A Strong Law of Large Numbers for Positive Random Variables *

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Abstract

In the spirit of the famous Komlós (1967) theorem, every sequence of nonnegative, measurable functions $\{f_n\}_{n\in\mathbb{N}}$ on a probability space, contains a subsequence which—along with all its subsequences—converges a.e. in Cesàro mean to some measurable $f_* : \Omega \to [0,\infty]$. This result of von Weizsäcker (2004) is proved here using a new methodology and elementary tools; these sharpen also a theorem of Delbaen & Schachermayer (1994), replacing general convex combinations by Cesàro means.

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1 Introduction

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider real-valued measurable functions $f_1, f_2, \cdots$. If these are independent and have the same distribution with $\mathbb{E}(|f_1|) < \infty$, the celebrated Kolmogorov strong law of large numbers ([18]; [19]; [11], p. 73) states that the "sample average" $(f_1 + \cdots + f_N)/N$ converges $\mathbb{P}-$a.e. to the "ensemble average" $\mathbb{E}(f_1) = \int_\Omega f_1 \,\mathrm{d}\mathbb{P}$, as $N \to \infty$. More generally, if $f_n(\omega) = f\big(T^{n-1}(\omega)\big)$, $n \geq 2$, $\omega \in \Omega$ are the images of an integrable function $f_1 : \Omega \to \mathbb{R}$ along the orbit of successive actions of a measure-preserving transformation $T : \Omega \to \Omega$, then the above sample average converges $\mathbb{P}-$a.e. to the conditional expectation $f_* = \mathbb{E}(f_1|\mathcal{I})$ of f_1 given the $\sigma-$algebra $\mathcal{I}$ of $T-$invariant sets, by the Birkhoff pointwise ergodic theorem ([11], p. 333).

A deep result of Komlós [20], already 55 years old but always very striking, says that such "stabilization via averaging" occurs within *any* sequence $f_1, f_2, \cdots$ of measurable, real-valued functions with $\sup_{n\in\mathbb{N}} \mathbb{E}(|f_n|) < \infty$. More precisely, there exist then an integrable function f_* and a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $(f_{n_1} + \cdots + f_{n_K})/K$ converges to f_*, $\mathbb{P}-$a.e. as $K \to \infty$; and the same is true for any further subsequence of this $\{f_{n_k}\}_{k\in\mathbb{N}}$.

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This result inspired further path-breaking work in probability theory ([12], [6], [7]) culminating with Aldous (1977), where exchangeability plays a crucial rôle. It, and its ramifications [9], [10] involving forward convex combinations, have been very useful in the field of convex optimization; more generally, when one seeks objects with specific properties, and tries to ascertain their existence using weak compactness arguments. Stochastic control, optimal stopping and hypothesis testing are examples of the former (e.g., [22], [16], [8], [17], [23]); the Doob-Meyer and Bichteler-Dellacherie theorems in stochastic analysis provide instances of the latter (e.g., [13], [2], [3]).

We develop here a very simple argument for the Komlós theorem, in the important special case of nonnegative $f_1, f_2, \cdots$ treated by von Weizsäcker (2004). The argument dispenses with boundedness in $\mathbb{L}^1$, at the cost of allowing the function f_* to take infinite values.

2 Background

We place ourselves on a given, fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and consider a sequence $f_1, f_2, \cdots$ of measurable, real-valued functions defined on it. We say that this sequence *converges hereditarily in* Cesàro *mean* to some measurable $f_* : \Omega \to \mathbb{R} \cup \{\pm\infty\}$, and write $f_n \xrightarrow[n\to\infty]{hC} f_*$, $\mathbb{P}-$a.e., if, for *every* subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ of the original sequence, we have

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} f_{n_k} = f_*\,, \qquad \mathbb{P}-\text{a.e.} \tag{2.1}$$

Clearly then, every other such sequence $g_1, g_2, \cdots$ which is *equivalent* to $f_1, f_2, \cdots$, in the sense of $\sum_{n\in\mathbb{N}} \mathbb{P}(f_n \neq g_n) < \infty$ (cf. [19]), also has this property.

In 1967, Komlós proved the following remarkable result. The argument in [20] is very clear, but also long and quite involved. Simpler proofs and extensions have appeared since (e.g., [25], [27]; [4]).

Theorem 2.1 (Komlós (1967)). *If the sequence $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{L}^1$, i.e., $\sup_{n\in\mathbb{N}} \mathbb{E}(|f_n|) < \infty$ holds, there exist an integrable $f_* : \Omega \to \mathbb{R}$ and a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$, which converges hereditarily in* Cesàro *mean to f_* :*

$$f_{n_k} \xrightarrow[k\to\infty]{hC} f_*\,, \qquad \mathbb{P}-\textit{a.e.} \tag{2.2}$$

This result was motivated by an earlier one, Theorem 2.2 right below. For the convenience of the reader, we provide in § 5.1 a simple proof (in the manner of [5], pp. 137-141) of that precursor result, which proceeds by extracting first a *martingale difference* subsequence. This crucial idea, which establishes a powerful link to martingale theory and simplifies the arguments, appears in this context for the first time in [20] (for related results, see [21]).

Theorem 2.2 (Révész (1965)). *If the sequence $\{f_n\}_{n\in\mathbb{N}}$ satisfies $\sup_{n\in\mathbb{N}} \mathbb{E}(f_n^2) < \infty$, there exist a function $g \in \mathbb{L}^2$ and a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$, such that $\sum_{k\in\mathbb{N}} a_k\big(f_{n_k} - g\big)$ converges $\mathbb{P}-$a.e., for any sequence $\{a_k\}_{k\in\mathbb{N}}$ of real numbers with $\sum_{k\in\mathbb{N}} a_k^2 < \infty$.*

It is clear that this property of the subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ is inherited by all *its* subsequences (just "stretch out" the a_k's accordingly, and fill out the gaps with zeroes).

In a related development, Delbaen & Schachermayer ([9], Lemma A1.1; [10]) showed with very simple arguments that, from every sequence $\{f_n\}_{n\in\mathbb{N}}$ of nonnegative, measurable functions, a sequence of forward convex combinations $g_n \in \text{conv}(f_n, f_{n+1}, \cdots)$, $n \in \mathbb{N}$ of its elements can be extracted, which converges $\mathbb{P}-$a.e. to a measurable $f_* : \Omega \to [0, \infty]$. This result was called "a somewhat vulgar version of KOMLÓS's theorem" in [10], and is implied by Theorem 3.1 below. Indeed, convergence for CESÀRO averages is much more precise than for unspecified convex combinations.

In several contexts, including optimization treated via convex duality, nonnegativity is often no restriction at all, but rather the natural setting (e.g., [22]; [23]; [14]; [15], Chapter 3 and Appendix). Then, in the presence of convexity, Lemma A1.1 in [9], or Theorem 3.1 here, are very useful analogues of Theorem 2.1: they lead to limit functions f_* in convex sets (such as the positive orthant in $\mathbb{L}^0$, or the unit ball in $\mathbb{L}^1$) which are not compact in the usual sense, but *are* "convexly compact" in the sense introduced by ŽITKOVIĆ [29].

3 Result

The purpose of this note is to prove with new and elementary tools the following version of Theorem 2.1, due to VON WEIZSÄCKER [28] and studied further in [26], § 5.2.3 of [14].

Theorem 3.1. *Given a sequence* $\{f_n\}_{n\in\mathbb{N}}$ *of* nonnegative*, measurable functions on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, there exist a measurable function* $f_* : \Omega \to [0, \infty]$ *and a subsequence* $\{f_{n_k}\}_{k\in\mathbb{N}}$ *of the original sequence, such that* (2.2) *holds.*

Our proof appears in Section 5; it is, we believe, not without methodological/pedagogical merit. We observe that the result imposes no restriction whatsoever on the functions $f_1, f_2, \cdots$, apart from measurability and nonnegativity. This comes at a price: the limiting function f_*, constructed here carefully in (4.3)–(4.6) below, can take the value $+\infty$ on a set of positive measure.

4 Preparation

We place ourselves in the setting of Theorem 3.1. The arguments that follow often necessitate passing to subsequences, and to diagonal subsequences, of a given $\{f_n\}_{n\in\mathbb{N}}$. To simplify typography, we denote frequently such subsequences by the same symbols, $\{f_n\}_{n\in\mathbb{N}}$.

For each integer $k \in \mathbb{N}$, we introduce now the truncated functions

$$f_n^{(k)} := f_n \cdot \mathbf{1}_{\{k-1 \le f_n < k\}}\,, \qquad n \in \mathbb{N} \tag{4.1}$$

and note the partition of unity $\sum_{k\in\mathbb{N}} f_n^{(k)} = f_n\,,\ \forall\, n \in \mathbb{N}\,.$

Lemma 4.1. *For the sequence of functions* $\{f_n\}_{n\in\mathbb{N}}$ *in Theorem 3.1, there exists a subsequence, denoted by the same symbols and such that, for every* $k \in \mathbb{N}$*, the functions of* (4.1) *converge to an appropriate measurable function* $f^{(k)} : \Omega \to [0, \infty)$*, in the sense*

$$f_n^{(k)} \xrightarrow[n\to\infty]{hC} f^{(k)}, \qquad \mathbb{P} - a.e. \tag{4.2}$$

For each fixed $k \in \mathbb{N}$, *this convergence holds also in* $\mathbb{L}^1$.

Proof (after [5], pp. 145–146): For arbitrary, fixed $k \in \mathbb{N}$, the sequence $\{f_n^{(k)}\}_{n\in\mathbb{N}}$ of (4.1) is bounded in $\mathbb{L}^\infty$, thus also in $\mathbb{L}^2$. Theorem 2.2 provides a function $f^{(k)} \in \mathbb{L}^2$ and a subsequence $\{f_{n_j}^{(k)}\}_{j\in\mathbb{N}}$ of

$\{f_n^{(k)}\}_{n\in\mathbb{N}}$, such that $\sum_{j\in\mathbb{N}}(f_{n_j}^{(k)} - f^{(k)})/j$ converges $\mathbb{P}-$a.e.; as mentioned right after Theorem 2.2, this is inherited by all subsequences of $\{f_{n_j}^{(k)}\}_{j\in\mathbb{N}}$, and the KRONECKER Lemma ([11], p. 81) gives

$$0 = \lim_{J\to\infty} \frac{1}{J} \sum_{j=1}^{J} \big(f_{n_j}^{(k)} - f^{(k)}\big) = \lim_{J\to\infty} \frac{1}{J} \sum_{j=1}^{J} f_{n_j}^{(k)} - f^{(k)}, \qquad \mathbb{P}-\text{a.e.}$$

We pass now to a diagonal subsequence, denoted $\{f_n\}_{n\in\mathbb{N}}$ again, and such that (4.2) holds for *every* $k \in \mathbb{N}$. The last claim follows by the dominated convergence theorem. $\square$

With these ingredients, we introduce the measurable function $f : \Omega \to [0, \infty]$ via

$$f := \sum_{k\in\mathbb{N}} f^{(k)}, \qquad \text{and consider the set} \quad A_\infty := \{f = \infty\}. \tag{4.3}$$

With the help of FATOU's Lemma, and the notation of (4.1)–(4.3), Lemma 4.1 gives then

$$\varliminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_n \geq f\,, \qquad \mathbb{P}-\text{a.e.} \tag{4.4}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_n = \infty = f\,, \qquad \mathbb{P}-\text{a.e.} \quad \text{on} \quad A_\infty \tag{4.5}$$

for a suitable subsequence (denoted by the same symbols) of the original sequence $\{f_n\}_{n\in\mathbb{N}}$, and for all further subsequences of this subsequence.

The inequality in (4.4) can easily be strict. Consider, for instance, $f_n \equiv n$, so that $f_n^{(k)} = 0$ holds in (4.1) for every fixed $k \in \mathbb{N}$ and all $n \in \mathbb{N}$ sufficiently large. We obtain $f^{(k)} = 0$ in (4.2), thus $f = 0$ in (4.3); and yet $\frac{1}{N}\sum_{n=1}^{N} f_n \to \infty$ as $N \to \infty$.

This preparation allows us to formulate a more technical and precise version of Theorem 3.1, Proposition 4.2 below, which implies it. The convention $\infty \cdot 0 = 0$ is employed here, and throughout.

Proposition 4.2. *Fix a sequence $\{f_n\}_{n\in\mathbb{N}}$ of nonnegative, measurable functions on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and recall the notation of* (4.1)–(4.3). *There exist then a subsequence, denoted again $\{f_n\}_{n\in\mathbb{N}}$, and a set $A \supseteq A_\infty$, such that*

$$f_n \xrightarrow[n\to\infty]{hC} f_* := \max\big(f,\, \infty \cdot \mathbf{1}_A\big)\,, \qquad \mathbb{P}-a.e. \tag{4.6}$$

We have $A = A_\infty$, thus also $f_* \equiv f$, when $\lim_{K\to\infty} \varlimsup_{n\to\infty} \mathbb{P}\big(f_n \geq K, f < \infty\big) = 0$.

This last condition is satisfied when $\big\{f_n\, \mathbf{1}_{\{f<\infty\}}\big\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{L}^0$, i.e., $\lim_{K\to\infty} \sup_{n\in\mathbb{N}} \mathbb{P}(f_n \geq K, f < \infty) = 0$ holds. A bit more stringently, if not only $\{f_n\}_{n\in\mathbb{N}}$ but also its solid, convex hull in $\mathbb{L}^0_+$, is bounded in $\mathbb{L}^0$, then $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{L}^1(\mathbb{Q})$ under some probability measure $\mathbb{Q} \sim \mathbb{P}$, and thus $\mathbb{P}(f < \infty) = 1$ (e.g., Proposition A.11 in [15]). Whereas, if $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{L}^1(\mathbb{P})$, i.e., $\kappa := \sup_{n\in\mathbb{N}} \mathbb{E}(f_n) < \infty$, then f in (4.3) is integrable, since $\mathbb{E}(f) \leq \kappa$ holds the from (4.4) and FATOU.

5 Proofs

We shall need a couple of auxiliary results. First, and always with the notation of (4.1)–(4.3), we note the following consequence of monotone and dominated convergence.

Lemma 5.1. *Suppose a set $D \subseteq \Omega \backslash A_\infty = \{f < \infty\}$ satisfies $\mathbb{E}\big(f\,\mathbf{1}_D\big) < \infty$. Then, for any given $\varepsilon \in (0,1)$, there exist an integer $K \in \mathbb{N}$ and a subsequence of the given sequence $\{f_n\}_{n\in\mathbb{N}}$ such that for it, and for any one of its subsequences (denoted again $\{f_n\}_{n\in\mathbb{N}}$), we have for all integers $L > K$:*

$$\lim_{n\to\infty} \mathbb{E}\big[\, f_n\, \mathbf{1}_{\{K \le f_n < L\}\cap D}\,\big] =: \lim_{n\to\infty} \mathbb{E}\Big[\, f_n^{[K,L)}\, \mathbf{1}_D\,\Big] < \varepsilon\,. \tag{5.1}$$

We are using throughout the notation

$$f_n^{[K,L)} := \sum_{k=K+1}^{L} f_n^{(k)} = f_n\, \mathbf{1}_{\{K \le f_n < L\}}\,, \qquad f_n^{[K,\infty)} := \sum_{k \ge K+1} f_n^{(k)} = f_n\, \mathbf{1}_{\{f_n \ge K\}}\,; \tag{5.2}$$

in an analogous manner $f^{\,[K,L)} := \sum_{k=K+1}^{L} f^{(k)}$, $f^{\,[K,\infty)} := \sum_{k\ge K+1} f^{(k)}$, and Lemma 4.1 gives

$$f_n^{[K,L)} \xrightarrow[n\to\infty]{hC} f^{\,[K,L)}\,, \qquad \text{both } \mathbb{P}\text{−a.e. and in } \mathbb{L}^1. \tag{5.3}$$

Secondly, we recall (4.5) and observe the following dichotomy.

Lemma 5.2. *In the setting of Proposition 4.2, consider any measurable set $B \supseteq \{f = \infty\}$ such that the property $f_n \xrightarrow[n\to\infty]{hC} \infty$ of (4.5) holds $\mathbb{P}$−a.e. on B. Then, either*

(i) there exist a set $C \supsetneq B$ with $\mathbb{P}(C) > \mathbb{P}(B)$ and a subsequence, still denoted $\{f_n\}_{n\in\mathbb{N}}$, with

$$f_n \xrightarrow[n\to\infty]{hC} \infty \qquad \textit{valid} \quad \mathbb{P} - \textit{a.e. on } C\,; \qquad \textit{or,} \tag{5.4}$$

(ii) the Cesàro *convergence $f_n \xrightarrow[n\to\infty]{hC} f < \infty$ holds $\mathbb{P} -$ a.e. on $\Omega \setminus B \subseteq \{f < \infty\}$.*

Under *Case (ii),* the set $B \supseteq A_\infty = \{f = \infty\}$ is maximal for the $\mathbb{P}$−a.e. property $f_n \xrightarrow{hC} \infty$: it cannot be "inflated" to a set $C \supsetneq B$, which satisfies (5.4) and has measure bigger than that of B. This leads eventually to Proposition 4.2, and thence to Theorem 3.1.

Before proving these two results, we dispense with the proof of Theorem 2.2; this is completely self-contained, and has nothing to do with either Lemma 5.1 or Lemma 5.2.

5.1 Proof of Theorem 2.2

Because $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{L}^2$, we can extract a subsequence that converges to some $g \in \mathbb{L}^2$ weakly in $\mathbb{L}^2$. Thus, it suffices to prove the result for a sequence $\{g_n\}_{n\in\mathbb{N}}$ bounded in $\mathbb{L}^2$, and with $g_n \to 0$ weakly in $\mathbb{L}^2$. We take such a sequence, then, and approximate each g_n by a *simple* function $h_n \in \mathbb{L}^2$ with $\|g_n - h_n\|_2 \le 2^{-n}$, $\forall\, n \in \mathbb{N}$. This gives, in particular,

$$\sum_{n\in\mathbb{N}} \big|g_n - h_n\big| < \infty\,, \quad \mathbb{P} - \text{a.e.}\,; \qquad h_n \to 0 \quad \text{weakly in } \mathbb{L}^2. \tag{5.5}$$

We construct now, by induction, a sequence $1 = n_1 < n_2 < \cdots$ of integers, such that

$$\big|\vartheta_k\big| < 2^{-k} \quad \text{holds } \ \mathbb{P} - \text{a.e., for} \quad \vartheta_k := \mathbb{E}\big(h_{n_k} \,\big|\, h_{n_1}, \cdots, h_{n_{k-1}}\big)\,, \quad k = 2, 3, \cdots, \tag{5.6}$$

as follows: The function $h_{n_1} = h_1$ is simple, thus so is $\mathbb{E}(h_n | h_1) = \sum_{j=1}^{J} \gamma_j^{(n)} \mathbf{1}_{A_j}$ with $A_1, \cdots, A_J$ a partition of the space, and $\mathbb{P}(A_j) > 0$, $\gamma_j^{(n)} := \big(1/\mathbb{P}(A_j)\big) \cdot \mathbb{E}\big(h_n\, \mathbf{1}_{A_j}\big)$. This last expectation tends to zero as $n \to \infty$ from (5.5), for every fixed j; so we can choose $n_2 > n_1 = 1$ with $\big|\gamma_j^{(n_2)}\big| < 2^{-2}$, for $j = 1, \cdots, J$; i.e., $\big|\vartheta_2\big| < 2^{-2}$, $\mathbb{P}$−a.e. Clearly, we can keep repeating this argument since, at each stage, $\big(h_{n_1}, \cdots, h_{n_{k-1}}\big)$ generates a finite partition of the space; and this way we arrive at (5.6).

The sequence $\{h_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{L}^2$, thus so is the martingale $X_k := \sum_{\ell=0}^{k} a_\ell \big(h_{n_\ell} - \vartheta_\ell\big)$, $k \in \mathbb{N}_0$, for any $\{a_k\}_{k\in\mathbb{N}_0} \subset \mathbb{R}$ with $\sum_{k\in\mathbb{N}} a_k^2 < \infty$. Martingale convergence theory ([11], p. 236) shows that the series $\sum_{k\in\mathbb{N}} a_k \big(h_{n_k} - \vartheta_k\big)$ converges $\mathbb{P}-$a.e. But we have also $\sum_{k\in\mathbb{N}} \big(|\vartheta_k| + |g_{n_k} - h_{n_k}|\big) < \infty$, $\mathbb{P}-$a.e. from (5.5)–(5.6), and deduce that $\sum_{k\in\mathbb{N}} a_k\, g_{n_k}$ converges $\mathbb{P}-$a.e., the claim of the theorem. □

5.2 Proof of Lemma 5.1

Let us call *"Lemma 5.1†"* the same statement as that of Lemma 5.1, except that (5.1) is now replaced by

$$\forall\ L = K+1, K+2, \cdots : \quad \mathbb{E}\Big[\, f_n^{[K,L)}\, \mathbf{1}_D \,\Big] < \varepsilon\,, \text{ for all but finitely many } n \in \mathbb{N}. \tag{5.7}$$

Claim: Lemma 5.1† implies Lemma 5.1. Let a subsequence of the original $\{f_n\}_{n\in\mathbb{N}}$ be given (denoted $\{f_n\}_{n\in\mathbb{N}}$ again), along with arbitrary $\varepsilon \in (0,1)$. Lemma 5.1† guarantees the existence of $K \in \mathbb{N}$, depending on ε and the subsequence, such that (5.7) holds for all integers $L \geq K+1$.

Choose $L = K+1$ first. From Lemma 5.1† and Bolzano-Weierstrass, (the current) $\{f_n\}_{n\in\mathbb{N}}$ has a subsequence for which the expectation in (5.7) converges, with limit $\leq \varepsilon/2$. Now choose $L = K+2$ and a subsequence of the last subsequence, for which the expectation in (5.7) converges and has limit $\leq \varepsilon/2$. Continuing in this manner, then diagonalizing, we obtain a subsequence that satisfies (5.7).

Proof of Lemma 5.1†. We argue by contradiction, assuming that $\{f_n\}_{n\in\mathbb{N}}$ has a subsequence for which Lemma 5.1† fails. Then there exists an $\varepsilon \in (0,1)$ with the property that, for every subsequence of $\{f_n\}_{n\in\mathbb{N}}$ and every $K \in \mathbb{N}$, there exists an integer $L > K$ such that

$$\mathbb{E}\Big[\sum_{k=K+1}^{L} f_n^{(k)}\, \mathbf{1}_D\Big] = \mathbb{E}\Big(f_n^{[K,L)}\, \mathbf{1}_D\Big) \geq \varepsilon \tag{5.8}$$

holds for infinitely many integers $n \in \mathbb{N}$. But this means that there is a subsequence, again denoted by $\{f_n\}_{n\in\mathbb{N}}$, *along which we have* (5.8) *for every* $n \in \mathbb{N}$; and, as a result, also

$$\mathbb{E}\Big[\sum_{k=K+1}^{L} \Big(\frac{1}{N}\sum_{n=1}^{N} f_n^{(k)}\Big)\, \mathbf{1}_D\Big] \geq \varepsilon\,, \qquad \forall\ N \in \mathbb{N}\,. \tag{5.9}$$

Now, all the truncated functions $f_n^{(k)}$ in (4.1), for $k = K+1, \ldots, L$ and $n \in \mathbb{N}$, take values on the "Procrustean bed" $\{0\} \cup [K, L)$; and $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f_n^{(k)} = f^{(k)}$ holds $\mathbb{P}-$a.e., for the selected subsequence and all its subsequences, on account of Lemma 4.1. Thus, $\mathbb{E}\big[\sum_{k=K+1}^{L} f^{(k)}\, \mathbf{1}_D\big] \geq \varepsilon$ from bounded convergence and (5.9); and the nonnegativity of these $f^{(k)}$'s implies also

$$\mathbb{E}\Big(\sum_{k\geq K+1} f^{(k)}\, \mathbf{1}_D\Big) = \mathbb{E}\Big(f^{\,[K,\infty)}\, \mathbf{1}_D\Big) \geq \varepsilon\,, \qquad \forall\ K \in \mathbb{N}\,. \tag{5.10}$$

The nonnegativity gives also $\lim_{K\to\infty} \uparrow \sum_{k=1}^{K} f^{(k)}\, \mathbf{1}_D = f\, \mathbf{1}_D$, both $\mathbb{P}-$a.e. and in $\mathbb{L}^1$. Since $\mathbb{E}\big(f\, \mathbf{1}_D\big) < \infty$ by assumption, $\mathbb{E}\big[f^{\,[K,\infty)}\, \mathbf{1}_D\big] < \varepsilon/2$ holds for all $K \in \mathbb{N}$ large enough. But this contradicts (5.10), and we are done. □

5.3 Proof of Lemma 5.2

We start by fixing $j \in \mathbb{N}$ and distinguishing two contingencies, with the definitions

$$D_j := \{f \leq j\} \backslash B\,, \qquad E_n^{[K,\infty)} := \big\{f_n^{[K,\infty)} \geq K\big\} \cap D_j = \big\{f_n \geq K\big\} \cap D_j\,, \tag{5.11}$$

$$\alpha := \lim_{K\to\infty}\, \overline{\lim_{n\to\infty}}\, \mathbb{P}\big(E_n^{[K,\infty)}\big) \ : \tag{5.12}$$

Contingency I: $\alpha > 0$.

Contingency II: $\alpha = 0$.

• Under **Contingency I**, we pass to a subsequence $\{f_n\}_{n\in\mathbb{N}}$ with $\mathbb{P}\big(E_n^{[n^2,\infty)}\big) \geq \alpha/2$, $\forall\ n \in \mathbb{N}$; and consider indicators $g_n := \mathbf{1}_{E_n^{[n^2,\infty)}}$, $n \in \mathbb{N}$, all of them supported on the set $\Omega \setminus B$. Arguing as in Lemma 4.1 we obtain a subsequence, still denoted $\{g_n\}_{n\in\mathbb{N}}$, with $g_n \xrightarrow[n\to\infty]{hC} g$, $\mathbb{P}-$a.e., for some $g : \Omega \to [0,1]$ with $\{g > 0\} \subseteq \Omega \backslash B$ and $\mathbb{E}(g) \geq \alpha/2$ by bounded convergence.

Thus, $f_n \xrightarrow[n\to\infty]{hC} \infty$ holds $\mathbb{P}-$a.e. on $\{g > 0\}$. This set has measure $\mathbb{P}\big(g > 0\big) = \mathbb{E}[\mathbf{1}_{\{g>0\}}] \geq \mathbb{E}(g) \geq \alpha/2$; we are under *Case (i)* of Lemma 5.2, with $C := \{g > 0\} \cup B$ and $\mathbb{P}(C) > \mathbb{P}(B)$.

• Now we pass to **Contingency II**. We fix $\varepsilon > 0$, $D_j = \{f \leq j\}\backslash B$, and apply Lemma 5.1 with this D_j to construct inductively a subsequence $\big\{n_m\big\}_{m\in\mathbb{N}}$, along with sequences $\big\{K_m\big\}_{m\in\mathbb{N}}$, $\big\{L_m\big\}_{m\in\mathbb{N}}$ of integers increasing to infinity and such that

$$\mathbb{P}\big(E_{n_m}^{[L_m,\infty)}\big) = \mathbb{P}\big(\big\{f_{n_m} \geq L_m\big\} \cap D_j\big) < 2^{-m} \tag{5.13}$$

$$\mathbb{E}\Big[\, f_{n_p}^{[K_m,L_p)}\, \mathbf{1}_{D_j} \Big] < 2^{-m}, \qquad \forall\ \ p = m, m+1, \cdots \tag{5.14}$$

hold for every $m \in \mathbb{N}$. With the choice (5.13), the sequences $\big\{f_{n_m} \cdot \mathbf{1}_{D_j}\big\}_{m\in\mathbb{N}}$ and $\big\{f_{n_m}^{[0,L_m)} \cdot \mathbf{1}_{D_j}\big\}_{m\in\mathbb{N}}$ are equivalent in the sense introduced in section 2, as the probability of their respective general terms being different is bounded from above by 2^{-m}. We claim that

$$f_{n_m} \cdot \mathbf{1}_{D_j} \xrightarrow[m\to\infty]{hC} f \cdot \mathbf{1}_{D_j}\,, \quad \mathbb{P}-\text{a.e.}; \tag{5.15}$$

and in view of the previous statement, this amounts to

$$f_{n_m}^{[0,L_m)} \cdot \mathbf{1}_{D_j} \xrightarrow[m\to\infty]{hC} f \cdot \mathbf{1}_{D_j}\,, \quad \mathbb{P}-\text{a.e.} \tag{5.16}$$

To prove (5.16), we start by observing that the sequence $\big\{f_{n_m}^{[0,L_m)}\cdot\mathbf{1}_{D_j}\big\}_{m\in\mathbb{N}}$ is *uniformly integrable*, thus bounded in $\mathbb{L}^1$, as $\sup_{\substack{p\in\mathbb{N}\\ p\geq m}} \mathbb{E}\Big[\, f_{n_p}^{[0,L_p)}\, \mathbf{1}_{D_j} \cdot \mathbf{1}_{\big\{f_{n_p}^{[0,L_p)} \geq K_m\big\}} \Big] < 2^{-m}$ holds on account of (5.14) for every $m \in \mathbb{N}$. Theorem 2.1 gives an integrable function $h : \Omega \to [0,\infty)$ with

$$f_{n_m}^{[0,L_m)} \cdot \mathbf{1}_{D_j} \xrightarrow[m\to\infty]{hC} h \cdot \mathbf{1}_{D_j}\,, \quad \mathbb{P}-\text{a.e.}, \tag{5.17}$$

and we need to argue that this h agrees with f from (4.3), $\mathbb{P}-$a.e. on D_j.

Indeed, for every $K \in \mathbb{N}$ and all m large enough, $\sum_{k=1}^{K} f_{n_m}\, \mathbf{1}_{\{k-1\leq f_{n_m}<k\}} = f_{n_m}^{[0,K)} \leq f_{n_m}^{[0,L_m)}$ holds, therefore $\sum_{k=1}^{K} f^{(k)} \cdot \mathbf{1}_{D_j} \leq h \cdot \mathbf{1}_{D_j}$ by letting $m \to \infty$, on account of (5.17) and Lemma 4.1. Passing now to the limit as $K \to \infty$ and recalling (4.3), we arrive at

$$f \cdot \mathbf{1}_{D_j} \ \leq\ h \cdot \mathbf{1}_{D_j}\,, \quad \mathbb{P}-\text{a.e.} \tag{5.18}$$

To obtain the inequality in the reverse direction, we take expectations. From the analogue of (5.17) $f_{n_m}^{[0,L_m\wedge K)} \cdot \mathbf{1}_{D_j} \xrightarrow[m\to\infty]{hC} h^{(K)} \cdot \mathbf{1}_{D_j}$, $\mathbb{P}-$a.e., with $0 \leq h^{(K)} \uparrow h$ as $K \to \infty$, uniform integrability, and Lemma 4.1, we have $\mathbb{E}\Big[\, f_{n_m}^{[0,L_m\wedge K)} \cdot \mathbf{1}_{D_j} \Big] \xrightarrow[m\to\infty]{hC} \mathbb{E}\big[\, h^{(K)} \cdot \mathbf{1}_{D_j} \big]$, therefore also

$$\mathbb{E}\big[\, h^{(K)} \cdot \mathbf{1}_{D_j} \big] = \lim_{M\to\infty} \frac{1}{M}\, \mathbb{E}\Bigg[\sum_{m=1}^{M} f_{n_m}^{[0,L_m\wedge K)} \cdot \mathbf{1}_{D_j} \Bigg] = \lim_{M\to\infty} \frac{1}{M}\, \mathbb{E}\Bigg[\sum_{m=1}^{M} \sum_{k=1}^{L_m\wedge K} f_{n_m}^{(k)} \cdot \mathbf{1}_{D_j} \Bigg]$$

$$\leq \lim_{M\to\infty} \frac{1}{M}\, \mathbb{E}\Bigg[\sum_{m=1}^{M} \Bigg(\sum_{k=1}^{K} f_{n_m}^{(k)} \Bigg) \cdot \mathbf{1}_{D_j} \Bigg] = \mathbb{E}\Bigg[\sum_{k=1}^{K} f^{(k)} \cdot \mathbf{1}_{D_j} \Bigg] \leq \mathbb{E}\big[\, f \cdot \mathbf{1}_{D_j} \big]$$

for every $K \in \mathbb{N}$. Letting $K \to \infty$, monotone convergence gives $\mathbb{E}\big[h \cdot \mathbf{1}_{D_j} \big] \leq \mathbb{E}\big[f \cdot \mathbf{1}_{D_j} \big]$; in conjunction with (5.18), this shows $f \cdot \mathbf{1}_{D_j} = h \cdot \mathbf{1}_{D_j}$, $\mathbb{P}-$a.e.; and on account of (5.17), it establishes (5.16), thus (5.15) as well. Finally, we let $j \to \infty$: we do this by extracting subsequences, successively for each $j \in \mathbb{N}$; then passing to a diagonal subsequence; obtaining (5.15) with D_j replaced by $D := \bigcup_{j \in \mathbb{N}} D_j = \{f < \infty\} \setminus B$; and deducing that we are in *Case (ii)* of Lemma 5.2. □

5.4 Proofs of Proposition 4.2 and Theorem 3.1

On the strength of Lemma 5.2 we construct, by exhaustion or transfinite induction arguments and as long as we are under the dispensation of its *Case (i),* an increasing sequence $B \subseteq B_1 \subseteq B_2 \subseteq \ldots$ of sets as postulated there, whose union $B_\infty := \bigcup_{j \in \mathbb{N}} B_j \supseteq B \supseteq \{f = \infty\}$ is maximal with the property (5.4) for an appropriate subsequence. But maximality means that, on the complement $\Omega \setminus B_\infty$ of this set, we must be in the realm of *Case (ii)* in Lemma 5.2. This establishes the first claim of Proposition 4.2 with $A = B_\infty \supseteq \{f = \infty\}$, thus also Theorem 3.1.

For the second claim of the Proposition, we note that equality holds right above, that is, $B_\infty = \{f = \infty\}$, if we are under Contingency II (i.e., $\alpha = 0$) in § 5.3 (proof of Lemma 5.2) and with $B = \{f = \infty\}$ in (5.11); a sufficient condition for this, is $\lim_{K \to \infty} \overline{\lim}_{n \to \infty} \mathbb{P}(f_n \geq K, f < \infty) = 0$. The claim now follows. □

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