UNIFORM MEASURES AND COSAKS SPACES

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INTRODUCTION

In this note we present a survey of results on uniform measures, based on the theory of Saks spaces and the dual concept - that of CoSaks spaces. The main result is that in the space of uniform measures, weak compactness is equivalent to compactness with respect to a natural strong topology. In the two extreme cases where the uniform space is uniformly discrete resp. where it is compact this theorem reduces to two familiar facts, namely Schur's Lemma on compactness in ℓ^1, respectively the fact that on the dual of a Banach space E, the $\sigma(E',E)$-compact sets coincide with those sets which are compact for $\tau_c(E',E)$, the topology of uniform convergence on the compact sets of E, and in fact our proofs use essentially the fact that the general case may be regarded as a combination of these two extreme cases. The theory of uniform measures has been developed mainly bei FEDOROVA [16] BEREZANSKII [3], DEAIBES [11], AZZAM [2], FROLIK [19] and PACHL [34] and the main result on compactness is due to PACHL whose proof is complicated and technical. In the topological case analogous results were obtained by BERRUYER and IVOL [4], BUCHWALTER [6], DUDLEY [14], LEGER and SOURY [29], ROME [38] and WHEELER [42]. Here the compactness result was obtained by ROME and WHEELER using partition of unity techniques. In the uniform case no such technique is available. Using our approach, a proof based on a gliding hump technique which reduces the result to Schur's Lemma presents itself very naturally. Here the notion of "uniform Lipschitz-tightness" is of central importance. It replaces the classical notion of uniform tightness for topological measure theory. The proof given is considerably simpler and more natural than that of PACHL.

We have taken the opportunity of presenting a sketch of a systematic development of the theory of uniform measures from the point of view of CoSaks spaces which we believe to be the natural and correct framework.
The general line of attack in this paper is to do the analysis in metric spaces and then lift to uniform spaces using formal manipulations with projective and inductive limits. For this reason we have been unable to resist the temptation of showing in the first two sections how the basic concepts of Lipschitz functions and uniform measures arise naturally from categorical considerations and how this serves to simplify and illuminate them. Readers with a distaste for category theory can skip these two sections after familiarising themselves with the notation introduced there.

The idea for writing this paper came from a very stimulating talk on the work of the Prague uniform space group given by Z. FROLIK in Linz, Summer 1978. A crucial role is played by the concept of a "compactology" which was introduced by WAELBROECK and systematically studied by BUCHWALTER and his coworkers.

§1. LIPSCHITZ SPACES

We begin with a brief exposition of results on Lipschitz functions. They are essentially due to ARENS and EELLS (who showed that every metric space can be isometrically embedded into a Banach space) resp. de LEEUW (who instigated the theory of Banach spaces of Lipschitz functions). In the following we shall combine these two treatments by showing that the construction of ARENS and EELLS can be made in a categorical fashion and that then many of the results of de LEEUW (and in particular, those that we shall require) follow from very general considerations.

We consider the category M_0 whose objects are metric spaces (X,d) with base point x_0 and radius ≤ 1 (i.e. $\sup\{d(x,x_0) : x \in X\} \leq 1$). The morphisms $f : (X,x_0) \to (Y,y_0)$ are contractions which respect the base point. If f is a Lipschitz mapping between metric spaces, Lip(f) denotes the Lipschitz constant of f (i.e. $\sup\{d_1(f(x),f(y))/(d(x,y) . x,y \in X, x \neq y)\}$

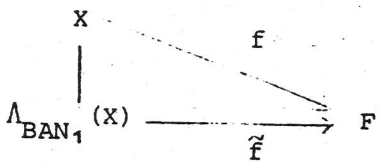
BAN_1 denotes the category of **real** Banach spaces with **linear** contractions as morphisms. There is a natural forgetful functor

$$\bigcirc : BAN_1 \to M_0$$

where if E is an object of BAN_1, $\bigcirc E$ is the unit ball of E with 0 as base point.

Now it follows from Freyd's adjoint functor theorem (see, e.g. MAC LANE [30]) that $\bigcirc$ has a left adjoint which we shall denote by Λ_{BAN_1}. This means that if (X,x_0) is an object of M_0, there is a Banach space $\Lambda_{BAN_1}(X)$ and a contraction from (X,x_0) into $\Lambda_{BAN_1}(X)$ with the following universal property:

> if $f : (X,x_0) \to E$ (E a Banach space) is a Lipschitz function with $Lip(f) \leq 1$ and $f(x_0) = 0$, then there is a unique linear contraction $\tilde{f} : \Lambda_{BAN_1}(X) \to E$ so that the following diagram commutes:



$\Lambda_{BAN_1}(X)$ is called the <u>free Banach space over</u> X. One can deduce from the above universal property that the mapping from X into $\Lambda_{BAN_1}(X)$ is isometric so that we can (and do) regard X as a subspace of $\Lambda_{BAN_1}(X)$. For if x,x' are points in X with $x \neq x'$, then the function

$$f : y \to d(y,x) - d(x,x_0)$$

is Lipschitz with $f(x_0) = 0$, $\mathrm{Lip}(f) \leq 1$. The extension $\tilde{f}$ of f is an element of $\Lambda_{BAN_1}(X)'$ with $\|\tilde{f}\| \leq 1$. Also

$$\tilde{f}(x'-x) = f(x') - f(x) = d(x',x) - d(x,x_0) - d(x,x) + d(x,x_0)$$
$$= d(x',x)$$

and so $\|x'-x\| \geq d(x',x)$.

Now if X is an object of M_0 we construct a Banach space $\mathrm{Lip}_0(X)$ as follows: the elements of $\mathrm{Lip}_0(X)$ are Lipschitz functions from X into $\mathbb{R}$ with the property that $f(x_0) = 0$. The norm is the Lipschitz norm

$$\|f\|_{L_0} : \sup\{ \frac{|f(x)-f(y)|}{d(x,y)} : x,y \in X,\ x \neq y \}$$

(i.e. $\|f\|_{L_0} = \mathrm{Lip}(f)$ - which is in this case a norm).

If we apply the above universal property to the Banach space $\mathbb{R}$ as target we see that $\mathrm{Lip}_0(X)$ is naturally isometric to $\Lambda_{BAN_1}(X)'$, the dual of $\Lambda_{BAN_1}(X)$, the isomorphism being the mapping $f \to \tilde{f}$ (the extension of f to $\Lambda_{BAN_1}(X)$). For the mapping is a linear isomorphism. On the other hand, it is norm decreasing since the extension does not increase the Lipschitz constant. Conversely its inverse - the restriction operator - is trivially norm-decreasing.

With this hindsight it is then easy to see how to construct $\Lambda_{BAN_1}(X)$ without having recourse to the Freyd theorem. We can embed X in $\mathrm{Lip}_0(X)'$ in the usual manner and the above consideration shows that this embedding is isometric. Then $\Lambda_{BAN_1}(X)$ has the closed linear span of X in $\mathrm{Lip}(X)'$ as a model.

More generally, we can define in the obvious way a Banach space $\mathrm{Lip}_0(X;F)$ for each Banach space F. As above it follows from the universal property that $L(\Lambda_{BAN_1}(X),F)$ and $\mathrm{Lip}_0(X;F)$ are naturally isometric for any Banach space. This implies the result that $\mathrm{Lip}_0(X;F)$ is a dual space if F is (cf. de LEEUW [28], JOHNSON [26]).

Now we reduce the case of metric space without base point to the above by the following trick. First we consider metric spaces with diameter ≤ 2 (of course every metric space is uniformly equivalent to such a space). Then we add a base point x_0 and extend the metric by defining $d(x_0,x) = 1$ $(x \in X)$. We denote this object of M_0 by $\tilde{X}$. Then if $f : X \to \mathbb{R}$ is Lipschitz with constant K the extension of f to $\tilde{X}$ obtained by putting $f(x_0) = 0$ is Lipschitz with constant K if and only if $|f| \leq K$ on X. Hence if $\mathrm{Lip}\ (X)$ denotes the Banach space of Lipschitz functions $f : X \to \mathbb{R}$ with the norm

$$\| \ \|_L : f \to \max\{\sup\{|f(x)| : x \in X\},\ \mathrm{Lip}\ (f)\}$$

then $\mathrm{Lip}(X)$ and $\mathrm{Lip}_0(\tilde{X})$ are isometric. Similarly, if F is a Banach space, $\mathrm{Lip}(X;F)$ (obvious definition) and $\mathrm{Lip}_0(\tilde{X};F)$ are isometric. Hence we obtain from the above that each metric space X with diameter ≤ 2 can be embedded isometrically and functorially in a Banach space $\Lambda_{BAN_1}(X)$ so that each Lipschitz function $f : X \to F$ (F a Banach space) with $\mathrm{Lip}(f) \leq 1$ and $\|f\|_\infty \leq 1$ i.e. $\|f\|_L \leq 1$ has a unique extension to a linear contraction $\tilde{f} : \Lambda_{BAN_1}(X) \to F$. Then we have natural isometries $L(\Lambda_{BAN_1}(X);F) = \mathrm{Lip}(X;F)$.

§2 FREE TOPOLOGICAL VECTOR SPACES:

Before constructing the space of uniform measures by duality in the next section, we show here how its existence is ensured by the Freyd adjoint functor since it is the solution of a universal problem, that of linearisation of bounded uniformly continuous functions (a fact which has been observed by several authors).

Again once one is assured of the existence of such a solution, it follows easily how one can construct it by duality. One sees thus how this construction fits into a general scheme of linearisation which has attracted some attention recently (see DOSTAL [13], JOHN [24], PTAK [35], RAIKOV [37], TOMAŠEK [41] and references there) and which include the construction of spaces of distributions, analytic functionals and various spaces of measures.

Consider the following categories:

a1) CREG - the completely regular spaces;

a2) UNIF - uniform spaces;

b1) LCS - locally convex spaces;

b2) WLCS - locally convex spaces with the weak topology;

b3) SS - Saks spaces (see COOPER [9] or §3 below);

b4) W -Waelbroeck spaces (see BUCHWALTER [5] or CIGLER [8])

Then there are natural forgetful functors from each of the categories in the b) list into the categories in the a) list (in the case of b3) and b4) we restrict to the unit ball before forgetting the linear structure). Each of these functors has (once more by Freyd's theorem) a left adjoint. We denote them by

$$\Lambda_{LCS}, \Lambda_{WLCS}, \Lambda_{SS}, \Lambda_{W} \text{ (for the functors on CREG)}$$

resp. $\Lambda^U_{LCS}, \Lambda^U_{WLCS}, \Lambda^U_{SS}, \Lambda^U_{W}$ (for the functors on UNIF).

A hat on the functor symbol (e.g. $\hat{\Lambda}_{LCS}$) will denote the composition of $\Lambda_?$ with the corresponding completion functor (in the b) list). For each of these Λ-functors, there are natural mappings $X \to \Lambda(X)$. It follows from the universal property that this mapping is an isomporphic embedding onto a closed subspace for the functors $\hat{\Lambda}_{LCS}, \hat{\Lambda}_{SS}, \hat{\Lambda}_{WLCS}, \hat{\Lambda}_{W}, \hat{\Lambda}^U_{LCS}, \hat{\Lambda}^U_{SS}$ as we shall now prove:

We begin with the topological case. The mapping is in each case injective since the bounded continuous functions separate the points of X. Hence we can regard X as a (settheoretical) subspace of $\Lambda_{LCS}(X)$ etc.

We now show that if A is closed in X then $A = \tilde{A} \cap X$ there $\tilde{A}$ is closed in $\Lambda_{LCS}(X)$. From this it follows that $X \to \Lambda_{LCS}(X)$ is a topological embedding. Since X is completely regular, there is a family $M \subseteq C^b(X)$ so that $A = \bigcap_{f \in M} f^{-1}(0)$. Then $\tilde{A} = \bigcap_{f \in M} \tilde{f}^{-1}(0)$ is the required set.

To show that X is closed in $\Lambda_{LCS}(X)$ we first note the following consequences of the universal property:

1) X is linearly independent in $\Lambda_{LCS}(X)$ (for if $\{x_1,\dots,x_n\}$ is a finite sequences of distinct elements of X, there is a continuous $f : X \to \mathbb{R}$ with $f(x_1) = 1$, $f(x_i) = 0$ $(i = 2,\dots,n)$. Then $\tilde{f}$ is an element of $\Lambda_{LCS}(X)'$ with $\tilde{f}(x_1) = 1$, $\tilde{f}(x_i) = 0$ $(i = 2,\dots,n)$ q.e.d.

2) $\Lambda_{LCS}(X)$ is the span of X. For by the uniqueness part of the universal property, the span L(X) is dense in $\Lambda_{LCS}(X)$. On the other hand, the extension $\tilde{I}$ of the natural injection $I : X \to L(X)$ (L(X) with the topology induced from $\Lambda_{LCS}(X)$) is a continuous linear mapping from $\Lambda_{LCS}(X)$ onto L(X) which is the identity when restricted to the dense subspace L(X). Hence it is the identity on $\Lambda_{LCS}(X)$ i.e. $L(X) = \Lambda_{LCS}(X)$.

We now show X is closed in $\Lambda_{LCS}(X)$. The injection $\bar{X} \to \Lambda_{LCS}(X)$ ($\bar{X}$ the closure of X in $\Lambda_{LCS}(X)$) satisfies the universal property for $\bar{X}$ and so we have $\Lambda_{LCS}(X) = \Lambda_{LCS}(\bar{X})$. But we have seen that X is a basis for $\Lambda_{LCS}(X)$ and of course, $\bar{X}$ is also a basis. This implies $X = \bar{X}$

Almost exactly the same proof shows that X is embedded as a closed subspace of $\Lambda_{WLCS}(X)$ and $\Lambda_{SS}(X)$. On the other hand, X is embedded as a topological subspace of $\Lambda_W(X)$, which will be closed only if X is compact.
We now consider the Λ^U functors. Here the proofs that $X \to \Lambda^U_{LCS}(X)$ is an injection and that X is a basis for $\Lambda^U_{LCS}(X)$ are exactly as above. We now show that the mapping is a uniform isomorphism. For this it suffices to show that if a uniformly bounded family $\{f_\alpha\}_{\alpha \in A}$ from X into $\mathbb{R}$ is uniformly equicontinuous for the original structure on X then it is uniformly equicontinuous for the structure induced from $\Lambda^U_{LCS}(X)$. But $\{f_\alpha\}$ induces in a natural way a mapping

$$f : X \to (f_\alpha(x))_{\alpha \in A}$$

from X into $\ell^\infty(A)$ and the mapping is uniformly continuous if and only if

$\{f_\alpha\}_{\alpha \in A}$ is uniformly equicontinuous. Hence if this is the case, f lifts to a continuous linear mapping $\tilde{f}$ from $\Lambda^U_{LCS}(X)$ into $\ell^\infty(A)$ and so the family $\{\tilde{f}_\alpha\}$ of its components is equicontinuous on $\Lambda^U_{LCS}(X)$. Hence the restrictions to X are uniformly equicontinuous q.e.d.

Exactly the same proof shows that X is uniformly isomorphic to a closed subspace of $\Lambda^U_{SS}(X)$. On the other hand the mappings $X \to \Lambda^U_{WLCS}(X)$, $X \to \Lambda^U_W(X)$ are not isomorphisms since $\Lambda^U_{WLCS}(X)$ induces the weak uniformity on X (i.e that defined by the uniformly continuous $\mathbb{R}$-valued functions on X) and $\Lambda^U_W(X)$ induces the precompact uniformity.

In particular, we can regard a completely regular space X as a subspace of $\hat{\Lambda}_{LCS}(X)$, $\hat{\Lambda}_{WLCS}(X)$, $\hat{\Lambda}_{SS}(X)$, $\Lambda_W(X)$.

Note that the closure of X in these spaces is

θX (the c-repletion - cf. BUCHWALTER [5] ;

υX (the realcompletion or real compactification - cf. e.g. BUCHWALTER [5] ;

cX (the topological completion c.f. ENGELKING [15] ;

βX (the Stone-Cech compactification c.f. ENGELKING [15];

respectively .

For readers who are unhappy at the use of Freyd's theorem we sketch briefly how these adjoints can be constructed directly. To be concrete, we construct Λ_{LCS}. If X is a completely regular space, we consider $\Lambda(X)$, the free vector space over X (i.e. the set of formal linear combinations of elements of X) and give it one of the following two (equivalent) structures:

a) the finest locally convex structure so that $X \to \Lambda(X)$ is continuous;

b) the projective structure induced by all mapping

$$\tilde{f} : \Lambda(X) \to E$$

where $\tilde{f}$ is the canonical extension of a continuous $f : X \to E$ (E a locally convex space).

The equivalence of these two structures is ensured by the fact that both by their very definitions, have the required universal property and this uniquely determines the topology of $\Lambda_{LCS}(X)$.

Now there exists a duality theory (for example, the duality between locally convex spaces and spaces with convex compactologies - see BUCHWALTER [5]) for each of the categories in the b) list and it follows once again from the universal properties that we have:

$$\hat{\Lambda}_{LCS}(X)' = C(X)$$
$$\hat{\Lambda}_{WLCS}(X)' = C(X)$$
$$\hat{\Lambda}_{SS}(X)' = C^b(X)$$
$$\Lambda_{W}(X)' = C^b(X)$$
$$\hat{\Lambda}^U_{LCS}(X)' = \mathcal{U}(X)$$
$$\hat{\Lambda}^U_{WLCS}(X)' = \mathcal{U}(X)$$
$$\hat{\Lambda}^U_{SS}(X)' = \mathcal{U}^b(X)$$
$$\Lambda^U_{W}(X)' = \mathcal{U}^b(X)$$

where $C(X)$ (resp. $\mathcal{U}(X)$) denotes the space of continuous (resp. uniformly continuous) real valued functions and a superscript b means "bounded". From this it easy to deduce the (more usual) definitions of the Λ-spaces as the duals of spaces of (uniformly) continuous functions with suitable structure. In particular it is now clear how we must define the space $\hat{\Lambda}^U_{SS}(X)$ of uniform measures on a uniform space X by duality and this is what we shall do in the next section.

§3 COSAKS SPACES

In this section we recall some definitions and constructions from COOPER [9]
A Saks space is a triple $(E,\| \|,\tau)$ where $(E,\| \|)$ is a normed space and τ is a locally convex topology on E so that $\bigcirc E$, the unit ball, is τ-closed and bounded. The Saks spaces form a category when we define morphisms from $(E,\| \|,\tau)$ into $(F,\| \|_1,\tau_1)$ to be linear norm contractions T so that $T|_{\bigcirc E}$ is τ-τ_1 continuous. The category of Saks spaces is complete and cocomplete i.e. possesses products, sums, subspaces and quotients. If $(E,\| \|,\tau)$ is a Saks space, we define a new locally convex topology, the mixed topology $\gamma(\| \|,\tau)$ (or γ for short) on E to be the finest locally convex topolgy on E which agrees with τ on $\bigcirc E$. Then $(E,\|.\|,\gamma)$ is also a Saks-space and we call it the <u>fine Saks space</u> associated with $(E,\|.\|,\tau)$.
A Saks-space $(E,\|.\|,\tau)$ is called <u>fine</u> if $\tau = \gamma(\|.\|,\tau)$. Note that every Saks space is isomorphic to the fine Saks-space associated to it, whence in every isomorphism-class of Saks spaces there is exactly one fine Saks-space. Also a linear map $T:(E,\|.\|,\tau) \to (F,\|.\|_1,\tau_1)$ is a Saks-morphism iff it is a norm-contraction and $\gamma(\|.\|,\tau)$ - $\gamma(\|.\|_1,\tau_1)$-continuous.
A Saks-space $(E,\|.\|,\tau)$ is complete if $\bigcirc E$ is τ-complete or equivalently if (E,γ) is a complete locally convex space.

We define the dual space E'_γ of a Saks-space $(E,\|.\|,\tau)$ to be $(E,\gamma)'$, which is a Banach-space with respect to the dual norm $\|.\|'$ of E. In fact we have an additional structure on E'_γ, namely the bornology $\mathcal{B}$ of τ-equicontinuous sets and these sets are relatively compact with respect to $\sigma(E'_\gamma,E)$. $\mathcal{B}$, the bornology of $\gamma(\|.\|,\tau)$-equicontinuous sets is what we shall call the $\|.\|$-<u>saturation</u> of $\mathcal{B}$, namely those balls C in E'_γ such that for every $\varepsilon > 0$ there is $B \in \mathcal{B}$ with $C \subseteq B + \varepsilon \bigcirc E'$.

Now if $(E,\|.\|,\gamma)$ is a complete fine Saks space then by Grothendieck's completeness theorem, c.f. SCHAEFER [40], we can recover $(E,\|.\|,\gamma)$ from $(E'_\gamma,\|.\|',\tilde{\mathcal{B}},\sigma(E',E))$ as the set of linear functionals on E'_γ such that the restriction to every member of $\mathcal{B}$ is $\sigma(E'_\gamma,E)$-continuous, equipped with the norm dual to $\|.\|'$ and the topology of uniform convergence on $\tilde{\mathcal{B}}$. We define the topology $\tilde{\sigma}$ on E' to be the finest locally convex topology that agrees with $\sigma(E'_\gamma,E)$ on the members of $\tilde{\mathcal{B}}$. Then $E = (E',\tilde{\sigma})'$.
This motivates the

<u>Definition:</u> A quadruple $(E,\|.\|,\tilde{\mathcal{B}},\tilde{\sigma})$ is called a <u>CoSaks space</u> if $(E,\|.\|)$ is a Banach-space, $\tilde{\mathcal{B}}$ is a bornology of $\|.\|$-bounded sets that is $\|.\|$-saturated (i.e. if $C \subseteq E$ is a ball such that $\forall \varepsilon > 0$ $C \subseteq B + \varepsilon \bigcirc E$ for some $B \in \tilde{\mathcal{B}}$, then $C \in \tilde{\mathcal{B}}$) and $\tilde{\sigma}$

is a locally convex Hausdorff-topology for which $\bigcirc E$ is closed and for which the members of $\breve{\mathcal{B}}$ are relatively compact in E, and which is the finest locally convex topology whose traces on the members of $\acute{\mathcal{B}}$ coincide with the traces of $\acute{\sigma}$,

$$T : (E,\|.\|,\tilde{\mathcal{B}},\tilde{\sigma}) \to (F,\|.\|_1,\tilde{\mathcal{B}}_1,\tilde{\sigma}_1)$$

is a CoSaks morphism if it is a linear norm-contraction, a bornological morphism with respect to $\acute{\mathcal{B}}$ and $\tilde{\mathcal{B}}_1$ and $\tilde{\sigma}$ - $\acute{\sigma}_1$ continuous:

It is clear from the above discussion that the definition is chosen so that the following proposition holds.

Proposition: The category of complete fine Saks spaces is dual to the category of CoSaks spaces.

Remark: The reader will perhaps have been irritated by the lack of symmetry in the definitions of Saks and CoSaks spaces. The reason for this lies in the fact that, in order to conform with the notation of [9], we have been forced to distinguish between a Saks space and its associated fine space althrough they are indistinguishable from the categorial point of view whereas in the definition of CoSaks spaces we have singled out one particular member of each isomorphism class.

We also note the following simple result:

Lemma: $\bigcirc E$ is $\acute{\sigma}$-bounded, and so $\tilde{\sigma}$ is coarser than the $\|.\|$-topology on E.

Proof: If there exists a $\tilde{\sigma}$-bounded sequence $\{x_n\}_{n=1}^{\infty}$ in $\bigcirc E$, then it is easy to see that for some increasing subsequence $\{n_k\}_{k=1}^{\infty}$ the sequence $\{k^{-1}x_{n_k}\}_{k=1}^{\infty}$ is not $\tilde{\sigma}$-bounded either. But as $\{k^{-1}x_{n_k}\}_{k=1}^{\infty}$ is $\|.\|$-precompact and is therefore contained in some closed ball B of saturated bornology $\tilde{\mathcal{B}}$ this contradicts the assumption that $\tilde{\sigma}$ induces a compact topology on every closed $B \in \tilde{\mathcal{B}}$.

In the following Proposition we examine further the relation between the bornologies of τ-equicontinuous sets and γ-equicontinuous sets.

Proposition: Let $(E,\|.\|,\tau)$ be a complete Saks-space, $\gamma = \gamma[\|.\|,\tau]$ the associated mixed topology. Let $(E,\|.\|,\tilde{\mathcal{B}},\tilde{\sigma})$ be the dual CoSaks-space and denote by $\mathcal{B}$ the bornology of τ-equicontinuous sets in E.

Then for a subset H of E the following are equivalent:

(i) H is $\tilde{\sigma}$-equicontinuous;

(ii) H is $\check{\sigma}$-equicontinuous on every member of $\mathcal{B}$;

(iii) H is $\|.\|$-bounded and $\check{\sigma}$-equicontinuous on every member of $\mathcal{B}$;

(iv) H is relatively γ-compact;

(iv)' H is relatively countably γ-compact;

(iv)'' H is γ-precompact;

(v) H is $\|.\|$-bounded and relatively τ-compact;

(v)' H is $\|.\|$-bounded and relatively countably τ-compact;

(v)'' H is $\|.\|$-bounded and τ-precompact

<u>Proof:</u> (i) <=> (ii) follows from the definition of $\tilde{\sigma}$ and (iv) <=> (iv)' <=> <=> (iv)'' <=> (v) <=> (v)' <=> (v)'' from the fact that (E,γ) is complete, γ-bounded sets are norm-bounded and that, on bounded subsets of E, γ coincides with τ.

(ii) => (iii) We only have to show that H is $\|.\|$-bounded and this follows from the uniform boundedness theorem since $\tilde{\mathcal{B}}$ covers E.

(iii) => (v)'' Fix a $\tilde{\sigma}$-compact $B \in \mathcal{B}$. If we consider H as a subset of C(B), the Banach space of continuous functions on B, then by Ascoli's theorem H is relatively compact in C(B) Since this holds for each $B \in \mathcal{B}$, H is τ-precompact.

(iv) => (ii) If we now fix a $\tilde{\sigma}$-compact $B \in \tilde{\mathcal{B}}$, then H is relatively compact in C(B) whence equicontinuous on B again by Ascoli's theorem.

Let us illustrate the situation:

<u>Examples:</u>

1) Let (Ω,Σ,μ) be a finite measure-space. $(L^\infty(\mu), \|.\|_\infty, \|.\|_1)$ is a complete Saks-space and γ is the Mackey-topology with respect to the duality of $L^\infty(\mu)$ and $L^1(\mu)$.

The dual fine CoSaks space $(E, \|.\|, \tilde{\mathcal{B}}, \tilde{\sigma})$ is the Banach-space $(L^1(\mu), \|.\|_1)$ equipped with the bornology of relatively $\sigma(L^1, L^\infty)$-compact balls and $\tilde{\sigma}$ is the topology of uniform convergence on the $\|.\|_\infty$-bounded and $\|.\|_1$-compact subsets of $L^\infty(\mu)$. Finally the bornology of τ-equicontinuous sets consists of the $\|.\|_\infty$-bounded balls in $L^1(\mu)$.

2) If $E = F'$ is a dual Banach-space, then $(E, \|.\|, \sigma(F',F))$ is a complete Saks-space and $\gamma(\|.\|, \sigma(F',F))$ is the topology of compact convergence. Whence the dual CoSaks space is the Banach-space $(F, \|.\|)$ with the bornology of $\|.\|$-relatively compact balls and σ is just the norm topology on F.

3) If S is a locally compact paracompact space, then $(C^b(S), \|.\|_\infty, \beta)$ is a complete fine Saks-space. The dual CoSaks space consists of the Banach-space $M^R(S)$ of Radon-measures on S equiped with the bornology of bounded, uniformly tight balls and $\check{\sigma}$ is the topology of uniform convergence on the $\|.\|_\infty$-bounded equicontinuous subsets of $C^b(S)$.

In this paper we will, in contrast to the last example, consider spaces of measures as Saks-spaces, defined as the dual of the space of bounded uniformly continuous functions with a suitable CoSaks structure.

More precisely let X be a uniform space. $(\mathcal{U}^b(X), \| \|_\infty)$ denotes the Banach-space (even algebra) of real-valued bounded, uniformly continuous functions on X. We consider the family $\mathcal{H}$ of all absolutely convex, uniformly bounded uniformly equicontinuous (abbreviated ueb) subsets of $\mathcal{U}^b(X)$. If $\mathcal{D}$ denotes the family of uniformly continuous pseudometrics on X then a bounded absolutely convex set H is in $\mathcal{H}$ if and only if H is pointwise dominated by some $d \in \mathcal{D}$ (in this the sense that for some $K > 0$,

$$|f(x) - f(y)| \leq Kd(x,y) \, (x,y \in X, \; f \in H)$$

Note that this means that H factorises over the appropriate metric space X_d and forms a bounded subset of $Lip(X_d)$ there. Then $(\mathcal{U}^b(X), \| \|_\infty, \mathcal{H}, \check{\sigma})$ is a CoSaks space where $\check{\sigma}$ is the finest locally convex topology that agrees on $\mathcal{H}$ with that of pointwise convergence on X.

For historical reasons, this topology is denoted by β_∞ (c.f. ROME [38], WHEELER [42]).

Now we define the space $M_u(X)$ of uniform measures on X to be the dual of this CoSaks space (so that it can also be regarded as the dual of the locally convex space $(\mathcal{U}^b(X), \beta_\infty)$).

By the above $M_u(X)$ has a natural fine Saks space structure $(M_u(X), \| \|, \gamma)$ where $\| \|$ is the dual norm to that of $\mathcal{U}^b(X)$ and γ is the topology of uniform convergence on the sets of $\mathcal{H}$.

By the Gelfand Naimark theorem we can embedd X (topologically) in a compact space $\check{X}$ (the Samuel compactification of X) so that $\mathcal{U}^b(X)$ and $C(\check{X})$ are naturally isometric by extension). Then every uniform measure can regarded as a Radon measure $\check{\mu}$ on $\check{X}$, and this identification is isometric i.e. the norm of μ in $M_u(X)$ is just the variation norm of $\check{\mu}$.

§4 APPROXIMATION OF UNIFORMLY CONTINUOUS FUNCTIONS BY LIPSCHITZ FUNCTIONS

In this section (X,d) is a metric space which we assume for convenience to have diameter ≤ 2. $(\mathcal{U}^b(X), \| \|_\infty)$ is the Banach space of bounded, uniformly continuous functions on X. It contains the space $\mathrm{Lip}(X)$ of course. If $\alpha \in \mathbb{R}_+$ we put

$$L_\alpha(X) := \{f \in \mathcal{U}^b(X) : \|f\|_\infty \leq 1 \text{ and } \mathrm{Lip}\, f \leq \alpha\}$$

in particular $L_1(X) = \mathrm{O\,Lip}\,(X)$.

<u>Lemma:</u> Let Y be a subset of X, g a function from Y into $[-1,1]$ with $\mathrm{Lip}(g) \leq \alpha$. Then there is an extension of g to a function $\tilde{g} : X \to [-1,1]$ with $\mathrm{Lip}\, \tilde{g} \leq \alpha$. Hence the restriction operator maps $\mathrm{O\,Lip}(X)$ onto $\mathrm{O\,Lip}(Y)$.

<u>Proof:</u> For $y \in Y$ define the function f_y on X

$$f_y(x) := g(y) - \alpha d(x,y)$$

and let $f(x) = -1 \vee \sup\{f_y(x) : y \in Y\}$.

<u>Proposition:</u> Let H be a ueb subset of $\mathcal{U}^b(X)$. Then for each $\varepsilon > 0$ there is an $n \in \mathbb{N}$ so that

$$H \subseteq nL_1(X) + \varepsilon \mathrm{O}\mathcal{U}^b(X).$$

i.e. in the language of §3, H is in the saturation of $\{n\mathrm{O\,Lip}(X)\}$ in $\mathcal{U}^b(X)$.

<u>Proof:</u> We may and do suppose $H \subseteq \mathrm{O}\mathcal{U}^b(X)$. Suppose that $0 < \varepsilon \leq 1$ and choose $\delta > 0$ so that $d(x,y) < \delta$ implies that $|f(x) - f(y)| < \varepsilon$ for each $f \in H$. Let $\{x_\alpha\}_{\alpha \in A}$ be a maximal family of points in X so that $d(x_\alpha, x_\beta) \geq \delta.\varepsilon$ for $\alpha \neq \beta$. If $f \in H$, let g be the restriction of f to $Y = \{x_\alpha\}$. Then $\mathrm{Lip}(g) \leq 2/\delta$ since $\dfrac{|g(x_\alpha) - g(x_\beta)|}{d(x_\alpha, x_\beta)} \leq 2.\delta^{-1}$.

Indeed if $d(x_\alpha, x_\beta) \geq \delta$ then the inequality is implied by $|g(x_\alpha) - g(x_\beta)| \leq 2$, while if $d(x_\alpha, x_\beta) < \delta$ we have $|g(x_\alpha) - g(x_\beta)| < \varepsilon$ and $d(x_\alpha, x_\beta) \geq \varepsilon.\delta$.

By the preceding Lemma, there is an extension $\tilde{g}$ of g to X with Lipschitz constant $\leq 2.\delta^{-1}$. Then $\|\tilde{g} - f\|_\infty < 3\varepsilon$. Indeed, for $x \in X$ there is, by the maximality of $\{x_\alpha\}$, an x_{α_0} so that $d(x_{\alpha_0}, x) < \varepsilon.\delta$. Then

$$\begin{aligned} |\tilde{g}(x) - f(x)| &\leq |\tilde{g}(x) - \tilde{g}(x_{\alpha_0})| + |\tilde{g}(x_{\alpha_0}) - f(x_{\alpha_0})| \\ &\quad + |f(x_{\alpha_0}) - f(x)| \\ &< \varepsilon.\delta.2\delta^{-1} + 0 + \varepsilon \\ &= 3\varepsilon \end{aligned}$$

Corollary: The sets $\{nL_1(X)\}_{n \in \mathbb{N}}$ generate the ueb compactology of $\mathcal{U}^b(X)$ in the sense of the saturation introduced in §3.

Corollary: Let γ be the topology on $M(X)$ of uniform convergence on the ueb sets. Then the restriction of γ to $OM_u(X)$ coincides with the topology of uniform convergence on $L_1(X)$ and is therefore induced by a norm on $M_u(X)$.

§5 UNIFORM MEASURES ON METRIC SPACES

In this section we show that if X is a complete metric space then the uniform measures on X are just the Radon (or tight) measures. We begin with the remark that if X is a uniform space and we regard $M_u(X)$ as a subspace of $M(\check{X})$ (cf. §3) then $M_u(X)$ inherits the order structure of $M(\check{X})$. In fact, $M_u(X)$ is a band in $M(\check{X})$ as the next result shows. Note that since the topology of $M_u(X)$ is not nicely related to the order structure (it is not locally solid) this result is not as evident as one might expect and in fact the corresponding result for "free uniform measures" (cf. PACHL [33]) is false. The result is well-known but we are unable to indicate the original reference. Our proof is essentially that of DEAIBES [11] (but we correct an error in his proof):

<u>Proposition:</u> Let $\mu \in M(\check{X})$. Then

$$\mu \in M_u(X) \text{ iff } \mu^+, \mu^- \in M_u(X).$$

<u>Proof:</u> We begin with the remark that if $\nu \in M(\check{X})$ then to show that $\nu \in M_u(X)$ it suffices to show that for every net $(f_\alpha)_{\alpha \in A}$ such that $\{f_\alpha\}$ is a ueb set and (f_α) tends to zero pointwise, $\lim_\alpha \nu(f_\alpha) = 0$. Furthermore, by considering (f_α^+) and (f_α^-) one sees that it suffices to suppose that $f_\alpha \geq 0$. Hence consider such a net (f_α) with $0 \leq f_\alpha \leq 1/2$ so that $f_\alpha \to 0$ pointwise. We show that if $\mu \in M_u(X)$ then $\mu^+(f\;) \to 0$.

Since $\mathcal{U}^b(X) = C(\check{X})$ we have the formula

$$\mu^+(1) = \sup\{\mu(g) : g \in \mathcal{U}^b(X),\ 0 \leq g \leq 1\}.$$

For $\varepsilon > 0$ choose $g \in \mathcal{U}^b(X)$ with $0 \leq g \leq 1$ and

$$\mu^+(1) \geq \mu(g) - \varepsilon.$$

Note that this implies that

$$\mu^+(1-g) \leq \varepsilon \text{ and } \mu^+(f) - |\mu(f)| \leq \varepsilon$$

for $0 \leq f \leq g$.

Now consider the family $(f_\alpha \wedge g)$. By assumption $\mu(f_\alpha \wedge g) \to 0$. We have the estimate $f_\alpha \leq (f_\alpha \wedge g) + (1-g)$ (consider the cases $g \leq 1/2$, $g \geq 1/2$). Hence

$$\mu^+(f_\alpha) \leq \mu^+(f_\alpha \wedge g) + \mu^+(1-g)$$
$$\leq |\mu(f_\alpha \wedge g)| + \varepsilon + \varepsilon,$$

which is less than 3ε for large α. Thus $\mu^+(f_\alpha) \to 0$ q.e.d.

We now show that if X is a complete metric space then the uniform measures on X coincide with the Radon measures. This result is well known but again we have been unable to trace the original source. Again our proof is essentially that of DEAIBES but we reproduce it since we encounter here in a natural way the concept of "Lipschitz-tightness" which will be essential in the sequel.

First some notation: if $K \subseteq X$ and $\alpha \in \mathbb{R}_+$, $L_{\alpha,K}$ will be the set of $f \in \mathcal{U}^b(X)$, with $\|f\|_\infty \leq 1$, $\mathrm{Lip}(f) \leq \alpha$ and $f \equiv 0$ on K. Also if $K \subseteq X$, $\eta > 0$ then $B(K,\eta) = \{y \in X : d(y,K) \leq \eta\}$.

Definition:

a) A measure $\mu \in M(\check{X})$ is called Lipschitz-tight (or L-tight for short) if

$$\lim_{K \in \mathcal{K}(X)} \sup\{|\mu(f)| : f \in L_{1,K}\} = 0$$

where $\mathcal{K}(X)$ denotes the family of compact subsets of X directed by inclusion.

b) a subset $H \subseteq M(\check{X})$ is called uniformly Lipschitz-tight if the above limit holds uniformly in $\mu \in H$.

Proposition: For $\mu \in M(\check{X})$ the following are equivalent:

1) $\mu \in M_u(X)$;

1)' $\mu^+, \mu^- \in M_u(X)$;

2) μ is L-tight;

2)' μ^+, μ^- are L-tight;

3) μ is tight i.e. for each $\varepsilon > 0$ there is a $K \in \mathcal{K}(X)$ so that if $f \in O\mathcal{U}^b(X)$ and $f = 0$ on K then $|\mu(f)| < \varepsilon$.

3)' μ^+ and μ^- are tight.

Proof:

1)' => 2)': The family $\{L_{1,K} : K \in \mathcal{K}(X)\}$ is ueb and tends to zero pointwise when K tends to X. Hence, by the definition of a uniform measure, if $\mu^+ \in M_u(X)$ then μ^+ is L-tight.

2)' ⟹ 3)': Note that if $\mu^+ \in M(\check{X})$ is L-tight then for any $\alpha \in \mathbb{R}_+$,

$$\lim_{K \in \mathcal{K}(X)} \{\sup|\mu^+(f)| \ : \ f \in L'_{\alpha,K}\} = 0.$$

(since $L_{\alpha,K}$ is contained in a multiple of $L_{1,K}$).
Then, if $\varepsilon > 0$, the L-tightness of $|\mu^+|$ implies that for each $n \in \mathbb{N}$, there is a $K_n \in \mathcal{K}(X)$ so that

$$|\mu^+|(X \setminus \check{B}(K_n, 1/n)) \leq \varepsilon/2^n$$

where $\check{B}(K,\eta) := \overline{B(K,\eta)}^{\check{X}}$ $\qquad (K \subseteq X, \eta > 0)$.

Indeed there is a function in $L_{n,K}$ which is 1 outside of $\check{B}(K_n,1/n)$ (when extended to $\check{X}$). For we can take the function

$$f : x \to n.d(K_n, X) \wedge 1$$

Then if $K := \bigcap_{n=1}^{\infty} B(K_n, 1/n)$, K is compact in X (see the Lemma below which is well known but which we prove for completeness) and satisfies required condition.

3)' ⟹ 1)': evident since the pointwise convergence of a ueb net implies compact convergence.

The equivalence of 1) and 1)' was proved in the previous Proposition, that of 3) and 3)' is evident as is the implication 2)' ⟹ 2). 2) implies 1) is a Corollary of the result of §4.

<u>Lemma:</u> Let (K_n) be a sequence in $\mathcal{K}(X)$. Then, with the notation of the above proof, $\bigcap_{n=1}^{\infty} \check{B}(K_n, 1/n) \subseteq X$.

<u>Proof:</u> (cf. ENGELKING [15], Th. 3.8.2.): Let $\check{x} \in \bigcap_{n=1}^{\infty} \check{B}(K_n, 1/n)$. $\check{B}(K_n, 1/n)$ can be covered by finitely many balls of radius $2/n$. Hence for each $n \in \mathbb{N}$, there is an $x_n \in X$ with $\check{x} \in \check{B}(x_n, 2/n)$. The function

$$f_n : x \to d(x_n, x) \wedge 1$$

can be continued to $\check{X}$ and $U_n := \check{f}_n^{-1}([0,3/n[)$ is an open neighbourhood of $\check{x}$ in $\check{X}$ whose trace in X has diameter at most $6/n$. Let $\mathcal{V}$ be the filter of closed neighbourhoods of $\check{x}$ in $\check{X}$. As $\mathcal{V}$ is finally contained in every U_n, the family $\{V \cap X\}_{V \in \mathcal{V}}$

forms a filter of closed subsets of X whose diameters tend to zero. By the completeness of (X,d), $\bigcap_{V \in \mathcal{V}} (V \cap X) \neq \phi$ and this latter set is of course exactly the point $\check{x}$. Hence $\check{x} \in X$.

Note that condition 3) above implies that μ defines a functional on $\mathcal{U}^b(X)$ which is continuous on the unit ball with respect to the topology of compact convergence. As $\mathcal{U}^b(X)$ is a dense subspace of $(C^b(X),\beta)$, the space of bounded continuous functions with the strict topology, by the Stone-Weierstraß theorem (cf. COOPER [9], p.84) μ extends to a unique continuous functional on $(C^b(X),\beta)$ i.e. to a Radon measure on the associated topological space. Hence the terminology "tight" is in agreement with the usage of topological measure theory.

§6 THE COMPACTNESS THEOREM FOR METRIC SPACES

We are now ready to prove the main result on compactness in $M_u(X)$ for the case where X is a complete metric space. The introduction of the concept of uniform Lipschitz tightness allows us to give a much shorter and more intuitive proof than the original one of PACHL. Of course, we use SCHUR's Lemma (i.e. the special case of a discrete space) in our proof.

Theorem (metric case): Let H be a subset of $M_u(X)$ (X a complete metric space). Then the following are equivalent:

1) H is relatively $\sigma(M_u, \mathcal{U}^b)$-compact;
1)' H is relatively countably $\sigma(M_u, \mathcal{U}^b)$-compact;
2) H is relatively γ-compact;
3) H is bounded and uniformly Lipschitz-tight.

Remark: Note that 2) is equivalent to all the conditions which are listed in the proposition of chapter 3.
Thus the following conditions (for example) are all equivalent to those of the above list:

- H is β_∞-equicontinuous
- H is equicontinuous on every ueb-set (with respect to the pointwise topology)
- H is $\|.\|$-bounded and equicontinuous on $L_1(X)$.

In particular, β_∞ is the Mackey topology for the duality $(\mathcal{U}^b(X), M_u(X))$.
We also remark that, contrary to the result of the previous paragraph, uniform L-tightness may not be replaced by uniform tightness (obvious definition): for if we take $X = \mathbb{R}$ with its usual metric and

$$\mu_n = \delta_n + \delta_{n+\frac{1}{n}}$$

then $\{\mu_n\}$ is uniformly Lipschitz tight but not uniformly tight.
Also the sequence $\{\sqrt{n}\mu_n\}$ of measures shows that the boundedness condition in 3) is indispensable.

Proof: 2) => 1) and 1) => 1)' are evident.

1)' => 3): If 1)' holds, then H is bounded and so we can suppose that $H \subseteq OM_u(X)$. If H is not L-tight we shall show how to construct $\eta > 0$, a sequence (μ_n) in H, a sequence (K_n) of compact sets in X so that

$$B(K_n,\eta) \cap B(K_m,\eta) = \phi \quad (n \neq m)$$

and a sequence (f_n) in $L_{\eta^{-1}}(X)$ so that $\mathrm{supp}(f_n) \subseteq B(K_n,\eta)$ and $|\mu_n(f_n)| \geq \eta$. Once this is done, we complete the proof as follows: for any sequence (λ_n) in $\bigcirc \ell^\infty$, the unit ball of ℓ^∞, $\Sigma\lambda_n f_n$ is in $L_{\eta^{-1}}(X)$. Hence

$$T : (\lambda_n) \to \Sigma\lambda_n f_n$$

is a CoSaks morphism from ℓ^∞ in $U^b(X)$ (we regard ℓ^∞ as $U^b(\mathbb{N})$, $\mathbb{N}$ with the discrete metric). Then the transposed operator

$$T' : M_u(X) \to \ell^1$$

sends H into a relatively countably $\sigma(\ell^1,\ell^\infty)$-compact set and so, by Schur's Lemma, into a relatively norm compact set in ℓ^1. But this contradicts the fact that $|T'\mu_n(e_n)| \geq \eta$, e_n dending the n-th unitvector in ℓ^∞.

We now show how to construct the above sequences inductively. By assumption there is an η $(0 < \eta \leq 1)$ so that for every compact set K there is an $f_K \in L_{1,K}$ and $\mu_K \in H$ with $|\mu_K(f_K)| \geq 4\eta$

Define $\tilde{f}_K$ by

$$\tilde{f}_K : x \longrightarrow \begin{cases} 0 & \text{if } |f_K(x)| \leq 2\eta \\ f_K - 2\eta & \text{if } f_K(x) \geq 2\eta \\ f_K + 2\eta & \text{if } f_K(x) \leq -2\eta \end{cases}$$

Then $f_K \in L_{1,B(K,2\eta)}$ is such that $|\mu_K(\tilde{f}_K)| \geq 2\eta$. We can now proceed with the construction. First we find a $g_1 \in L_1(X)$ and $\mu_1 \in H$ with $|\mu_1(g_1)| \geq 2\eta$. Since μ_1 is a Radon measure on X we can find a compact set K_1 so that $|\mu_1|(X \backslash K_1) \leq \eta$. Hence

$$f_1 : x \to [g_1(x) \wedge (1-\eta^{-1}d(x,K_1))] \vee [-1+\eta^{-1}d(x,K_1)]$$

is a member of $L_{\eta^{-1}}(x)$ whose support is contained in $B(K_1,\eta)$ and is such that $|\mu_1(f_1)| \geq \eta$

At the n-the step let $g_n \in L_{1,B_n}$ $(B_n := B(K_1 \cup \ldots \cup K_{n-1}, 2\eta))$ and $\mu_n \in H$ be such that $|\mu_n(g_n)| \geq 2\eta$

Choose a compact $K_n \subseteq X \backslash B_n$ so that

$$|\mu_n|(X \setminus B_n \cup K_n)) \leq \eta$$

Note that $d(K_n, K_m) \geq 2\eta$ for $m < n$.

Again define

$$f_n : x \to [g_n(x) \wedge (1-\eta^{-1}d(x,K_n))] \vee [-1+\eta^{-1}d(x,K_n)]$$

Then this is a member of $L_{\eta^{-1}}(X)$ whose support is contained in $B(K_n,\eta)$ and which is such that $|\mu_n(f_n)| \geq \eta$.

This completes the induction step and so the proof of 1) => 3).

3) => 2): By the remark following the theorem, it suffices to show that H is equicontinuous on $L_1(X)$. If $(f_\alpha)_{\alpha \in I}$ is a net in $L_1(X)$ tending pointwise to $f \in L_1(X)$, then $(f_\alpha - f)_{\alpha \in I}$ is a net in $2L_1(X)$ which tends to zero uniformly on compact sets. Thus for each $K \in \mathcal{K}(X)$ and $\varepsilon > 0$ there is α_0 so that for $\alpha \geq \alpha_0$

$$\{f_\alpha - f\} \in 2.L_{1,K} + \varepsilon.\ \mathcal{O}\mathcal{U}^b(X),$$

whence, by the $\|.\|$-boundedness of H, $\mu(f_\alpha) \to \mu(f)$ uniformly in $\mu \in H$.

Using the above proof, we also get:

<u>Corollary:</u> Let (μ_n) be a weak Cauchy sequence in $M_u(X)$. Then (μ_n) is γ-Cauchy and so γ-convergent.

<u>Proof:</u> If $\{\mu_n\}_{n=1}^\infty$ is not relatively γ-compact, then one constructs as above $\eta > 0$, a sequence $\{f_k\}_{k=1}^\infty \subseteq L_{\eta^{-1}}(X)$ with disjoint support and a subsequence $\{\mu_{n_k}\}_{k=1}^\infty$ such that

$$|\mu_{n_k}(f_k)| \geq \eta.$$

Defining again an operator T from ℓ^∞ to $\mathcal{U}^b(X)$ by

$$T : (\lambda_k) \to \sum_{k=1}^\infty f_k$$

we obtain a CoSaks-morphism. The transposed operator

$$T' : M_u(X) \to \ell^1$$

sends $\{\mu_{n_k}\}_{k=1}^\infty$ to a weak Cauchy-sequence and $|T'\mu_{n_k}(e_k)| \geq \eta$ where e_k denotes the k-th unit-vector in ℓ^∞. Again this is contradictary to Schur's lemma, since for a relatively $\|.\|$-compact set K in ℓ^1

$$\lim_{k\to\infty} \sup\{|\langle e_k, x\rangle| : x\in K\} = 0.$$

Corollary: Let T be a linear mapping from $\mathcal{U}^b(X)$ into a weakly compactly generated (in particular, separable or reflexive) Fréchet space F. Then T is β_∞-continuous if and only if it has a β_∞-closed graph.

Remark: In particular, if T is continuous for any locally convex Hausdorff topology on F which is coarser than the original topology (e.g. a weak topology defined by a total subset of F'), then T is continuous.

Proof of the Corollary: Note that we now know that $(\mathcal{U}^b(X), \beta_\infty)$ is a Mackey sapce whose dual is weakly sequentially complete. The result now follows from a closed graph theorem of KALTON and MARQUINA (see e.g. COOPER [9], p. 60).

Despite the remark after the statement of the theorem we do have equivalence of tightness and uniform tightness for positive measures.

Proposition: If H $\subseteq M_u^+(X)$, then the conditions of the theorem are equivalent to

4) H is bounded and uniformly tight.

Proof: 4) implies 3) is evident (even without the positivity assumption).
3) implies 4): since $L_{1,K}$ contains a function which is 1 on $X\backslash B(K,1)$, 3) implies that for $\varepsilon > 0$ there is a $K_1 \in \mathcal{K}(X)$ so that $\check{\mu}(\check{X}\backslash\check{B}(K,1)) \leq \varepsilon/2$ for $\mu\in H$ ($\check{X}$ the Samuel compactification of X and $\check{\mu}$ the Radon measure on $\check{X}$ corresponding to μ).
Similarly for $n\in\mathbb{N}$ there is a $K_n\in\mathcal{K}(X)$ so that

$$\check{\mu}(\check{X}\backslash\check{B}(K_n, 1/n)) \leq \varepsilon/2^n$$

for $\mu\in H$ since $L_{n,K}\subseteq nL_{1,K}$).
Putting $K := \bigcap \check{B}(K_n, \frac{1}{n})$ we obtain a compact subset of X so that

$$\mu(X\backslash K) = \check{\mu}(\check{X}\backslash K) \leq \varepsilon \qquad (\mu\in H).$$

§7 LIFTING TO UNIFORM SPACES

Let X be a uniform space. As in §3 we denote by $\mathcal{D}$ the family of all uniformly continuous pseudometries on X which are bounded by 2. Then we have the natural representation

$$\hat{X} = \varprojlim\{\hat{X}_d : d \in \mathcal{D}\}$$

where, for d $\mathcal{D}$, $\hat{X}_d$ represents the associated complete metric space and $\hat{X}$ is the completion of X. More generally, if $\mathcal{D}_1$ is a subset of $\mathcal{D}$ which is closed under pointwise suprema and generates the uniformity of X, then once again $\{\hat{X}_d : d \in \mathcal{D}_1\}$ forms a projective system and $\hat{X}$ is its projective limit.

As noted above the bornology of ueb sets in $\mathcal{U}^b(X)$ consists of those balls which factor through $\mathcal{U}^b(X_d)$ for some $d \in \mathcal{D}$ and form a Lipschitz bounded set there The set $\mathcal{B}$ of those balls which factor in this way over some $\mathcal{U}^b(X_d)$ $(d \in \mathcal{D}_1)$ is, in general, a proper subfamily of the ueb sets but they are linked in the sense that the ueb bornology is exactly the $\|\ \|_\infty$-saturation $\tilde{\mathcal{B}}$ of $\mathcal{B}$ (the proof of this assertion is an easy adaptation of the arguments of §4).

If $\mathcal{D}_p$ denotes the family of all uniformly continuous precompact pseudometrics in $\mathcal{D}$ then, for $d \in \mathcal{D}_p$, $\hat{X}_d$ is compact. The projective limit $\check{X} = \varprojlim\{\hat{X}_d : d \in \mathcal{D}_p\}$ is compact and X embeds homeomorphically (as a topological space) into $\check{X}$. It is easily checked that $\check{X}$ is just the Samuel-compactification of X i.e. the spectrum of the Banach-algebra $\mathcal{U}^b(X)$.

Now if X, $\mathcal{D}_1$ are as above we can obtain projective and inductive representations of $M_u(X)$ and $\mathcal{U}^b(X)$ as follows:

$$\{\mathcal{U}^b(X_d) \to \mathcal{U}^b(X_{d_1}) : d \le d_1, \quad d,d_1 \in \mathcal{D}_1\}$$

forms an inductive system of CoSaks spaces while

$$\{M_u(X_{d_1}) \to M_u(X_d) : d \le d_1, \ d,d_1 \in \mathcal{D}_1\}$$

forms a projective system of Saks-spaces.

If $\mathcal{U}^b(X_d)$ is considered as subspace of $\mathcal{U}^b(X)$, the bornologies of Lipschitz-boun-bounded sets in $\mathcal{U}^b(X_d)$ (or equivalently of ueb-sets in $\mathcal{U}^b(X_d)$), as d ranges through $\mathcal{D}_1$ generate the bornology of ueb-sets in $\mathcal{U}^b(X)$ (in the sense of $\|.\|$-saturation). Thus it is clear that

$$\mathcal{U}^b(X) = \varinjlim\{\mathcal{U}^b(X_d) : d \in \mathcal{D}_1\}$$
$$= \varinjlim\{\mathcal{U}^b(X_d) : d \in \mathcal{D}\}$$

the injective limit being taken in the category of CoSaks spaces and

$$M_u(X) = \varprojlim\{M_u(X_d) : d \in \mathcal{D}_1\}$$
$$= \varprojlim\{M_u(X_d) : d \in \mathcal{D}\}$$

this time the projective limit in the category of Saks spaces.

We remark that if we form $\varinjlim(\mathcal{U}^b(X_d) : d \in \mathcal{D}_p)$, we again get the Banach space $(\mathcal{U}^b(X), \|.\|_\infty)$ while the bornology now consists of the relatively $\|.\|_\infty$-compact balls in $\mathcal{U}^b(X)$.

Using this formalism it is now easy to lift results of §5,§6 to uniform spaces simply by observing that the appropriate statements hold in every component of the projective limit. We gather together the most important results.

Proposition: Let X be a uniform space. A functional $\mu \in (\mathcal{U}^b(X), \|\ \|_\infty)'$ is a member of $M_u(X)$ iff its image on every $\hat{X}_d$ ($d \in \mathcal{D}_1$ or equivalently $d \in \mathcal{D}$) is a uniform measure, i.e. a Radon-measure by the results of §6.

Theorem (uniform case): If X is a uniform space, H a subset of $M_u(X)$ then the following are equivalent:

1) H is relatively $\sigma(M_u, \mathcal{U}^b)$-compact;

1)' H is relatively countably $\sigma(M_u, \mathcal{U}^b)$-compact;

2) H is relatively γ-compact;

3) the image of H in every $M_u(\hat{X}_d)$, $d \in \mathcal{D}$ is relatively γ-compact;

3)' H is $\|.\|$-bounded and its image in every $M_u(\hat{X}_d)$, $d \in \mathcal{D}_1$ is relatively γ-compact;

4) H is bounded and its image in every $M_u(\hat{X}_d)$, $d \in \mathcal{D}$ is uniformly L-tight;

4)' H is bounded and its image in every $M_u(\hat{X}_d)$, $d \in \mathcal{D}_1$, is uniformly L-tight;

Proof: A subset H in a projective limit of Saks-spaces is γ-compact (respectively weakly compact) iff it is $\|.\|$-bounded and all its projections into the component spaces are γ-compact (respectively weakly compact). c.f. COOPER 9 ,pp.10,16.

From this remark and the corresponding theorem for the metric case the theorem is easily deduced.

Corollary: Let X be a uniform space. Every $\sigma(M_u(X), \mathcal{U}^b(X))$ Cauchy-sequence (μ_n) converges in the γ-topology.

Proof: For every $d \in \mathcal{D}$ the image of $\{\mu_n\}_{n=1}^{\infty}$ is weakly Cauchy in $M_u(\hat{X}_d)$, whence γ-convergent in $M_u(\hat{X}_d)$ by the result for the metric case. So $\{\mu_n\}_{n=1}^{\infty}$ is γ-convergent in $M_u(X)$.

Exactly as in the metric case one also derives the two following results för the uniform case.

Proposition: Let X be a uniform space and T be a linear mapping from $\mathcal{U}^b(X)$ into a weakly compactly generated Fréchet space. Then T is β_∞-continuous iff it has a β_∞-closed graph.

Proposition: A subset $H \subseteq M_u^+(X)$ is relatively γ-compact iff it is $\|.\|$-bounded and its image in every $M_u^+(X_d)$ ($d \in \mathcal{D}$ or, equivalently, $d \in \mathcal{D}_1$) is uniformly tight.

Following PACHL [34] , we now indicate briefly how the concept of uniform measure embraces many important classes of measures:

I. Separable measures:

If X is a completely regular space, we can regard it as a uniform space with the fine uniformity i.e. the finest uniformity compatible with its topology. Then $\mathcal{U}^b(X) = C^b(X)$ and the corresponding CoSaks structure on $C^b(X)$ is that of the bounded, equicontinuous subsets of $C^b(X)$. As mentioned in the introduction the corresponding topology on $C^b(X)$ has been studied by WHEELER and the corresponding space of measures by various authors from various points of view.

II. σ-additive abstract measures:

Now let (Ω, Σ) be a measure space i.e. Σ is a σ-algebra on Ω. In addition, we assume that Σ separates the points of Ω. For each countable partition $A = (A_n)$ of Ω by sets of Σ, we define a pseudometric d_A by putting

$$d_A(x,y) = \begin{cases} 1 & \text{if } x,y \text{ do not belong to the same } A_n \\ 0 & \text{otherwise} \end{cases}$$

The associated metric space is just $\mathbb{N}$ with the discrete metric. We can regard Ω as a uniform space with the structure induced by these pseudo-metrics (such uniform spaces have been studied by HAGER under the name "measurable uniform spaces" in [20]). Then $\mathcal{U}^b(\Omega)$ is the space $B(\Omega)$ of bounded measurable functions on Ω (for every bounded measurable function can be uniformly approximated by countably (even finitely) valued functions). From the equation $M_u(\Omega) = S \varprojlim_A M_u(\Omega_{d_A})$ we see that an element of $M_u(\Omega)$ is a bounded set function on Σ with the property that for each measurable partition (A_n) $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} (A_n)$ i.e. it is a σ-additive measure.

III. Cylindrical measures:

Let E,F be vector spaces in duality. We denote by $\mathcal{J}(F)$ the family of finite dimensional subspaces of F. They form an inductive system (ordered by inclusion) and so by duality we get a projective system of finite dimensional spaces (which we can regard as quotients of E) whose projective limit is F^*. We can then give E a uniform structure as a subset of this projective limit (i.e. this is just a complicated way of talking about the $\sigma(E,F)$-uniformity on E). The corresponding space of uniform measures is denoted by $M_{CYL}(E)$ - the space of cylindrical measures on E. Now the above complications begin to pay off because we can write

$$M_{CYL}(E) = S \varprojlim \{M_u(E/G^o) : G \in \mathcal{J}(F)\}$$

and so an element of $M_{CYL}(E)$ can be regarded as a projective limit of Radon measures (in the category of Saks spaces) on finite dimensional quotients of E and this is the normal definition of a cylindrical measure.

§8 VECTOR-VALUED MEASURES AND ORLICZ-PETTIS TYPE THEOREMS

In this section we indicate briefly how the classical Orlicz-Pettis theorem can be interpreted as the statement that the dual $(\mathcal{U}^b(\Omega,\beta_\infty)$ of a space of measures is a Mackey space.

As in example II of §7 let Σ be a σ-algebra of subsets of Ω and let

$$m : \Sigma \to E$$

be a weakly σ-additive set-function from Σ to a locally convex space E. By the Dieudonné-Grothendieck-theorem (c.f. DIESTEL-UHL [12], th. I.3.3), m has bounded range and so we may, if E is quasicomplete, define the integration operator

$$T_m : \mathcal{U}^b(\Omega) \to E$$

$$T_m : (\chi_A) \to m(A)$$

where T_m is a continuous operator from $(\mathcal{U}^b(\Omega),\beta_\infty)$ into $(E,\sigma(E,E'))$. As $(\mathcal{U}^b(\Omega),\beta_\infty)$ is a Mackey space T_m is continuous with respect to the original topology of E.

If $\{A_n\}_{n=1}^\infty$ decreases to ϕ in Σ, then $\{\chi_{A_n}\}_{n=1}^\infty$ is a ueb -set tending pointwise to zero, whence $\{T(\chi_{A_n})\}_{n=1}^\infty$ tends to zero in the original topology of E. Thus we have proved the Orlicz-Pettis-theorem:

"A weakly σ-additive measure on a σ-algebra is strongly σ-additive".

Now assume E is a weakly compactly generated Fréchet-space, F is an E-total subset of E' and

$$m : \Sigma \to E$$

is a $\sigma(E,F)$ countably additive measure. Again by Dieudonné-Grothendieck m has bounded range and we may define an integration operator

$$T_m : \mathcal{U}^b(\Omega) \to E$$

$$T_m : \chi_A \to m(A)$$

where by the closed-graph theorem proved in §7, T_m is β_∞-continuous, which implies as above that m is strongly σ-additive.

R E F E R E N C E S

[1] R.F. ARENS, J. EELLS On embedding uniform and topological spaces Pac.J.Math. 6(1956) 397-403

[2] N. AZZAM Mesures sur les espaces uniformes, Prépublications Université St.-Etienne 1974

[3] I.A. BEREZANSKII Measures on uniform spaces, Trans.Moscow Math.Soc. 19(1968) 1-40

[4] J. BERRUYER, B. IVOL Espaces de mesures et compactologies, Publ.Dép. Math. Lyon 9.1(1972) 1-36

[5] H.BUCHWALTER Topologies, bornologies et compactologies (Lyon, undated)

[6] Quelques curieuses topologies sur $M_\sigma(T)$ et $M_\beta(T)$ Ann.Inst.Fourier (to appear).

[7] H. BUCHWALTER, R. PUPIER Complétion d'un espace uniforme et formes linéaires, C.R.Acad.Sc.Paris A273 (1971) 96-98

[8] J. CIGLER Funktoren auf Kategorien von Banachräumen (Lecture Notes, Vienna 1974)

[9] J.B. COOPER Saks spaces and applications to functional analysis (Amsterdam, 1978)

[10] J.B. COOPER, W. SCHACHERMAYER Saks spaces and vector valued measures (Institutsbericht 98, Linz, 1978)

[11] A. DEAIBES Espaces uniformes et espaces de mesures, Publ.Dép. Math. Lyon 12.4(1975) 1- 166

[12] J. DIESTEL, J.J. UHL The theory of vector measures (Providence 1977)

[13] M.A. DOSTAL Some recent results on topological vector spaces (Springer Lecture Notes 384 (1974) 20-91).

[14] R.M. DUDLEY Convergence of Baire measures, Studia Math. 27(1966) 252-268

[15] R. ENGELKING Outline of general topology (Amsterdam, 1968)

[16] V.L. FEDOROVA Linear functionals and the Daniell integral on spaces of uniformly continuous functions, Usp.Math. Nauk 133.1 (1967) 172-173 (Russian)

[17] Z. FROLIK Mesures uniformes, C.R. Acad.Sc.Paris A277 (1973) 105-108

[18] Z. FROLIK — Représentation de Riesz des mesures uniformes C.R.Acad.Sc.Paris, A277 (1973) 163-166

[19] — Metric-fine uniform spaces (Seminar Abstract Analysis - Prague, 1974)

[20] A. HAGER — Measurable uniform spaces, Fund.Math. 77(1972) 51-73

[21] R.G.HAYDON — Sur les espaces M(T) et $M^{\infty}(T)$, C.R.Acad.Sc.Paris 275(1972) A989-991

[22] J.R. ISBELL — Algebras of uniformly continuous functions, Ann. Math. 68(1958) 96-125

[23] — Uniform spaces (Providence, 1964)

[24] K. JOHN — Differentiable manifolds as topological linear spaces, Math.Ann. 186 (1970) 177-190

[25] J. JOHNSON — Lipschitz function spaces for arbitrary metrics, Bull.Amer.Math.Soc. 78 (1972) 702-706

[26] — Banach spaces of Lipschitz functions and vector valued Lipschitz functions, Bull.Amer.Math.Soc. 75 (1969) 1334-1338.

[27] R.B. KIRK — Topologies on spaces of Baire measures, Trans.Amer. Math.Soc. 184 (1973) 1-29

[28] K. de LEEUW — Banach spaces of Lipschitz functions, Studia Math. 21 (1961) 55-66

[29] C. LEGER, P. SOURY — Le convexetopologique des probabilités sur un espace topologique, J.Math.Pure et Appl. 50 (1971) 363-425

[30] S. MAC LANE — Categories for the working mathematican (Berlin, 1971)

[31] E.G. MANES — A characterisation of dual spaces, Func.Anal. 17 (1974) 292-295

[32] E. MICHAEL — A short proof of the Arens-Eells embedding theorem Proc.Amer.Math.Soc.15(1964) 415-416

[33] J. PACHL — Free uniform measures, Comm.Math.Univ.Carol.15 (1974) 541-553

[34] — Compactness in spaces of uniform measures, Trans. Amer.Math.Soc. (to appear)

[35] V. PTAK Algebraic extensions of topological spaces, Proc.of a Symposium, Berlin 1967 - VEB Berlin (1969, 179-188)

[36] R. PUPIER Méthodes fonctorielles en topologies générale Thése, Univ.Lyon I (1971)

[37] D.A. RAIKOV Free locally convex spaces, Mat.Sbornik 63 (1964) 582-590

[38] M. ROME L'espace $M^{\infty}(T)$, Publ.Dép.Math. Lyon 9-1 (1972) 37-60

[39] A.K. ROY Extreme points and linear isometries of the Banach space of Lipschitz functions, Can.J.Math. 20(1968) 1150-1164

[40] H.H. SCHAEFER Topological vector spaces (New York, 1966)

[41] S. TOMAŠEK On certain classes of Λ-structures I, II, Czech.Math.J. 20(1970) 1-18, 19-33

[42] R.F. WHEELER The strict topology, separable measures and paracompactness, Pac.J.Math. 47 (1973) 287-302

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