C^{2,0}} \leq \eta$. Then it follows from (A.9) that

$$(4.20) \quad \begin{aligned} \frac{1}{T} \left(\log(V_T^G) - \log(V_T^{G_T^*}) \right) &\geq \frac{1}{T} \left(-2M\eta - \left(\frac{M}{2} d^2\eta + \frac{M^3}{2} d^2\eta \right) \max_i [\mu^i, \mu^i]_T \right) \\ &=: -K_T. \end{aligned}$$

Fix $\epsilon > 0$ and note that by assumption (4.19) and continuity of $T \mapsto \frac{1}{T} [u^i, u^i]_T$ on $[1, \infty)$, $\sup_{T \in [1, \infty)} \frac{1}{T} [\mu^i, \mu^i]_T$ can be bounded by some constant. Therefore we can choose η sufficiently small such that $K_T \leq \epsilon$ for all $T \geq 1$.

Denote by $B = B_\eta(G_T^*)$ the $\|\cdot\|_{C^{2,0}}$ -ball with radius η in $\mathcal{G}^{M,\alpha}$ which has m -measure at least $\delta > 0$, where δ only depends on η . We then may estimate using Jensen’s inequality and (4.20)

$$\begin{aligned} \left(\frac{V_T^{M,\alpha}(\nu)}{V_T^{G_T^*}} \right)^{\frac{1}{T}} &= \left(\frac{\int_{\mathcal{G}^{M,\alpha}} V_T^G m(dG)}{V_T^{G_T^*}} \right)^{\frac{1}{T}} \geq \left(\frac{\int_{B_\eta(G_T^*)} V_T^G m(dG)}{V_T^{G_T^*}} \right)^{\frac{1}{T}} \\ &\geq \delta^{\frac{1}{T}-1} \frac{\int_{B_\eta(G_T^*)} (V_T^G)^{\frac{1}{T}} m(dG)}{(V_T^{G_T^*})^{\frac{1}{T}}} \geq \delta^{\frac{1}{T}} e^{-K_T} \geq \delta^{\frac{1}{T}} e^{-\epsilon}. \end{aligned}$$

Letting $T \rightarrow \infty$ for any given ϵ (which determines η and in turn δ) yields the assertion. \square

To compare the asymptotic performance with that of the log-optimal portfolio, we optimize over portfolio maps in $\mathcal{F}\mathcal{G}^{M,\alpha}$ and suppose henceforth that $(\mu_t)_{t \geq 0}$ is of the form (4.8). Under Assumption 4.8 and from Theorem 4.9 define

$$(4.21) \quad \begin{aligned} \widehat{\pi}^{M,\alpha} &:= \arg \max_{\pi^G \in \mathcal{F}\mathcal{G}^{M,\alpha}} \left(\int_{\Delta^d} \left(\frac{\pi^G(x)}{x} \right)^\top c(x) \lambda(x) \varrho(dx) \right. \\ &\quad \left. - \frac{1}{2} \int_{\Delta^d} \left(\frac{\pi^G(x)}{x} \right)^\top c(x) \left(\frac{\pi^G(x)}{x} \right) \varrho(dx) \right) \end{aligned}$$

and the corresponding wealth process $\widehat{V}^{M,\alpha}$ by $\widehat{V}^{M,\alpha} = V^{\widehat{\pi}^{M,\alpha}}$, whenever $\widehat{\pi}^{M,\alpha}$ is well defined. As

$$\sup_{\pi^G \in \mathcal{F}\mathcal{G}^{M,\alpha}} \mathbb{E} \left[\log(V_T^{\pi^G}) \right]$$

yields $\widehat{\pi}^{M,\alpha}$ as optimizer for all $T > 0$, $\widehat{V}^{M,\alpha}$ corresponds to the log-optimal portfolio among functionally generated portfolios with generating function in $\mathcal{G}^{M,\alpha}$.

Theorem 4.12. *Let $M, \alpha > 0$ be fixed and let $(\mu_t)_{t \geq 0}$ be a stochastic process of the form (4.8) satisfying Assumption 4.8. Moreover, suppose that*

$$(4.22) \quad \int_{\Delta^d} c^{ii}(x) \varrho(dx) < \infty, \quad \text{for all } i \in \{1, \dots, d\},$$

$$(4.23) \quad \int_{\Delta^d} \max_{i \in \{1, \dots, d\}} |(c(x)\lambda(x))^i| \varrho(dx) < \infty.$$

Consider a probability measure m on $\mathcal{G}^{M, \alpha}$ with full support and set $\nu = F_* m$ with F defined in (4.5). Then

$$(4.24) \quad \liminf_{T \rightarrow \infty} \frac{1}{T} \log V_T^{*, M, \alpha} = \liminf_{T \rightarrow \infty} \frac{1}{T} \log V_T^{M, \alpha}(\nu) = \lim_{T \rightarrow \infty} \frac{1}{T} \log \widehat{V}_T^{M, \alpha}, \quad \mathbb{P}\text{-a.s.}$$

where $\widehat{V}_T^{M, \alpha}$ denotes the log-optimal portfolio among $\mathcal{F}\mathcal{G}^{M, \alpha}$ -maps defined via (4.21), $V^{*, M, \alpha}$ and $V^{M, \alpha}(\nu)$ are defined pathwise in (4.4) and (4.7) respectively.

Proof. We first note that $\widehat{\pi}^{M, \alpha}$ is well-defined. Indeed, the map

$$\begin{aligned} G &\mapsto \int_{\Delta^d} \left(\frac{\pi^G(x)}{x} \right)^\top c(x) \lambda(x) \varrho(dx) - \frac{1}{2} \int_{\Delta^d} \left(\frac{\pi^G(x)}{x} \right)^\top c(x) \left(\frac{\pi^G(x)}{x} \right) \varrho(dx) \\ &= \int_{\Delta^d} \left(\frac{\nabla G(x)}{G(x)} \right)^\top c(x) \lambda(x) \varrho(dx) - \frac{1}{2} \int_{\Delta^d} \left(\frac{\nabla G(x)}{G(x)} \right)^\top c(x) \left(\frac{\nabla G(x)}{G(x)} \right) \varrho(dx) \end{aligned}$$

is continuous from $(\mathcal{G}^{M, \alpha}, \|\cdot\|_{2,0})$ to \mathbb{R} . This together with compactness of $\mathcal{G}^{M, \alpha}$ with respect to $\|\cdot\|_{2,0}$ imply the well-definedness of $\widehat{\pi}^{M, \alpha}$.

Note also that (4.22) and (4.23) as well as the conditions on G imply the assumptions of the ergodic theorem (Theorem 4.9). Hence, we have for each $\pi^G \in \mathcal{F}\mathcal{G}^{M, \alpha}$ the \mathbb{P} -a.s. limit

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log V_T^{\pi^G} = L^{\pi^G}.$$

In particular,

$$(4.25) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \log \widehat{V}_T^{M, \alpha} = \sup_{\pi^G \in \mathcal{F}\mathcal{G}^{M, \alpha}} L^{\pi^G} =: L^{M, \alpha}$$

holds \mathbb{P} -a.s.

Due to (4.22), we can now apply Theorem 4.11 which implies the first equality in (4.24). Moreover, we have by the definition of $V_T^{*, M, \alpha}$ for each fixed T the inequality

$$(4.26) \quad \frac{1}{T} \log(\widehat{V}_T^{M, \alpha}) \leq \frac{1}{T} \log(V_T^{*, M, \alpha}), \quad \mathbb{P}\text{-a.s.}$$

Using (4.25), (4.26) and Theorem 4.11, we thus have \mathbb{P} -a.s.,

$$(4.27) \quad L^{M, \alpha} = \lim_{T \rightarrow \infty} \frac{1}{T} \log(\widehat{V}_T^{M, \alpha}) \leq \liminf_{T \rightarrow \infty} \frac{1}{T} \log(V_T^{*, M, \alpha}) = \liminf_{T \rightarrow \infty} \frac{1}{T} \log(V_T^{M, \alpha}(\nu)).$$

On the other hand, by the definition of $(\widehat{V}_t^{M, \alpha})_{t \geq 0}$ as log-optimizer within the class $\mathcal{F}\mathcal{G}^{M, \alpha}$

$$(4.28) \quad \mathbb{E}[\log(V_T^{M, \alpha}(\nu))] \leq \sup_{\pi^G \in \mathcal{F}\mathcal{G}^{M, \alpha}} \mathbb{E}[\log(V_T^{\pi^G})] = \mathbb{E}[\log(\widehat{V}_T^{M, \alpha})]$$

holds. Concerning the first inequality, note that the universal portfolio to build the wealth $V_T^{M, \alpha}(\nu)$ is given by (4.6). By the time-homogenous Markovianity it is thus

sufficient to dominate the left hand side of (4.28) by taking the supremum over elements in $\mathcal{FG}^{M,\alpha}$.

Combining now (4.28), Theorem 4.9 and (4.27) yields,

$$\begin{aligned} \mathbb{E}[\liminf_{T \rightarrow \infty} \frac{1}{T} \log(V_T^{M,\alpha}(\nu))] &\leq \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}[\log(V_T^{M,\alpha}(\nu))] \\ &\leq \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}[\log(\widehat{V}_T^{M,\alpha})] \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \log(\widehat{V}_T^{M,\alpha}) \\ &\leq \liminf_{T \rightarrow \infty} \frac{1}{T} \log(V_T^{M,\alpha}(\nu)), \quad \mathbb{P}\text{-a.s.}, \end{aligned}$$

where the first inequality follows from Fatou's lemma. From this we see that

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log(V_T^{M,\alpha}(\nu))$$

is \mathbb{P} -a.s. constant and equal to $\lim_{T \rightarrow \infty} \frac{1}{T} \log(\widehat{V}_T^{M,\alpha})$. Hence the assertion is proved. \square

As in the previous section we can formulate a result not depending explicitly on the constant M on α . Setting $\alpha = \frac{1}{M}$ we choose for $M = 1, 2, 3, \dots$ a measure m^M on $\mathcal{G}^{M, \frac{1}{M}}$ with full support. Define $m = \sum_{M=1}^{\infty} 2^{-M} m^M$ and the process $V(\nu)$ by

$$V_T(\nu) = \int_{\bigcup_{M=1}^{\infty} \mathcal{G}^{M, \frac{1}{M}}} V_T^G m(dG).$$

In order to compare the performance with the one of the global log-optimal portfolio, whenever it is functionally generated, we combine the above results with Proposition 4.7.

Corollary 4.13. *Let $(\mu_t)_{t \geq 0}$ be a stochastic process of form (4.8) satisfying Assumption 4.8. Moreover, suppose that λ and c satisfy (4.22) and*

$$(4.29) \quad \lambda(x) = \frac{\nabla \widehat{G}(x)}{\widehat{G}(x)},$$

$$(4.30) \quad L = \frac{1}{2} \int_{\Delta^d} \frac{\nabla \widehat{G}(x)}{\widehat{G}(x)} c(x) \frac{\nabla \widehat{G}(x)}{\widehat{G}(x)} \varrho(dx) < \infty$$

for some concave function $\widehat{G} \in C^2(\bar{\Delta}^d)$. Then we have \mathbb{P} -a.s.

$$(4.31) \quad \lim_{M \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{1}{T} \log(V_T^{*,M, \frac{1}{M}}) = \lim_{T \rightarrow \infty} \frac{1}{T} \log(V_T(\nu)) = \lim_{T \rightarrow \infty} \frac{1}{T} \log(\widehat{V}_T) = L.$$

Proof. Note first that L is well defined due to (4.30). Furthermore, note that for every $\varepsilon > 0$, there exists some $M > 0$ and some function $G \in \mathcal{G}^{M, \frac{1}{M}}$ such that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log(V_T^G) \geq \lim_{T \rightarrow \infty} \frac{1}{T} \log(\widehat{V}_T) + \varepsilon.$$

Indeed this simply follows from continuity of $G \mapsto V^G$ as asserted in Lemma 4.4 and by choosing $G \in \mathcal{G}^{M, \frac{1}{M}}$ close enough with respect to the $\|\cdot\|_{C^{2,0}}$ to the optimizing function $\widehat{G} \in C^2(\bar{\Delta}^d)$ whose generated portfolio yields \widehat{V} due to (4.29) and

Proposition 4.7. By Theorem 4.12, we can therefore conclude (following the proof of Corollary 3.11) that

$$(4.32) \quad \lim_{M \rightarrow \infty} \liminf_{T \rightarrow \infty} \frac{1}{T} \log(V_T^{*,M, \frac{1}{M}}) = \liminf_{T \rightarrow \infty} \frac{1}{T} \log(V_T(\nu)) = \lim_{T \rightarrow \infty} \frac{1}{T} \log(\widehat{V}_T) = L$$

holds true. As Theorem 4.11 implies that

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log V_T(\nu) = \lim_{M \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \log V_T^{*,M, \frac{1}{M}},$$

the assertion is proved if

$$(4.33) \quad \limsup_{T \rightarrow \infty} \frac{1}{T} (\log V_T(\nu) - \log \widehat{V}_T) = \limsup_{T \rightarrow \infty} \frac{1}{T} \log \left(\frac{V_T(\nu)}{\widehat{V}_T} \right) = 0, \quad \mathbb{P}\text{-a.s.}$$

By the considerations of Section 4.2.3 (see also [2, Propostion 4.3]), it follows that $(\frac{V_t(\nu)}{\widehat{V}_t})_{t \geq 0}$ is a non-negative supermartingale. It converges \mathbb{P} -a.s. to a finite limit as $t \rightarrow \infty$. This in turn implies (4.33) and proves the statement. \square

Finally, a similar result can be obtained by restricting the log-optimal portfolio to the class of C^2 -functionally generated portfolios without imposing the drift condition in Proposition 4.7. We denote by V_T^{fun} the wealth process of the log-optimal portfolio among concave C^2 -functionally generated portfolios, i.e., π^{fun} is defined as in (4.21), however by taking the arg max over all concave C^2 -functionally generated portfolios.

Corollary 4.14. *Let $(\mu_t)_{t \geq 0}$ be a stochastic process of form (4.8) satisfying Assumption 4.8. Moreover, suppose that (4.22) and (4.23) hold true. Then*

$$(4.34) \quad \lim_{M \rightarrow \infty} \liminf_{T \rightarrow \infty} \frac{1}{T} \log V_T^{*,M, \frac{1}{M}} = \liminf_{T \rightarrow \infty} \frac{1}{T} \log V_T(\nu) = \lim_{T \rightarrow \infty} \frac{1}{T} \log V_T^{fun}, \quad \mathbb{P}\text{-a.s.}$$

Proof. The proof is the same as the first part of Corollary 4.13 up to (4.32). Note that we cannot get rid of the lim inf because the supermartingale argument from the proof of Corollary 4.13 does not hold. \square

APPENDIX A. PROOFS OF CERTAIN RESULTS AND LEMMAS

Proof of Theorem 2.1. Fix $T > 0$ and the trajectory $s = (s_t^1, \dots, s_t^d)_{t=0}^T \in (\mathbb{R}^d)^{T+1}$. For fixed s the function $b \mapsto V_T(b)(s)$ is continuous on $\bar{\Delta}^d$. Hence there is $\bar{b} = \bar{b}(s) \in \bar{\Delta}^d$ such that

$$(A.1) \quad V_T^*(s) = V_T(\bar{b})(s).$$

In fact, condition (2.8) implies that the sequence of functions $(b \mapsto \frac{1}{T} \log V_T(b))_{T=1}^\infty$ is Lipschitz on $\bar{\Delta}^d$, uniformly in $T \in \mathbb{N}$ and s satisfying (2.8) for some fixed constants $C > c > 0$.

Indeed, consider the distance on $\bar{\Delta}^d$ defined by $\|b - \tilde{b}\|_1 = \sum_{j=1}^d |b^j - \tilde{b}^j|$. Then we may estimate

$$\left| \frac{1}{T} \log V_T(b) - \frac{1}{T} \log V_T(\tilde{b}) \right| \leq (\log(C) - \log(c)) \|b - \tilde{b}\|_1.$$

For $\epsilon > 0$ we may therefore define $\delta := \frac{\epsilon c}{C} > 0$ such that, for every δ -neighborhood $U(\bar{b})$ around any $\bar{b} \in \bar{\Delta}^d$ we have

$$\frac{1}{T} \log V_T(b) \geq \frac{1}{T} \log V_T(\bar{b}) - \epsilon,$$

for every $b \in U(\bar{b})$. If the probability measure ν has full support, we also may find $\eta = \eta(\epsilon, c, C) > 0$ such that each such δ -neighborhood $U(\bar{b})$, where \bar{b} runs through $\bar{\Delta}^d$, satisfies $\nu(U(\bar{b})) > \eta$. Using (A.1) we therefore may conclude, similarly as in (A.8), that (2.7) holds true, uniformly in $s = (s_t^1, \dots, s_t^d)_{t=0}^\infty$ satisfying (2.8) for some fixed constants $C > c > 0$. \square

Proof of Lemma 3.8. Let $\hat{\pi} : \Delta^d \rightarrow \bar{\Delta}^d$ be the optimizer of (3.3) and define, for $0 < \epsilon < 1$,

$$\pi_\epsilon = (1 - \epsilon)\hat{\pi} + \epsilon \left(\frac{1}{d}, \dots, \frac{1}{d} \right).$$

Note that π_ϵ takes values in $\bar{\Delta}_\epsilon^d$ (see Definition 3.1), which is crucial for the subsequent arguments and the reason why we do not directly work with $\hat{\pi}$. Also note that, for $p \in \bar{\Delta}_\epsilon^d$, we have

$$(A.2) \quad \left\langle p, \frac{y}{x} \right\rangle = \sum_{j=1}^d p^j \frac{y^j}{x^j} \geq \frac{\epsilon}{d},$$

for $x, y \in \Delta^d$, as at least one of the terms $\frac{y^j}{x^j}$ is greater than or equal to one.

The average performance L^{π_ϵ} defined via (3.7) for the portfolio map π_ϵ is still almost as good as the optimal average performance $L \equiv L^{\hat{\pi}}$:

$$(A.3) \quad \begin{aligned} L^{\pi_\epsilon} &= \int_{\Delta^d} \left[\int_{\Delta^d} \log(\langle \pi_\epsilon(x), \frac{y}{x} \rangle) \varrho(x, dy) \right] d\varrho(x) \\ &\geq \int_{\Delta^d} \left[\int_{\Delta^d} \log((1 - \epsilon)\langle \hat{\pi}(x), \frac{y}{x} \rangle) \varrho(x, dy) \right] d\varrho(x) \\ &\geq L + \log(1 - \epsilon). \end{aligned}$$

To approximate π_ϵ by a Lipschitz function π_{Lip} taking its values in $\bar{\Delta}_\epsilon^d$, we need some preparation. By Assumption 3.6 we can find $\delta > 0$ such that, for $A \subseteq \Delta^d$,

$$(A.4) \quad \int_A \left[\int_{\Delta^d} (\log(\frac{\epsilon}{d}) - \log(\langle \pi_\epsilon(x), \frac{y}{x} \rangle)) \varrho(x, dy) \right] d\varrho(x) > -\epsilon,$$

provided that $\varrho[A] < \delta$. In particular, we may find $\eta > 0$ such that

$$(A.5) \quad \int_{\Delta^d \setminus \bar{\Delta}_\eta^d} \left[\int_{\Delta^d} (\log(\frac{\epsilon}{d}) - \log(\langle \pi_\epsilon(x), \frac{y}{x} \rangle)) d\varrho(x, y) \right] d\varrho(x) > -\epsilon.$$

Now we find a Lipschitz function $\pi_{Lip} : \Delta^d \rightarrow \bar{\Delta}_\epsilon^d$ such that

$$(A.6) \quad \|\pi_{Lip}(x) - \pi_\epsilon(x)\|_1 = \sum_{j=1}^d |\pi_{Lip}(x)^j - \pi_\epsilon(x)^j| < \eta\epsilon^2,$$

for all $x \in \Delta^d \setminus A$, where the exceptional set A satisfies $\varrho[A] < \delta$. Indeed, the functions from $\mathbb{R}^d \rightarrow \bar{\Delta}_\epsilon^d$ which are continuously differentiable in a neighborhood of Δ^d are dense with respect to the $L^1(\mathbb{R}^d, \varrho; \mathbb{R}^d)$ -norm. Let M be a Lipschitz constant for π_{Lip} such that $M^{-1} \leq \epsilon$.

To estimate $L^{\pi_{Lip}} - L^{\pi_\epsilon}$ we argue separately on the sets $\Delta^d \setminus \bar{\Delta}_\eta^d$, $A \cap \bar{\Delta}_\eta^d$ and $\bar{\Delta}_\eta^d \setminus A$. To start with the latter set note that, for $x \in \bar{\Delta}_\eta^d$ and $y \in \Delta^d$ we have that the function

$$p \mapsto \langle p, \frac{y}{x} \rangle = \sum_{i=1}^n p^i \frac{y^i}{x^i}, \quad p \in \bar{\Delta}^d,$$

is Lipschitz on $\bar{\Delta}^d$ with Lipschitz constant bounded by $(\frac{\eta}{d})^{-1}$. From (A.6) we get

$$(A.7) \quad \int_{\bar{\Delta}_\eta^d \setminus A} \left[\int_{\Delta^d} (\log(\langle \pi_{Lip}(x), \frac{y}{x} \rangle) - \log(\langle \pi_\epsilon(x), \frac{y}{x} \rangle)) d\varrho(x, y) \right] d\varrho(x) \\ \geq -(\eta \cdot \epsilon^2) \left(\frac{\eta}{d}\right)^{-1} \left(\frac{\epsilon}{d}\right)^{-1} \geq -d^2 \epsilon.$$

The term $(\frac{\epsilon}{d})^{-1}$ above comes from the fact that $\langle \pi_{Lip}(x), \frac{y}{x} \rangle$ as well as $\langle \pi_\epsilon(x), \frac{y}{x} \rangle$ takes values in $[\frac{\epsilon}{d}, \infty[$ and the function $z \mapsto \log(z)$ is Lipschitz on this set with constant $(\frac{\epsilon}{d})^{-1}$.

As regards the set $A \cap \bar{\Delta}_\eta^d$ we obtain from (A.2) and (A.4) the estimate

$$(A.8) \quad \int_{A \cap \bar{\Delta}_\eta^d} \left[\int_{\Delta^d} (\log(\langle \pi_{Lip}(x), \frac{y}{x} \rangle) - \log(\langle \pi_\epsilon(x), \frac{y}{x} \rangle)) d\varrho(x, y) \right] d\varrho(x) \geq -\epsilon$$

and a similar estimate holds true for the set $\Delta^d \setminus \bar{\Delta}_\eta^d$ by (A.5). Hence, we obtain from (A.3), (A.7), and (A.8)

$$L^{\pi_{Lip}} \geq L + \log(1 - \epsilon) - d^2 \epsilon - 2\epsilon.$$

As $\epsilon > 0$ is arbitrary, we have proved Lemma 3.8. \square

Proof of Lemma 4.1. This follows from the fact that the embedding from $C^{2,\alpha}(\bar{\Delta}^d) \rightarrow C^{2,\alpha'}(\bar{\Delta}^d)$ is compact for $\alpha' < \alpha$ (see e.g., [12, Satz 2.42]). This means in particular that any bounded set in $C^{2,\alpha}(\bar{\Delta}^d)$ is totally bounded in $C^{2,0}(\bar{\Delta}^d)$, thus relatively compact. To prove compactness it thus suffices to prove that $\mathcal{G}^{M,\alpha}$ is closed. Take a sequence $G^n \in \mathcal{G}^{M,\alpha}$ converging to G with respect to the $\|\cdot\|_{C^{2,0}}$ norm. Then, we can estimate $\|G\|_{C^{2,\alpha}}$ by

$$\|G\|_{C^{2,\alpha}} = \|G\|_{C^{2,0}} + \max_{|\mathbf{k}|=2} \sup_{x \neq y} \frac{|D^{\mathbf{k}}G(x) - D^{\mathbf{k}}G(y)|}{\|x - y\|^\alpha} \\ \leq \|G - G^n\|_{C^{2,0}} + \|G^n\|_{C^{2,0}} \\ + \max_{|\mathbf{k}|=2} \sup_{x \neq y} \frac{|D^{\mathbf{k}}G(x) - D^{\mathbf{k}}G^n(x)| + |D^{\mathbf{k}}G^n(x) - D^{\mathbf{k}}G^n(y)| + |D^{\mathbf{k}}G^n(y) - D^{\mathbf{k}}G(y)|}{\|x - y\|^\alpha}$$

for any $n \in \mathbb{N}$. Letting $n \rightarrow \infty$ and using the fact that $\|G^n - G\|_{C^{2,0}} \rightarrow 0$ yields $\|G\|_{C^{2,\alpha}} \leq M$. Similarly, we obtain $G \geq \frac{1}{M}$. This together with the fact that G is concave as a limit of concave functions proves $G \in \mathcal{G}^{M,\alpha}$ and thus in turn compactness of $\mathcal{G}^{M,\alpha}$ with respect to $\|\cdot\|_{C^{2,0}}$. \square

Proof of Lemma 4.4. For $G, \tilde{G} \in \mathcal{G}^{M,\alpha}$, we have

$$\begin{aligned} \log(V_T^G) - \log(V_T^{\tilde{G}}) &= \log(G(\mu_T)) - \log(\tilde{G}(\mu_T)) - (\log(G(\mu_0)) - \log(\tilde{G}(\mu_0))) \\ &\quad - \int_0^T \left(\sum_{i,j} \frac{D^{ij}G(\mu_t)}{2G(\mu_t)} - \frac{D^{ij}\tilde{G}(\mu_t)}{2\tilde{G}(\mu_t)} \right) d[\mu^i, \mu^j]_t \\ &= \log(G(\mu_T)) - \log(\tilde{G}(\mu_T)) - (\log(G(\mu_0)) - \log(\tilde{G}(\mu_0))) \\ &\quad - \sum_{i,j} \int_0^T \left(\frac{D^{ij}G(\mu_t) - D^{ij}\tilde{G}(\mu_t)}{2G(\mu_t)} + \frac{D^{ij}\tilde{G}(\mu_t) (\tilde{G}(\mu_t) - G(\mu_t))}{2\tilde{G}(\mu_t)G(\mu_t)} \right) d[\mu^i, \mu^j]_t. \end{aligned}$$

Hence, using the fact that $\|\tilde{G}\|_{C^{2,0}} \leq M$ as well as $G \geq \frac{1}{M}$ and $\tilde{G} \geq \frac{1}{M}$ and that $z \mapsto \log(z)$ is Lipschitz continuous on $[\frac{1}{M}, \infty)$ with constant M , we obtain the estimate

(A.9)

$$\begin{aligned} |\log(V_T^G) - \log(V_T^{\tilde{G}})| &\leq 2M\|G - \tilde{G}\|_{C^{2,0}} \\ &\quad + \left(\frac{M}{2}d^2\|G - \tilde{G}\|_{C^{2,0}} + \frac{M^3}{2}d^2\|G - \tilde{G}\|_{C^{2,0}} \right) \max_i [\mu^i, \mu^i]_T. \end{aligned}$$

This proves the asserted continuity. \square

REFERENCES

- [1] A. D. Banner, R. Fernholz, and I. Karatzas. Atlas models of equity markets. *The Annals of Applied Probability*, 15(4):2296–2330, 2005.
- [2] D. Becherer. The numeraire portfolio for unbounded semimartingales. *Finance and Stochastics*, 5(3):327–341, 2001.
- [3] A. Blum and A. Kalai. Universal portfolios with and without transaction costs. *Machine Learning*, 35(3):193–205, 1999.
- [4] M. R. Brod. Generating the universal portfolio. 2014. Master’s thesis, Eidgenössische Technische Hochschule, Zürich.
- [5] C. Cuchiero. Polynomial processes in stochastic portfolio theory. *Forthcoming in Stochastic processes and their applications*, DOI: 10.1016/j.spa.2018.06.007, 2018.
- [6] J. E. Cross and A. R. Barron. Efficient Universal Portfolios for Past-Dependent Target Classes *Mathematical Finance*, 13(2):245–276, 2003.
- [7] R. Cont and D.-A. Fournie. A functional extension of the Itô formula. *Comptes Rendus Mathématique*, 348(1):57–61, 2010.
- [8] R. Cont and D.-A. Fournié. Functional Itô calculus and stochastic integral representation of martingales. *The Annals of Probability*, 41(1):109–133, 2013.
- [9] T. M. Cover. Universal portfolios. *Math. Finance*, 1(1):1–29, 1991.
- [10] T. M. Cover and E. Ordentlich. Universal portfolios with side information. *Information Theory, IEEE Transactions on*, 42(2):348–363, 1996.
- [11] F. Delbaen and W. Schachermayer. The no-arbitrage property under a change of numéraire. *Stochastics Stochastics Rep.*, 53(3-4):213–226, 1995.
- [12] M. Dobrowolski. *Angewandte Funktionalanalysis: Funktionalanalysis, Sobolev-Räume und elliptische Differentialgleichungen*. Springer-Verlag, 2010.
- [13] B. Dupire. Functional Itô calculus. *Bloomberg portfolio research paper*, 2009. 2009-04.
- [14] A. Eberle. Markov processes. *Lecture notes, University Bonn*, 2016.
- [15] D. Fernholz and I. Karatzas. On optimal arbitrage. *Ann. Appl. Probab.*, 20(4):1179–1204, 2010.
- [16] D. Fernholz and I. Karatzas. Probabilistic aspects of arbitrage. In *Contemporary quantitative finance*, pages 1–17. Springer, Berlin, 2010.
- [17] E. R. Fernholz. *Stochastic portfolio theory*, volume 48 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 2002. Stochastic Modelling and Applied Probability.

- [18] R. Fernholz and I. Karatzas. Relative arbitrage in volatility-stabilized markets. *Annals of Finance*, 1(2):149–177, 2005.
- [19] R. Fernholz and I. Karatzas. Stochastic portfolio theory: an overview. *Handbook of numerical analysis*, 15:89–167, 2009.
- [20] H. Föllmer. Calcul d'Itô sans probabilités. *Séminaire de probabilités de Strasbourg*, 15:143–150, 1981.
- [21] E. Hazan and S. Kale. An online portfolio selection algorithm with regret logarithmic in price variation. *Mathematical Finance*, 25(2):288–310, 2015.
- [22] H. Hulley and M. Schweizer. On minimal market models and minimal martingale measures. In *Contemporary Quantitative Finance*, pages 35–51. Springer, 2010.
- [23] T. Ichiba and M. Brod. Some aspects of universal portfolios. 2014. Presentation, Banff 2014.
- [24] T. Ichiba, V. Papathanakos, A. Banner, I. Karatzas and R. Fernholz. Hybrid atlas models. *Ann. Appl. Probab.*, 21(2): 609–644, 2011.
- [25] F. Jamshidian. Asymptotically optimal portfolios. *Mathematical Finance*, 2(2):131–150, 1992.
- [26] I. Karatzas and C. Kardaras. The numéraire portfolio in semimartingale financial models. *Finance Stoch.*, 11(4):447–493, 2007.
- [27] C. Kardaras and Robertson, S. Robust maximization of asymptotic growth. *Ann. Appl. Probab.*, 22(4), 1576–1610, 2012.
- [28] D. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, pages 904–950, 1999.
- [29] B. Li and S. C. Hoi. Online portfolio selection: A survey. *ACM Computing Surveys (CSUR)*, 46(3):35, 2014.
- [30] S. Pal and T.-K. L. Wong. The geometry of relative arbitrage. *Mathematics and Financial Economics*, 10(3):263–293, 2016.
- [31] A. Schied, L. Speiser, and I. Voloshchenko. Model-free portfolio theory and its functional master formula. *preprint arXiv:1606.03325*, 2016.
- [32] M. Schweizer. On the minimal martingale measure and the Föllmer-Schweizer decomposition. *Stochastic Anal. Appl.*, 13(5):573–599, 1995.
- [33] T.-K. L. Wong. Universal portfolios in stochastic portfolio theory. *preprint arXiv:1510.02808*, 2015.

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