On some Classical Measure - Theoretic Theorems for non-σ-complete Boolean Algebras.

by

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ABSTRACT We investigate those non-σ-complete Boolean algebras for which the theorems of Vitali-Hahn-Saks, Nikodym, Orlicz-Pettis and Grothendieck hold and how the validity of these theorems is interrelated.

INTRODUCTION

If F is a σ-algebra the following theorems hold for F:

Vitali-Hahn-Saks (VHS): A sequence $\{\mu_n\}_{n=1}^{\infty}$ of finitely additive, bounded, real valued measures on F, such that $\{\mu_n(A)\}_{n=1}^{\infty}$ converges for every $A \in F$, is uniformly exhaustive.

Nikodym (N): A family M of finitely additive, bounded, real valued measures on F, such that $\{\mu(A) : \mu \in M\}$ is bounded for every $A \in F$, is uniformly bounded.

Orlicz-Pettis (OP): Every measure $\mu : F \to X$ with values in a Banach space X which is σ-additive with respect to the weak topology of X is σ-additive with respect to the norm topology of X.

Grothendieck (G): Let $B(F)$ be the Banach space of bounded F-measurable functions equipped with the sup-norm. A sequence $\{\mu_n\}_{n=1}^{\infty}$ in $B(F)^*$, that converges weak*, converges weakly, i.e. $B(F)$ is a Grothendieck space.

Rosenthal (R): Every continuous linear map from $B(F)$ to a Banach space X, which is not weakly compact, fixes a copy of l^{∞}, i.e. $B(F)$ is a Rosenthal - space.

For proofs we refer to [D-U 77] and for an account of the historic development of these important and beautiful theorems from Vitali's paper in 1907 on we refer to [F 76].

As regards (VHS) and (N), these theorems used to be stated originally in terms of σ-additive measures. T.Andô [A 61] was the first to realize that these theorems have their proper setting in the framework of finitely additive measures and since then many improvements and generalisations were obtained. So the natural question as to whether these theorems could be extended to non-σ-complete Boolean algebras arose.

As regards (G), it was observed by Grothendieck that on the dual of $L^{\infty}(\mu)$ weak and weak* sequential convergence coincide. Lindenstrauss [L 64] proved that for a C(K)-space this property is equivalent to a certain extension property of separably valued operators (see th. 5.1 below). He raised the question of characterizing this property, which we call (G), in terms of the topological properties of K. Since then considerable progress has been made in the study of Grothendieck spaces but they still remain mysterious objects. A good reference is [D 73], in which a list of open problems is stated.

We take the validity of the theorems in question as a definition (analogue to calling a locally convex space barelled if the Banach - Steinhaus theorem holds on it): We say that a Boolean algebra F satisfies (VHS), (N), (OP), (G) or (R) if the corresponding theorem holds on F.

First we note that an arbitrary Boolean algebra need not satisfy these properties. Let $\Phi(\mathbb{N})$ be the algebra of finite and cofinite subsets of $\mathbb{N}$ and let δ_n denote the Dirac measure located at $\{n\}$. The sequence $\{\delta_n\}_{n=1}^{\infty}$ provides a counter-example to (VHS), while the set $\{n(\delta_n - \delta_{n+1}) : n = 1,2,3,...\}$ provides a counterexample to (N). Concerning (OP) consider the measure μ from $\Phi(\mathbb{N})$ into c_o (the Banach space of null-sequences), whose coordinates are the scalar measures $\delta_n - \delta_{n+1}$. For (G) note that $B(\Phi(\mathbb{N}))$ may be identified with the Banach space c of convergent sequences and that $\{\delta_n\}_{n=1}^{\infty}$ (i.e., the evaluation in the n'th coordinate) is a sequence in c^* which converges weak* but not weakly. Finally note, that the identity on c is not weakly compact but c does not contain a copy of l^{∞}, whence F does not have (R).

On the other hand, there exist classes of non-σ-complete algebras which have the above stated properties ([S 68], [D 78], [F76]).

It was realized that there are strong interrelations between these properties of a Boolean algebra. (The reader probably has noticed that the counterexamples in the case of $\Phi(\mathbb{N})$ are essentially the same.) So it was asked ([S 68], [D-F-H 75], [F 76]) whether the four properties (VHS), (G), (N) and (OP) are equivalent for a Boolean algebra.

The only known equivalence result was due to Diestel, Faires and Huff [D-F-H 75]: A Boolean algebra has (VHS) iff it has (G) and (N).

It is proved that (G) implies (OP) (theorem 6.5 below). Whether (G) implies (VHS) (or equivalently (N)) remains open. None of the other possible implications between (VHS), (G), (N) and (OP) holds as is shown by a series of examples (see § 3 below). As regards (R) we can only show the (trivial) observation that (R) implies (G). Nevertheless we include property (R) in this paper as we feel that it has its proper setting in this framework.

As regards the organisation of the paper: After a preliminary § 1 we give in § 2 definitions and present the above mentioned Diestel - Faires - Huff theorem. We also show that F has (N) iff the normed space of F-measurable simple functions is barreled, which allows some sharpenings of theorems of Dieudonné - Grothendieck and Seever. Finally we show that in the definition of (OP) one may reduce to the case of bounded σ-additive measures.

In § 3 we give examples. For instance the algebra J of Jordan-measurable subsets of [0,1] has (N) and (OP) but not (R), (VHS) and (G). However, there is a quotient algebra of $\mathcal{J}$ which does not have (OP). Also the Stone space of $\mathcal{J}$ has the property that every infinite closed subset contains a copy of $\beta\mathbb{N}$.

In § 4 we investigate two special classes of Boolean algebras:

1) If $\mathcal{F}$ is a Boolean subalgebra of a σ-complete algebra Σ which is not too far away from Σ (the condition is that the Banach space $B(\mathcal{F})$ of bounded $\mathcal{F}$-measurable functions is a countable intersection of closed hyperplanes in $B(\Sigma)$), then $\mathcal{F}$ has all our properties.

2) On the other hand, if $\mathcal{F}$ is a countable union of a strictly increasing sequence $\{\mathcal{F}_n\}_{n=1}^{\infty}$ of Boolean algebras (these objects arise for example in the theory of martingales), then $\mathcal{F}$ does not satisfy (VHS), (G), (N) or (R).

In § 5 we investigate in detail property (G). It is shown that (G) is equivalent to a condition very similar to (OP) (stated, however, in terms of finitely additive measures). We also characterize (G) in terms of convex weak*-compact subsets of $B(\mathcal{F})^*$. Finally, in § 6 we prove that (G) implies (OP).

My warmest thanks go to Joseph Diestel. Without him this article never would have been written. He mentioned the open questions to me that are partially solved in this paper he gave me the unpublished preprint [D-F-H 75] and I had the opportunity of some stimulating conversations with him. I also thank Barbara Faires for her kind collaboration. Finally, I am greatly indebted to the referee for many valuable comments, simplifications of proofs and modifications of the poor style in which the first version of this paper was written.

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§ 1. PRELIMINARIES

In this paper F is a Boolean algebra with the operations of sup, inf and complementation denoted by $\vee, \wedge$ and C. The smallest and largest element of F will be denoted by $\emptyset$ and Ω. The Stone representation theorem (see [S 71a] for example) states that F has a unique representation as the field of clopen sets of a totally disconnected compact Hausdorff space, the "Stone representation space" of F, which will also be denoted by Ω. Note, however, that F always has many representations as a field of sets (the Stone representation is one such representation, unique only in the above stated sence).

If $\{A_i\}_{i\in I}$ is a family of elements of F we write $\bigvee_{i\in I} A_i$ for the smallest element in F that majorises all A_i, if such an element exists. (A necessary and sufficient condition for the existence is that, if F is Stone represented, the closure of the union of the A_i is open). F is σ-complete (resp. complete) if all countable (resp. all) suprema exist in F.

The letters X,Y,Z will denote Banach spaces, which - only for simplicity - are assumed real, and X^*, Y^*, Z^* the topological duals. A function $\mu : F \to X$ is called a measure, if it is additive, i.e. whenever A_1, A_2 in F are disjoint (this means $A_1 \wedge A_2 = \emptyset$), then $\mu(A_1 \vee A_2) = \mu(A_1) + \mu(A_2)$.

A measure μ is called bounded, if $Rg(\mu) = \{\mu(A) : A \in F\}$ is bounded and is called σ-additive (resp. weakly σ-additive) if for every sequence of mutually disjoint elements $\{A_n\}_{n=1}^{\infty}$, s.t. $\bigvee_{n=1}^{\infty} A_n$ exists in F, we have $\sum_{n=1}^{\infty} \mu(A_n) = \mu(\bigvee_{n=1}^{\infty} A_n)$, the sum converging in the norm topology (resp. in the weak topology).

Denote by $B(F)$ the Banach algebra of real valued bounded F-measurable functions, equipped with the sup norm. $B(F)$ may be obtained in an abstract way as the completion of the normed algebra $B_s(F)$ of simple functions, i.e. expressions of the form $\sum_{i=1}^{n} \lambda_i \chi_{A_i}$, with the obvious norm and algebra operations. A less formal approach is to note that $B(F)$ may be naturally identified with the Banach algebra $C(\Omega)$ of continuous functions on the Stone space Ω.

The dual $B(F)^*$ may be represented as the space of all real valued finitely additive measures μ on F with finite variation norms $\|\mu\| = \sup \{\sum_{i=1}^{n} |\mu(A_i)| : \{A_i\}_{i=1}^{n}$ partition of $F\}$. Note that a real valued measure has bounded variation norm iff it is bounded, as the following estimate shows ([D-S 58], III.1.5.).

$$\sup \{|\mu(A)| : A \in F\} \leq \|\mu\| \leq 2 \sup \{|\mu(A)| : A \in F\}.$$

Another representation of $B(F)^*$ is to interpret it as the space of Radon measures on the Stone space Ω.

Our notation follows [D-U 77] with the following exception. A measure $\mu : F \to X$ is called "exhaustive" (in [D-U 77] "strongly additive") if, for every sequence $\{A_n\}_{n=1}^{\infty}$ of mutually disjoint members of F, $\|\mu(A_n)\|$ tends to zero. A family M of X - valued measures is called "uniformly exhaustive" if $\|\mu(A_n)\|$ tends to zero uniformly in $\mu \in M$. The following result gives the connection between uniform exhaustivity and weak compactness in $B(F)^*$. Although this result is wellknown, I am unable to give a reference for the exact result that we need and shall therefore sketch a proof.

<u>1.2 Proposition.</u> A subset M of $B(F)^*$ is relatively weakly compact iff M is uniformly exhaustive and bounded on the members of F.

<u>Proof:</u> We consider $B(F)^*$ as the space of Radon - measures on the Stone space Ω.

If M is relatively weakly compact then M is bounded and by Dunford's characterization of weak compactness in spaces of σ-additive measures ([D-U 77], th.IV.2.5) there is a positive Radon - measure η on Ω such that M is uniformly absolutely continuous with respect to η. So for every disjoint sequence $\{A_i\}_{i=1}^{\infty}$ of clopen sets in Ω, $\eta(A_i) \to 0$, whence $\mu(A_i) \to 0$ uniformly in $\mu \in M$.

Conversely suppose M to be uniformly exhaustive and bounded on members of F. Then M is a bounded subset of $B(F)^*$. Indeed, if this were not the case one could - by the assumption of pointwise boundedness of M on the members of F - easily construct a disjoint sequence $\{A_i\}_{i=1}^{\infty}$ such that

$$\limsup_{i\to\infty} \{|\mu(A_i)| : \mu \in M\} = \infty,$$

a contradiction to the uniform exhaustivity of M.

As M is bounded we may apply corr. I. 5. 4 of [D-U 77] to find a control-measure $\eta \in B(F)^*$ for M, i.e. a positive Radon-measure η on Ω such that.

$$\lim_{\eta(A)\to 0} \sup \{|\mu(A)| : \mu \in M\} = 0,$$

where A ranges through F. With the help of lemma I.5.1 of [D-U 77] we again may apply Dunford's characterization of weak compactness to infer that M is relatively weakly compact in $B(F)^*$.

□

§ 2 DEFINITIONS AND A THEOREM OF DIESTEL, FAIRES AND HUFF

2.1 Definition. F has (VHS) if one of the following equivalent conditions is satisfied:

VHS_1 A sequence $\{\mu_n\}_{n=1}^{\infty}$ in $B(F)^*$ which converges pointwise on F, (i.e., for all $A \in F$, $\{\mu_n(A)\}_{n=1}^{\infty}$ converges), is uniformly exhaustive.

VHS_2 A sequence $\{\mu_n\}_{n=1}^{\infty}$ in $B(F)^*$ which converges with respect to the $\sigma(B(F)^*, B_s(F))$ - topology converges weakly, i.e. with respect to $\sigma(B(F)^*, B(F)^{**})$.

Proof: The equivalence is a direct consequence of proposition 1.2 and the observation that a relatively weakly compact sequence that converges with respect to $\sigma(B(F)^*, B_s(F))$ converges weakly.

□

2.2 Definition. A Banach space X is called a Grothendieck space if a sequence in X^* which converges weak* to zero converges weakly.

2.3.Definition. $\mathcal{F}$ has (G) if one of the following equivalent conditions is satisfied:

G_1 A bounded sequence $\{\mu_n\}_{n=1}^{\infty}$ in $B(\mathcal{F})^*$, which converges pointwise on $\mathcal{F}$, is uniformly exhaustive.

G_2 $B(\mathcal{F})$ is a Grothiendieck space

Proof: $B_s(\mathcal{F})$ is a dense subspace of $B(\mathcal{F})$. Whence a bounded $\sigma(B(\mathcal{F})^*, B_s(\mathcal{F}))$ - convergent sequence is $\sigma(B(\mathcal{F})^*, B(\mathcal{F}))$ -convergent. Conversely it follows from the uniform boundedness principle that a $\sigma(B(\mathcal{F})^*, B(\mathcal{F}))$ - convergent sequence is bounded. Whence the equivalence of G_1 and G_2 is again a consequence of proposition 1.2 .

□

2.4 Definition. $\mathcal{F}$ has (N) if one of the following equivalent conditions is satisfied:

N_1 A subset M of $B(\mathcal{F})^*$ which is pointwise bounded on $\mathcal{F}$(i.e., for all $A \in \mathcal{F}$ we have $\sup\{|\mu(A)| : \mu \in M\} < \infty$) is uniformly bounded.

N_2 The normed space $B_s(\mathcal{F})$ (with the supremum norm) is barreled.

Proof: Since pointwise boundedness of M on $\mathcal{F}$ is the same as boundedness with respect to the duality $\langle B_s(\mathcal{F}), B(\mathcal{F})^*\rangle$, the equivalence is a particular case of IV.5.2 of [S 71].

□

2.5 Theorem [D-F-H 75]: (VHS) $\Longleftrightarrow$ (G) $\oplus$ (N), i.e. $\mathcal{F}$ has (VHS) iff it has (G) and (N).

Proof: $VHS_1 \Rightarrow G_1$ and $G_1 + N_1 \Rightarrow VHS_1$ are equivalent. So it remains to prove (VHS) $\Rightarrow$ (N). Suppose non(N) and (VHS). Then we can find a sequence $\{\mu_n\}_{n=1}^{\infty}$ in $B(\mathcal{F})^*$ which is pointwise bounded on $\mathcal{F}$ but such that $\|\mu_n\| \to \infty$. Defining

$$\lambda_n = \|\mu_n\|^{-1/2} \mu_n$$

we exhibit a sequence tending pointwise to zero on $\mathcal{F}$.
but such that $\|\lambda_n\| \to \infty$, a contradiction to (VHS).

□

Despite its simplicity, the characterization of (N) by condition N_2 (which is also essentially contained in [D-F-H 75]) gives more insight into the nature of property (N). For example the following theorems of Seever and Dieudonné - Grothendieck may be considerably sharpened by using the "open mapping theorem technique" connected with the notion of barreledness.

<u>2.6 Theorem ([D-U 77], th. I. 3.3 and I.3.4):</u>

The following properties of $\mathcal{F}$ are equivalent:

(i) $B_s(\mathcal{F})$ is barreled

(ii) Given any continuous linear map T from a Banach space X to $B(\mathcal{F})$ such that $T(X)$ contains $B_s(\mathcal{F})$, T is onto $B(\mathcal{F})$.

(iii) Every linear map T from $B_s(\mathcal{F})$ to a Banch space X with closed graph is continuous.

(iv) Every measure μ from $\mathcal{F}$ to a Banach space X, such that $x^* \circ \mu$ is bounded for each x^* in some total (i.e. point-separating) subset Γ of X^*, is bounded.

<u>Proof:</u> (i)$\Longleftrightarrow$(iii): follows by ([S 71], IV.8.5 and IV.8.6).

(iii) $\Rightarrow$ (iv): If μ is given as in (iv) μ extends in a natural way to a linear map $T : B_s(\mathcal{F}) \to X$ by putting $T(\chi_A) = \mu(A)$. For every $x^* \in \Gamma$, $x^* \circ T$ is a bounded linear functional on $B_s(\mathcal{F})$. This means that T is continuous with respect to the norm-topology on $B_s(\mathcal{F})$ and the $\sigma(X, \Gamma)$ topology on X so its graph is closed in $B_s(\mathcal{F}) \times X$. By assumption T is continuous, which implies the result.

(iv) $\Rightarrow$ (ii): Let $T : X \to B(\mathcal{F})$ as in (ii) be given. Factoring X by $\mathrm{Ker}(T)$, we may assume T to be injective. Denote by OX the unit-ball of X. We shall show that $T(OX)$ contains a zero-neighbourhood of $B(\mathcal{F})$, which will prove (ii). As $B_s(\mathcal{F})$ is dense in $B(\mathcal{F})$, it will suffice to show that $T(OX) \cap B_s(\mathcal{F})$ contains a zero-neighbourhood of $B_s(\mathcal{F})$. Indeed, by the usual argument used in the proof of the open mapping theorem (c.f. [S 71], lemma III.2 or [J 74] theorem 22.4), this implies that $T(OX)$ already contains a zero-neighbourhood of $B(\mathcal{F})$. As every member of the unit-ball of $B_s(\mathcal{F})$ is a convex combination of indicator - functions, we only have to show that $\{T^{-1}(\chi_A) : A \in \mathcal{F}\}$ is a bounded subset of X. In order to deduce this from (iv) define the measure

$$\mu : \mathcal{F} \longmapsto X$$
$$A \longmapsto T^{-1}(\chi_A).$$

Putting $\Gamma = T^*(B(\mathcal{F})^*)$, which is a total subset of X^*, we are exactly in the situation of (iv) and may conclude that the range of μ is bounded, thus proving (ii).

(ii) $\Longrightarrow$ (i): If $B_s(F)$ is not barreled, there exists a lower semicontinuous seminorm, p say, on $B_s(F)$ which is not continuous. Therefore the norm $q = p + \|\cdot\|_\infty$ is strictly stronger than $\|\cdot\|_\infty$ on $B_s(F)$. Let X be the completion of $(B_s(F), q)$. As q is stronger than $\|\cdot\|_\infty$, X embeds continuously into $B(F)$. Moreover $X \neq B(F)$ since otherwise q would be equivalent to $\|\cdot\|_\infty$. The identity $X \to B_s(F)$ contradicts (ii).

□

Remark: In the above theorem the word "Banach space" may be replaced by "Ptak space", whence in particular by (locally convex) "Fréchet space".

An interesting aspect of property (N) is that it furnishes natural examples of non-complete normed barreled spaces with closed subspaces which are not barreled. For example let Σ be a σ-algebra and F a subalgebra which does not have (N), e.g. $\Sigma = \{$all subsets of $\mathbb{N}\}$ and $F = \Phi(\mathbb{N})$. Then $B_s(F)$ is a closed non-barreled subspace of the barreled space $B_s(\Sigma)$, which is a non-complete normed space.

2.7 Definition: F has (OP) if one of the following equivalent conditions is satisfied:

OP_1 A measure μ from F to a Banach space X, which is weakly σ-additive is σ-additive (i.e., for the norm topology).

OP_2 A bounded measure μ from F to a Banach space X, which is weakly σ-additive is σ-additive.

Clearly $OP_1 \Rightarrow OP_2$. But the implication $OP_2 \Rightarrow OP_1$ is not obvions since σ-additive (even real valued) measures on algebras are not bounded in general. This is due to the fact that there are algebras on which no non-trivial countable suprema exist (see 3.14 below). On such an algebra any measure (equivalently any linear map from $B_s(F)$ to X) is σ-additive, (as the requirements are empty) but there are unbounded ones among them.

We now prove $OP_2 \Rightarrow OP_1$. Suppose $\mu : F \to X$ is weakly σ-additive and let $\{A_n\}_{n=1}^{\infty}$ be a disjoint sequence in F such that $\bigvee_{n=1}^{\infty} A_n$ exists in F. Then

$$\text{there is } N \text{ s.t. } \mu \text{ restricted to } F \wedge \bigvee_{n=N}^{\infty} A_n \text{ is bounded.} (*)$$

Assuming (*) for the moment we may apply OP_2 to μ restricted to $F \wedge \bigvee_{n=N}^{\infty} A_n$ and infer that $\sum_{n=N}^{\infty} \mu(A_n)$ is strongly convergent in X. This gives $OP_2 \Rightarrow OP_1$.

To prove (*), assume that it is not true. We shall construct inductively a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of integers and elements $B_k \in F$ such that

$$B_k \subseteq \bigvee_{l=n_k+1}^{n_{k+1}} A_l$$

and

$$\|(\mu(B_k)\| \to \infty.$$

Let

$$C_k = \bigvee_{l=n_k+1}^{n_{k+1}} A_l \setminus B_k.$$

Since $B_1 \vee C_1 \vee B_1 \vee C_2 \vee \ldots = \bigvee_{n=1}^{\infty} A_n \in F$ the corresponding series $\mu(B_1) + \mu(C_1) + \mu(B_2) + \ldots$ converges weakly to $\mu(\bigvee_{n=1}^{\infty} A_n)$ by the weak σ-additivity of μ. Hence $\mu(B_K)$ tends to zero weakly and cannot be unbounded; a contradiction.

Induction: Let $n_1 = 0$ and suppose $n_1,\ldots,n_k$ and $B_1,\ldots,B_{k-1}$ defined. By assumption there exists $D_k \subseteq \bigvee_{l=n_k+1}^{\infty} A_l$ s.t. $\|\mu(D_k)\| > k$. Find x_k^* in X^* s.t. $\|x_k^*\| = 1$ and $|x_k^* \circ \mu(D_k)| > k$. By the σ-additivity of $x_k^* \circ \mu$ we may find n_{k+1} s.t. $|x_k^* \circ \mu(D_k \wedge \bigvee_{l=n_k+1}^{n_{k+1}} A_l| > k$, whence $\|\mu(D_k \wedge \bigvee_{l=n_k+1}^{n_{k+1}} A_l)\| > k$. Putting $B_k = D_k \wedge \bigvee_{l=n_k+1}^{n_{k+1}} A_l$, we complete the induction step, thus finishing the proof of $OP_2 \Longrightarrow OP_1$.

□

2.8 Definition: We call a Banach space X a Rosenthal space if every non weakly compact operator $T : X \to Y$ fixes a copy of l^∞, i.e. there is a continuous linear map $j : l^\infty \to X$ such that $T \circ j$ is an isomorphism into X.

2.9 Definition: We say that F has (R) if $B(F)$ is a Rosenthal space.

2.10 Proposition: (R) $\Longrightarrow$ (G), i.e. an algebra F having (R) has (G).

Proof: It is immediate from the definition that the Banach space $B(F)$ is a Grothendieck space iff every map $T : B(F) \to c_0$ is weakly compact (see th.5.1 below).

But as c_o does not contain an isomorphic copy of l^{∞}.
it is plain that (R) implies (G).

□

To finish the section we state the following result on the stability of the properties in question. For the definition of subalgebras, quotients and ideals of a Boolean algebra we refer to [H 63] or [S 62].

2.11 Proposition.

a) None of the proposition (VHS),(G),(N),(OP) or (R) is inherited by sub-algebras.

b) (VHS),(G),(N) and (R) are inherited by quotient-algebras, while (OP) is not.

Proof:

a) This follows from the example in the introduction.
$\Phi(\mathbb{N})$ is a sub-algebra of the complete Boolean algebra $P(\mathbb{N})$ of all subsets of $\mathbb{N}$.

b) Let I be an ideal in F and $G = F/I$ the quotient algebra. Note that $B(G)$ is in a natural way a quotient of $B(F)$, and $B_s(G)$ a quotient of $B_s(F)$.
As the property of being a Grothendieck, Rosenthal or a barreled space is inherited by (separated) quotient spaces, we see that (R),(G) and (N) are inherited by quotient algebras and by theorem 2.5 we get the result for (VHS) too.
The fact that (OP) is not inherited by quotients will be shown in the forthcoming example (prop. 3.9 below).

□

§ 3 EXAMPLES

3.1 Let $\mathcal{J}$ denote the family of Jordan - measurable sets in $[0,1]$, i.e. the $A \subseteq [0,1]$ such that $bd(A) = \bar{A} \setminus A^{o}$ is of Lebesgue measure 0. $\mathcal{J}$ forms a field of subsets of $[0,1]$ and $B(\mathcal{J})$ is the Banach space of bounded Riemann - integrable functions on $[0,1]$.

We shall show that $\mathcal{J}$ does not have (G) (and therefore (VHS) and (R) neither) but does have (N) and (OP). Finally we construct a quotient algebra of $\mathcal{J}$ which does not have (OP).

3.2 Proposition. $\mathcal{J}$ does not have (G).

Proof: Define a sequence $\{\mu_n\}_{n=1}^{\infty}$ in $B(\mathcal{J})^*$ by

$$\mu_1 = \delta_{\{\frac{1}{2}\}},$$

$$\mu_2 = \frac{1}{2}\left(\delta_{\{\frac{1}{4}\}} + \delta_{\{\frac{3}{4}\}}\right),$$

$$\mu_3 = \frac{1}{4}\left(\delta_{\{\frac{1}{8}\}} + \delta_{\{\frac{3}{8}\}} + \delta_{\{\frac{5}{8}\}} + \delta_{\{\frac{7}{8}\}}\right),$$

....

$$\mu_n = 2^{-n+1} \sum_{k=1}^{2^{n-1}} \delta_{\{\frac{2k-1}{2^n}\}}, \text{ etc.,}$$

where δ denotes the usual Dirac-measure. We shall show that for $A \in \mathcal{J}$, $\mu_k(A)$ converges to $m(A)$, the Lebesgue measure of A. This is clear for intervals and therefore for finite unions of intervals. If $A \in \mathcal{J}$ is arbitrary, then $A^{o} = \bigcup_{n=1}^{\infty} G_n$, where G_n are disjoint open intervals and $C\,\overline{A}$, the complement of the closure of A, is of the form $C\,\overline{A} = \bigcup_{n=1}^{\infty} H_n$, where the H_n are also disjoint open intervals.

As A is Jordan-measurable, $m(A) = m(A^{o}) = \sum_{n=1}^{\infty} m(G_n) = m(\overline{A}) = 1 - \sum_{n=1}^{\infty} m(H_n)$. Let $\varepsilon > 0$ be given and choose N such that $\sum_{n=1}^{N} m(G_n) + \sum_{n=1}^{N} m(H_n) > 1 - \varepsilon$. Then let K be such that $\forall k > K$

$$| \, m \, (\bigcup_{n=1}^{N} G_n) - \mu_k \, (\bigcup_{n=1}^{N} G_n) | < \varepsilon$$

and $| \, m \, (\bigcup_{n=1}^{N} H_n) - \mu_k \, \bigcup_{n=1}^{N} H_n) | < \varepsilon$

Then $\forall k > K$ we have

$$\mu_k(A) \geq \mu_k(\bigcup_{n=1}^{N} G_n) \geq m(\sum_{n=1}^{N} G_n) - \varepsilon \geq m(A) - 2\varepsilon,$$

and $$\mu_k(A) \leq 1 - \mu_k(\bigcup_{n=1}^{N} H_n) \leq 1 - m(\bigcup_{n=1}^{N} H_n) + \varepsilon \leq m(A) + 2\varepsilon,$$

so $|m(A) - \mu_k(A)| < 2\varepsilon$.

But the sequence $\{\mu_k\}_{k=1}^{\infty}$ is not uniformly exhaustive as the sets $\{\{\frac{1}{2}\}\},\{\{\frac{1}{4}\} , \{\frac{3}{4}\}\},\{\{\frac{1}{8}\},\{\frac{3}{8}\},\{\frac{5}{3}\},\{\frac{7}{8}\}\},\ldots,$form a family of disjoint sets $\{A_k\}_{k=1}^{\infty}$ in J such that $\mu_1(A_k) = \delta_{k,1}$.

This readily shows, that J does not have (G).

□

3.3 Proposition. J has (N).

Proof: Suppose there is a sequence $\{\mu_n\}_{n=1}^{\infty} \in B(J)^*$ such that $\{\mu_n(A)\}_{n=1}^{\infty}$ is bounded $\forall A \in J$ and $\|\mu_n\| \to \infty$. By compactness of $[0,1]$ there is $t_o \in [0,1]$ such that, for every $k \in \mathbb{N}$

$$\{|\mu_n|(\,]t_o - \tfrac{1}{k}, t_o + \tfrac{1}{k}[\, \cap [0,1])\}_{n=1}^{\infty}$$

is unbounded.

Now we adapt the proof of Nikodym's theorem as presented in [D-U 77]: We may find a partition (E_1,F_1) of $[0,1]$ into disjoint members of J and an integer n_1 such that

$$|\mu_{n_1}(E_1)|,\ |\mu_{n_1}(F_1)| > 2.$$

At least one of

$$\inf_{k\in\mathbb{N}} \sup_{n\in\mathbb{N}} \sup_{E\in J} |\mu_n(E \cap E_1 \cap]t_o - \tfrac{1}{k}, t_o + \tfrac{1}{k}[)|$$

and
$$\inf_{k\in\mathbb{N}} \sup_{n\in\mathbb{N}} \sup_{E\in J} |\mu_n(E \cap F_1 \cap]t_o - \tfrac{1}{k}, t_o + \tfrac{1}{k}[)|$$

is infinite. If the former is infinite, set $S_1 = E_1$ and $T_1 = F_1$; otherwise set $S_1 = F_1$ and $T_1 = E_1$. In any case there is an $n_2 > n_1$ and (E_2,F_2), a disjoint partition of $S_1 \cap]t_o - \frac{1}{2}, t_o + \frac{1}{2}[$, such that

$$|\mu_{n_2}(E_2)|,|\mu_{n_2}(F_2)| > 3 + |\mu_{n_2}(T_1)|.$$

Now at least one of

$$\inf_{k\in\mathbb{N}} \sup_{n\in\mathbb{N}} \sup_{E\in J} |\mu_n(E \cap E_2 \cap]t_o - \tfrac{1}{k}, t_o + \tfrac{1}{k}[)|$$

and
$$\inf_{k\in\mathbb{N}} \sup_{n\in\mathbb{N}} \sup_{E\in J} |\mu_n(E \cap F_2 \cap]t_o - \tfrac{1}{k}, t_o + \tfrac{1}{k}[)|$$

is infintie. If the former, set $S_2 = E_2$ and $T_2 = F_2$; otherwise, set $S_2 = F_2$ and $T_2 = E_2$.

Continue in this fashion, obtaining a sequence $\{T_n\}_{n=1}^{\infty}$ of pairwise disjoint members of J, such that $T_n \subseteq]t_o - \frac{1}{n}, t_o + \frac{1}{n}[$ and a strictly increasing sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ such that, for each $k \geq 1$,

$$|\mu_{n_k}(T_k)| > \sum_{j=1}^{k-1} |\mu_{n_k}(T_j)| + k + 1.$$

Now make the crucial observation: for any subsequence $\{T_{k_l}\}_{l=1}^{\infty}$, $\bigcup_{l=1}^{\infty} T_{k_l} \in J$. Indeed

$$\overline{\bigcup_{l=1}^{\infty} T_{k_l}} \subseteq \bigcup_{l=1}^{\infty} \overline{T_{k_l}} \cup \{t_o\}$$

and $$\left(\bigcup_{l=1}^{\infty} T_{k_l}\right)^o \supseteq \bigcup_{l=1}^{\infty} T_{k_l}^o \quad .$$

So $$\overline{\bigcup_{l=1}^{\infty} T_{k_l}} \setminus \left(\bigcup_{l=1}^{\infty} T_{k_l}\right)^o \subseteq$$

$$\left(\overline{\bigcup_{l=1}^{\infty} T_{k_l}} \cup \{t_o\}\right) \setminus \bigcup_{l=1}^{\infty} T_{k_l}^o \subseteq$$

$$\bigcup_{l=1}^{\infty} \left(\overline{T_{k_l}} \setminus T_{k_l}^o\right) \cup \{t_o\}.$$

This implies $\bigcup_{l=1}^{\infty} T_{k_l} \in J$. So we may form any union $\bigcup_{l=1}^{\infty} T_{k_l}$ without leaving J and the sequel of the proof of [D-U 77] carries over word for word.

□

<u>Remark</u>: We have proved proposition 3.3 for the Jordan-measurable sets with respect to Lebesgue-measure on [0,1]. But the only essential things, that were needed, was the compactness argument and the fact that the point $\{t_o\}$ has Lebesgue - measure zero. Having this in mind one can easily adapt the above proof to more general circumstances: Let S be a completely regular topological space and m

a σ-finite positive Radon-measure on S without atoms (i.e. for $t \in S$, $m(\{t\}) = 0$). The algebra $J_m(X)$ of subsets A of X with $m(\bar{A} \setminus A^o) = 0$ has (N).

But note, that the situation changes drastically if m has atoms. For example let $m = \delta_{\{0\}}$ be the Dirac-measure at zero on $\mathbb{R}$. Then $J_m(X)$ are the subsets A of $\mathbb{R}$ such that $\{0\} \notin \bar{A} \setminus A^o$. In other words $A \in J_m(X)$ iff $\{0\}$ is either an interior point of A or of CA. So the sequence $\mu_n = n. \ (\delta\{\frac{1}{n}\} - \delta\{\frac{1}{n-1}\})$ is unbounded in norm while $\{\mu_n(A)\}_{n=1}^{\infty}$ is finally zero for each $A \in J_m(X)$.

Let us now state two corollaries of popositon 3.3, which we formulate for Lebesgue-measure on $[0,1]$ but which of course may be generalized as above. It is interesting that the field of finite unions of intervalls is not sufficient for the forthcoming corollary to hold.

<u>3.4 Corollary:</u> A subset B of $L^1[0,1]$ is bounded iff for every $A \in J$ the set $\{\int_A f \, dm : f \in B\}$ is bounded.

□

<u>3.5 Corollary:</u> The space of simple Riemann-integrable functions on $[0,1]$, equipped with the supremum norm, is barreled.

□

<u>3.6 Definition.</u> ([D 78]): An algebra F is called <u>up-down semi-complete</u> if for every disjoint sequence $\{A_n\}_{n=1}^{\infty}$ in F such that $\bigvee_{n=1}^{\infty} A_n$ exists in F, $\bigvee_{k=1}^{\infty} A_{n_k}$ also exists in F for every subsequence $\{n_k\}_{k=1}^{\infty}$.

For example if Ω is an F-space or, equivalently $\mathcal{F}$ satisfies Seever's interpolation property then $\mathcal{F}$ is up-down semi-complete [S 68]. Dashiell [D 78] has shown that an "up-down semi-complete" algebra which satisfies a mild additional lattice property satiesfies (R), (VHS), (G) and (N), and has applied these results to lattices of Baire functions.

It is clear that we have also

3.7 Proposition. An up-down semicomplete algebra $\mathcal{F}$ satisfies (OP).

□

3.8 Proposition. $\mathcal{J}$ is up-down semi-complete.

Proof: Let $\{A_n\}_{n=1}^{\infty}$ be a disjoint sequence in $\mathcal{J}$ such that $\bigvee_{n=1}^{\infty} A_n \in \mathcal{J}$. First note that, since the one-point-sets $\{t\}$ belong to $\mathcal{J}$, $\bigvee_{n=1}^{\infty} A_n$, the minimal set in $\mathcal{J}$ majorizing all A_n, coincides with the set-theoretic union $\bigcup_{n=1}^{\infty} A_n$. By adding $[0,1] \setminus \bigcup_{n=1}^{\infty} A_n$ to the sequence $\{A_n\}_{n=1}^{\infty}$ we may suppose $\bigcup_{n=1}^{\infty} A_n = [0,1]$. Whence

$$\sum_{n=1}^{\infty} m(A_n) = \sum_{n=1}^{\infty} m(A_n^{o}) = 1.$$

For a subsequence $\{n_k\}_{k=1}^{\infty}$ the boundary of the set $\bigcup_{k=1}^{\infty} A_{n_k}$ is given by $Bd = \overline{\bigcup_{k=1}^{\infty} A_{n_k}} \setminus (\bigcup_{k=1}^{\infty} A_{n_k})^{o}$. For every $n \in \mathbb{N}$, $Bd \cap A_n^{o} = \phi$. Indeed, if n occurs in $\{n_k\}_{k=1}^{\infty}$ then $A_n^{o} \subseteq (\bigcup_{k=1}^{\infty} A_{n_k})^{o}$, if it does not, then $\overline{\bigcup_{k=1}^{\infty} A_{n_k}} \cap A_n^{o} = \phi$. This shows that $m(Bd) = 0$, i.e., $\bigcup_{k=1}^{\infty} A_{n_k}$ belongs to $\mathcal{J}$ and it is evidently the Boolean supremum of the sequence $\{A_{n_k}\}_{k=1}^{\infty}$.

□

Remark: This answers the question, raised in ([D 78]), if up-down semi-completness implies (R) in the negative. $\mathcal{J}$ also provides a negative answer to the question, raised in [S 68] and [D-F-H 75], if (N) and (VHS) are equivalent. Finally I want to note that proposition 3.8 of course also generalizes to the hypothesis in the remark following proposition 3.3.

3.9 Proposition. There is a quotient-algebra of $\mathcal{J}$ that does not satisfy (OP) and therefore cannot be up-down semi-complete. Whence neither (OP) nor up-down semi-completness is inherited by quotients.

Proof: Let I be the ideal in $\mathcal{J}$ of sets that contain no dyadic point other that 0 or 1. Consider $\{\mu_n\}_{n=1}^{\infty}$ as defined in the proof of proposition 3.2 and note that the μ_n all vanish on I.

So, if $\tilde{\mathcal{J}}$ denotes the quotient-algebra $\mathcal{J}/I$, the μ_n are well defined on $\tilde{\mathcal{J}}$. Define

$$\mu : \tilde{\mathcal{J}} \longmapsto c_o$$

$$\tilde{A} \longmapsto \{\mu_{2n}(\tilde{A}) - \mu_{2n-1}(\tilde{A})\}_{n=1}^{\infty}.$$

It follows from the proof of 3.2 that μ takes its values in c_o. Evidently μ is weakly σ-additive as on the bounded sets of c_o the weak topology concides with the coordinate-wise topology.

Define $A_1 = \{\frac{1}{2}\}, \ldots, A_n = \{\frac{1}{2^n}, \frac{3}{2^n}, \ldots, \frac{2^n-1}{2^n}\}, \ldots,$ which are elements of $\mathcal{J}$ and denote by $\tilde{A}_n$ their images in $\tilde{\mathcal{J}}$. Then $\bigvee_{n=1}^{\infty} \tilde{A}_n = \widetilde{[0,1]}$ but $\sum_{n=1}^{\infty} \mu(\tilde{A}_n)$ of course does not converge strongly in c_o.

□

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3.10 Corollary: (N) $\not\Rightarrow$ (OP).

Proof: The algebra $\tilde{J}$, constructed in 3.9, has (N) (by 3.3 and 2.11) but not (OP).

□

Summing up: (N) implies neither (VHS), nor (G), nor (OP). Also (OP) implies neither (VHS) nor (G). That (OP) does not imply (N) will be shown in 3.16. Unfortunately, we leave open the question of wether (G) implies (VHS).

We still show another property of the Stone space of J.

3.11 Proposition. Let Ω be the Stone space of J and D an infinite subset of Ω. Then there is a sequence $\{x_n\}_{n=1}^{\infty}$ in D such that the closure of $\{x_n\}_{n=1}^{\infty}$ in Ω is homeomorphic to $\beta\mathbb{N}$, the Stone - Cech - compactification of $\mathbb{N}$. Hence every infinite closed subset of Ω contains a copy of $\beta\mathbb{N}$.

Proof: It will be convenient to make the following convention. If A is in J, we write A, if we consider it as a member of the field of subsets of $[0,1]$, and write $\tilde{A}$, if we consider it as a clopen subset of Ω.

By the compactness of $[0,1]$ there is for $x \in \Omega$ a unique "localisation point", i.e. a point $t \in [0,1]$ such that for every neighbourhood V of t, $V \in J$, x is contained in $\tilde{V}$. Hence we can choose a sequence $\{x_n\}_{n=1}^{\infty}$ in D so that the localisation points t_n of x_n converge to some $t_o \in [0,1]$. It is easy to construct a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and a disjoint sequence $\{A_k\}_{k=1}^{\infty}$ in J

such that $x_{n_k} \in \tilde{A}_k$ and each A_k is contained in $]t_o - \frac{1}{k}, t_o + \frac{1}{k}[$. Relabel x_{n_k} by x_k.

If N_1, N_2 is a partition of $\mathbb{N}$, let $B_1 = \cup\{A_k : k \in N_1\}$ and $B_2 = \cup\{A_k : k \in N_2\}$. Note that B_1 and B_2 are disjoint members of $\mathcal{J}$. Hence $\tilde{B}_1$ and $\tilde{B}_2$ are disjoint clopen sets in Ω such that $\{x_k\}_{k\in N_1} \subseteq B_1$ and $\{x_k\}_{k\in N_2} \subseteq B_2$. In particular

$$\overline{\{x_k\}_{k\in N_1}} \cap \overline{\{x_k\}_{k\in N_2}} = \phi,$$

which proves that $\overline{\{x_k\}_{k=1}^{\infty}}$ is homeomorphic to $\beta\mathbb{N}$.

□

<u>3.12 Definition.</u> An algebra F has property (n-σ) if no non-trivial countable suprema exist in F, i.e., for no sequence $\{A_n\}_{n=1}^{\infty}$ of mutually disjoint, non-zero elements in F, the supremum $\bigvee_{n=1}^{\infty} A_n$ exists in F.

The following proposition is trivial.

<u>3.13 Proposition.</u> (n-σ) $\Rightarrow$ (OP), i.e. if F has (n-σ), F has (OP). □

It is well known that the quotient algebra of $P(\mathbb{N})$ modulo the ideal of finite subsets of $\mathbb{N}$ or, equivalently, the algebra of clopen subsets of $\beta\mathbb{N} \setminus \mathbb{N}$ has property (n-σ) (c.f. [S 71 a], prop. 16.5.6.). However, this algebra also verifies all our other properties (as $\beta\mathbb{N} \setminus \mathbb{N}$ is an F-space [S 68]). So we have to consider more general cases.

3.14 Proposition: Let X be a set and F a field of subsets of X. Suppose F contains all countable sets and let I be the ideal of finite sets. Then F/I has property (n-σ).

Proof: (c.f. [S 71 a] 16.5.6.) Let $\tilde{A}_1 \leq \tilde{A}_2 \leq \tilde{A}_3 \quad \ldots$ be a strictly increasing sequence in $\tilde{F} = F/I$ and suppose $\tilde{A} = \bigcup_{n=1}^{\infty} \tilde{A}_n$ exists in $\tilde{F}$. Choose $A_1, A_2, \ldots$ and A to be representants of $\tilde{A}_1, \tilde{A}_2, \ldots$ and $\tilde{A}$ in F. Of course we may choose the $\{A_n\}_{n=1}^{\infty}$ to be increasing in F too. Then, for every $n \in \mathbb{N}$, $A_n \setminus A$ is finite, while $A_{n+1} \setminus A_n$ is infinite (the $\tilde{A}_n$ are strictly increasing). Therefore there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in A s.t. $x_n \in A_{n+1} \setminus A_n$. Let $B = A \setminus \{x_n\}_{n=1}^{\infty}$, which belongs to F, and let $\tilde{B}$ be its image in $\tilde{F}$. Then $\tilde{A}_n \leq \tilde{B}$ for every $n \in \mathbb{N}$, while $\tilde{B}$ is strictly smaller than $\tilde{A}$, a contradiction.

□

3.15 Let F be the field of subsets of $[0,1]$ generated by the dyadic intervals $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ and the countable sets. Let I be the ideal of finite sets and $\tilde{F} = F/I$ the quotient algebra. By 3.14 $\tilde{F}$ has property (n-σ) and therefore, by 3.13, (OP).

3.16 Proposition. $\tilde{F}$ does not have (N) or (G). Hence (OP) $\not\Rightarrow$ (N).

Remark: The second part of the proposition is the result promised after 3.10.

Proof: For $n \geq 1$ define the measures

$$\mu_n = n^2 . (\chi_{[0,\frac{1}{n}]} . m - \chi_{[\frac{1}{n},\frac{2}{n}]} . m)$$

on F (m denoting the Lebesgue measure). Clearly $\{\mu_n(A)\}_{n=1}^{\infty}$ is eventually zero for every $A \in F$, while $\|\mu_n\| = 2n$ tends to infinity. As the μ_n all vanish on I, they factor through $\tilde{F}$, which readily shows that $\tilde{F}$ does not have (N). Considering $\nu_n = n^{-1} . \mu_n$, we may conclude that $\tilde{F}$ does not have (G) either.

□

3.17 In the above example the property (OP) was implied by the fact, that because of (n-σ) every measure is trivially σ-additive (regardless with respect to which topology). Another way of finding algebras having (OP) is to construct examples of algebras F on which no real valued measure (exept $\mu = 0$) is σ-additive.
Indeed a Banach valued measure on F, which is weakly σ-additive, then is necessarily identically zero.

It is an old result (c.f. [H 63], lemma 15.4.) that the complete Boolean algebra of regular open subsets of [0,1] has the property that every σ-additive real valued measure on it vanishes identically. Accidentally, this algebra being complete satisfies all our properties. But there are other algebras with this property.

3.18 Proposition. Let F be the algebra of clopen subsets of $\{0,1\}^{\mathbb{N}}$. Every σ-additive real valued measure on F vanishes identically. Hence F has (OP).

Proof: Let $\mu \in B(F)^*$, $\mu \neq 0$, and suppose for simplicity $\mu \geq 0$. As $B(F)$ may be identified with $C(\Delta)$, the space of continuous functions on $\Delta = \{0,1\}^{\mathbb{N}}$, μ is the restriction of a positive Radon-measure on Δ, denoted $\bar{\mu}$, to the clopen sets.

If $\bar{\mu}$ has atoms, there exists $x_o \in \Delta$ such that $\bar{\mu}(\{x_o\}) > 0$. Represent $\Delta \setminus \{x_o\}$ as a countable disjoint union of clopen sets $\{A_n\}_{n=1}^{\infty}$. Note that $\bigvee_{n=1}^{\infty} A_n = \Delta$ (the supremum being taken in F). But

$$\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \bar{\mu}(A_n) = \bar{\mu}(\Delta \setminus \{x_o\}) < \bar{\mu}(\Delta) = \mu(\Delta);$$

therefore μ is not σ-additive.

If $\bar{\mu}$ has no atom, let $\{x_n\}_{n=1}^{\infty}$ be a dense sequence in Δ and choose inductively clopen neighbourhoods A_n of x_n s.t. $\mu(A_n) < 2^{-n}\cdot \mu(\Delta)$. Putting $B_n = A_n \setminus (A_1 \vee \ldots \vee A_{n-1})$, we again obtain a disjoint sequence $\{B_n\}_{n=1}^{\infty}$ in F, s.t. $\bigvee_{n=1}^{\infty} B_n = \Delta$, while $\sum_{n=1}^{\infty} \mu(B_n) < \mu(\Delta)$.

For the case of signed μ, the above argument adapts easily.

□

On the other hand, it is plain to show that F does not have (N) and (G). (Compare the proof of 3.16, noting that F may also be represented as the field of subsets of $[0,1[$ generated by the dyadic intervals $[\frac{k}{2^n}, \frac{k+1}{2^n}[$.

Hence again we get an example of an algebra having (OP) but none of our other properties.

§ 4 SOME SPECIAL CLASSES OF BOOLEAN ALGEBRAS

In the first part of this section we exhibit a relatively broad class of Boolean algebras satisfying all our properties, while in the second part we exhibit another relatively broad cladd, that satisfies non of them exept possibly (OP).

4.1 There have been constructed many examples of non-σ-complete algebras F, on which (R) and (VHS) hold ([A 62],[I-S 63], [S 68]). One takes for examples $\beta\mathbb{N}$ and glues together 2 points of $\beta\mathbb{N} \setminus \mathbb{N}$. The quotient space K thus obtained does not have nice disconnectedness-properties (it is not even an F-sapce), while the space $C(K)$ may be identified with a closed hyperplane of l^∞ and is therefore isomorphic to l^∞. The situation carries over to the respective Boolean algebras of clopen sets. We do not carry out the details here, as we shall consider below a more general case.

4.2 Definition. An algebra F has property (E) if for every sequence $\{A_n\}_{n=1}^\infty$ of mutually disjoint elements of F there is a subsequence $\{A_{n_k}\}_{k=1}^\infty$ such that for every subsequence $\{A_{n_{k_l}}\}_{l=1}^\infty$ the supremum $\bigvee_{l=1}^\infty A_{n_{k_l}}$ exists in F.

4.3 Proposition. (E) $\Rightarrow$ (R) and (E) $\Rightarrow$ (VHS), i.e. an algebra having (E) has (R) and (VHS).

Proof: Just copy the proofs of (R) and (VHS) in the case of σ-complete Boolean algebras (c.f. [D-U 77], th.I.4.2. and th.I.4.8.), except passing once more to a subsequence in the respective proofs.

□

4.4 Proposition. Let Σ be a σ-complete Boolean algebra and F a subalgebra such that $B(F)$ may be represented as a countable intersection of hyperplanes of $B(\Sigma)$. Then F has (E).

Proof: We adapt one of the many proofs that c_o is not complemented on l^∞ (c.f. [J 74]. prop. 29.19.).
By hypothesis there is a sequence $\{\mu_i\}_{i=1}^{\infty}$ in $B(\Sigma)^*$ such that

$$B(F) = \bigcap_{i=1}^{\infty} \{\mu_i^{-1}(\{0\})\}.$$

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of mutually disjoint elements of F, let Σ_1 be the sub-σ-algebra of Σ generated by $\{A_n\}_{n=1}^{\infty}$ and denote by $\bar{\mu}_i$ the restriction of μ_i to Σ_1 and by $|\bar{\mu}_i|$ the variation - measure of $\bar{\mu}_i$. Clearly for every $n \in \mathbb{N}$ and $i \in \mathbb{N}$, $\bar{\mu}_i(A_n) = |\bar{\mu}_i|(A_n) = 0$.

Let $\{B_\alpha\}_{\alpha\in I}$ be an uncountable family of infinite subsets of $\mathbb{N}$ such that, for $\alpha_1 \neq \alpha_2$, $B_{\alpha_1} \cap B_{\alpha_2}$ is a finite subset of $\mathbb{N}$ (c.f. [J 74], lemma 29.18., for example). For $\alpha \in I$ let $C_\alpha = \bigvee_{n\in B_\alpha} A_n$, an element of Σ_1. For $i \in \mathbb{N}$ and $\alpha_1 \neq \alpha_2$ we have

$$\begin{aligned} |\bar{\mu}_i|(C_{\alpha_1}) + |\bar{\mu}_i|(C_{\alpha_2}) &= |\bar{\mu}_i|(C_{\alpha_1} \vee C_{\alpha_2}) + |\bar{\mu}_i|(C_{\alpha_1} \wedge C_{\alpha_2}) \\ &= |\bar{\mu}_i|(C_{\alpha_1} \vee C_{\alpha_2}), \end{aligned}$$

because $C_{\alpha_1} \wedge C_{\alpha_2}$ is a finite union of the A_n's and therefore $|\bar{\mu}_i|$ vanishes on it.

It follows that $\forall i \geq 1$ and $\forall \varepsilon > 0$ there are only finitely many $\alpha \in I$ such that $|\bar{\mu}_i|(C_\alpha) > \varepsilon$. So there are at most countable many $\alpha \in I$ such that for some $i \geq 1, |\bar{\mu}_i|(C_\alpha) \neq 0$.

Let α_o be such that $|\bar{\mu}_i|(C_{\alpha_o}) = 0$ for every $i \in \mathbb{N}$ and let $\{n_k\}_{k=1}^{\infty}$ be an increasing enumeration of the set of integers in B_{α_o} For every subsequence $\{n_{k_l}\}_{l=1}^{\infty}$ and every $i \in \mathbb{N}$, we have $|\bar{\mu}_i|(\bigvee_{l=1}^{\infty} A_{n_{k_l}}) = 0$, whence $\bar{\mu}_i(\bigvee_{l=1}^{\infty} A_{n_{k_l}}) = \mu_i(\bigvee_{l=1}^{\infty} A_{n_{k_l}}) = 0$.
In other words the characteristic function of $\bigvee_{l=1}^{\infty} A_{n_{k_l}}$ belongs to every $\mu_i^{-1}(\{0\})$, whence $\bigvee_{l=1}^{\infty} A_{n_{k_l}}$ belongs to F.

□

4.5 We now consider a class of Boolean algebras, that behaves badly with respect to the properties we are interested in, namely, algebras which are unions of a strictly increasing sequence of subalgebras. These objects arise naturally in the context of martingales. It is a standard exercise which can be found in most probability text-books, that these algebras may fail to be σ-complete. It was even shown [B-H 76], that these algebras are never σ-complete. Our approach furnishes an easy proof of this curious fact as well as stronger results. In fact, they never satisfy (N) or (G) (and therefore neither (VHS) nor (R)). On the other hand, they may verify (OP).

Let us first show the latter assertion. The algebra $\mathcal{F}$ considered at 3.18 may be represented as $\bigcup_{n=1}^{\infty} \mathcal{F}_n$, where $\mathcal{F}_n$ is the algebra of subsets of Δ depending only on the first n coordinates. We have seen in 3.18 that $\mathcal{F}$ has (OP).

<u>4.6 Proposition.</u> Let $\mathcal{F}$ be a Boolean algebra and suppose $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$, where $\mathcal{F}_n$ is a strictly increasing sequence of subalgebras of $\mathcal{F}$. Then $\mathcal{F}$ does not have (N) nor (G).

For the proof we shall need some lemmas.

<u>4.7 Lemma:</u> Let $(X, \| \ \|)$ be a normed space and suppose $X = \bigcup_{n=1}^{\infty} X_n$ where $\{X_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of closed subspaces of X. Then $(X, \| \ \|)$ is not barreled.

<u>Proof:</u> Let B_n be the ball in X_n around zero with radius $\frac{1}{n}$ and let T be the closed convex hull of $\{B_n\}_{n=1}^{\infty}$. Then T is a barrel but not a neighbourhood of zero in X.

□

<u>4.8 Lemma:</u> ([D-S 58], II.3.12.): Let Y be a subspace of a normed space X. Let $x \in X$ be such that

$$\inf\{|y-x| : y \in Y\} = d > 0.$$

Then there is $x^* \in X^*$ with

$$\langle x,x^* \rangle = 1; \quad \|x^*\| = d^{-1}; \quad \langle y,x^* \rangle = 0 \quad \text{for } y \in Y.$$

□

4.9 Lemma: Let $F = \bigcup_{n=1}^{\infty} F_n$, where F_n is a strictly increasing sequence of subalgebras. Let $\mu \in B(F)^*$, $\mu \geq 0$. Given F_m and $\varepsilon > 0$ there exists $k \in \mathbb{N}$, $k > m$ and $E_k \in F_k \setminus F_m$ so that $\mu(E_k) < \varepsilon$.

Proof: Represent F as the algebra of clopen subsets of the Stone - representation space. Then μ may be extended to a Radon-measure $\bar{\mu}$ on the Borel field of Ω. Decompose $\bar{\mu}$ into its atomic and diffuse part i.e.

$$\mu = \sum_{n=1}^{N} \alpha_n \cdot \delta_{t_n} + \mu_d,$$

where N is zero, a finite number or $+\infty$, $\{t_n\}_{n=1}^{N}$ are points of Ω, $\{\alpha_n\}_{n=1}^{\infty}$ positive scalars and μ_d is a diffuse measure on Ω. Given F_m and $\varepsilon > 0$ we can find a partition of Ω into disjoint clopen sets $\{A_i\}_{i=1}^{p}$ and points $s_i \in A_i$ so that

$$\bar{\mu}(A_i \setminus \{s_i\}) < \varepsilon. \qquad i = 1,..,p.$$

Find n so large that $\{A_i\}_{i=1}^{p} \in F_n$, take $k > \max(n,m)$ and $B_k \in F_k \setminus F_{k-1}$. For some i_o, $1 \leq i_o \leq p$, $B_k \cap A_{i_o} \notin F_{k-1}$ (otherwise B_k would belong to F_{k-1}). If $s_{i_o} \notin B_k \cap A_{i_o}$, let $E_k = B_k \cap A_{i_o}$, if not, let $E_k = C\, B_k \cap A_{i_o}$. In any case $E_k \in F_k \setminus F_{k-1}$ and

$$\mu(E_k) < \varepsilon.$$

□

Proof of prop. 4.6: As regards (N), remark that $B_s(F) = \bigcup_{n=1}^{\infty} B_s(F_n)$ cannot be barreled by 4.7, whence

$\mathcal{F}$ does not have property N_2 of definition 2.4.

To show that $\mathcal{F}$ does not have (G) we shall define inductively a sequence $\{\mu_i\}_{i=1}^{\infty}$ of measures on $\mathcal{F}$, and a strictly increasing sequence $\{n_i\}_{i=1}^{\infty}$ such that

$$\mu_{i|\mathcal{F}_{n_{i-1}}} \equiv 0 \ ; \quad \|\mu_i\| \leq 2$$

and $\{\mu_i\}_{i=1}^{\infty}$ is not uniformly exhaustive. This contradicts (G).

<u>Induction:</u> Take $\mathcal{F}_{n_o} = \mathcal{F}_1$, $\mathcal{F}_{n_1} = \mathcal{F}_2$, $E_1 \in \mathcal{F}_{n_1} \setminus \mathcal{F}_{n_o}$. Since

$$\inf\{\|f - \chi_{E_1}\| : f \in B_s(\mathcal{F}_{n_o})\} = 1/2$$

by lemma 4.8, we may find $\mu_1 \in B(\mathcal{F})^*$ such that

$$\mu_1(E_1) = 1 \ ; \ \|\mu_1\| = 2 \ ; \ \mu_1 \equiv 0 \quad \text{on} \quad \mathcal{F}_{n_o}.$$

Suppose $(E_1,..,E_{k-1})$, $(\mathcal{F}_{n_o},...,\mathcal{F}_{n_{k-1}})$ and $(\mu_1,..,\mu_{k-1})$ have been already chosen. Define

$$\nu_k = \sum_{i=1}^{k-1} 2^{-i} \ |\mu_i| .$$

Applying 4.9 we may find $n_k > n_{k-1}$, $E_k \in \mathcal{F}_{n_k} \setminus \mathcal{F}_{n_{k-1}}$, $\nu_k\ (E_k) < k^{-1}$. Applying 4.8 we may find $\mu_k \in B(\mathcal{F})^*$ such that

$$\mu_k(E_k) = 1; \ \|\mu_k\| = 2 \ ; \ \mu_k \equiv 0 \text{ on } \mathcal{F}_{n_{k-1}}.$$

This finishes the inductive construction.

It remains to show that the sequence $\{\mu_k\}_{k=1}^{\infty}$

is not uniformly exhaustive. Consider

$$\nu = \sum_{i=1}^{\infty} 2^{-i} \, |\mu_i| .$$

All μ_k are ν-continuous $(\nu(A_i) \to 0 \Rightarrow \mu_k(A_i) \to 0)$ for each k separately. It is known that $\{\mu_k\}$ is uniformly exhaustive iff $\{\mu_k\}$ is uniformly ν-continuous (see e.g. [D 72], II.6.2.). However we have

$$\nu(E_k) = \nu_k(E_k) + \sum_{i=k}^{\infty} 2^{-i} |\mu_i|(E_k) \to 0,$$

with k to infinity, while $\mu_k(E_k) = 1$ for all k.

□

§ 5 THE GROTHENDIECK - PROPERTY

We first recall known results.

5.1 Theorem: For a Boolean algebra F the following are equivalent.

(i) F has (G).

(ii) Every continuous operator $T : B(F) \to c_o$ is weakly compact.

(ii)' Every continuous operator $T : B(F) \to X$ into a weakly compactly generated (abbreviated WCG) Banach space X is weakly compact.

(iii) If $T_n : B(F) \to X$ is a sequence of weakly compact operators into a Banach space X, converging in the strong operator topology, then the limit $T = \lim T_n$ is weakly compact.

(iii)' As (iii), replacing "strong operator topology" by "weak operator topology".

(iv) If $T : B(\mathcal{F}) \to X$ is a continuous operator into a separable Banach space X and $B(\mathcal{F})$ is a subspace of a Banach space Z, then there is a norm - preserving extension of T to Z.

(v) If $T : B(\mathcal{F}) \to X$ is a continuous operator to a Banach space X, which is not weakly compact, then T fixes a copy of $C[0,1]$, i.e. there is a subspace of $B(\mathcal{F})$ isomorphic to $C[0,1]$ on which T reduces to an isomorphism.

The equivalence of (i) to (iii)' may be found in [D 73] and the equivalence of (i) and (iv) in [L 64]. The remarkable result that (i) implies (v) is due to Diestel and Seifert [D-S 78], while it is plain that (v) implies (ii).

We now show that property (G) is equivalent to a property very similar to (OP) (a result of the type "weak implies strong") but which may be stated in terms of finitely additive measures.

<u>5.2 Theorem.</u> $\mathcal{F}$ has (G) iff

(vi) For every bounded measure $\mu : \mathcal{F} \to X$, such that there is some bounded positive measure η on $\mathcal{F}$ with $x^* \circ \mu \ll \eta$ for every $x^* \in X^*$ it follows that $\mu \ll \eta$.

Here $x^* \circ \mu \ll \eta$ (resp. $\mu \ll \eta$) means that for $\varepsilon > 0$ there is $\delta > 0$ such that $\mu(A) < \delta$ implies $|x^* \circ \mu(A)| < \varepsilon$ (resp., $\|\mu(A)\| < \varepsilon$).

<u>Proof:</u> (v) $\Rightarrow$ (vi) Suppose μ, η, X given as above. We represent F as the algebra of clopen sets of the Stone space Ω, whence η and $x^* o \mu$ extend to Radon measures on Ω. Let $T : C(\Omega) \to X$ be the integration operator associated to μ.

<u>Claim:</u> T is weakly compact.

Indeed, if it were not so by (v) we could find a subspace Z of $C(\Omega)$, isomorphic to $C[0,1]$, such that $T_{|Z}$ is an isomorphism onto its image. Denote $T_o = T_{|Z}$ and $Y = T_o(Z)$. By the Radon-Nikodym theorem the adjoint T^* transforms X^* into $L^1(\eta)$ which is a WCG space. On the other hand $T_o^*(Y^*) = Z^*$ is isomorphic to $M[0,1]$, where $M[0,1]$ is the space of Radon measures on $[0,1]$, which is not WCG. Consider the continuous quotient map $q : M(\Omega) \to Z^*$ and $L^1(\eta) \subset M(\Omega)$. We have $q(L^1(\eta)) = Z^*$. Indeed T^* factors through $L^1(\eta)$ and $q \circ T^*$ is onto Z^*. Hence Z^* must be WCG, a contradiction proving the claim.

Therefore $T : C(\Omega) \to X$ is weakly compact and T^* (ball (X^*)) is a weakly compact set in $L^1(\eta)$. Hence

$$\lim_{\eta(A)\to 0} |x^* o \mu (A)| = 0,$$

uniformly in $\|x^*\| \leq 1$, in other words

$$\lim_{\eta(A)\to 0} \|\mu(A)\| = 0.$$

(vi) ⇒ (ii): A continuous operator $T : B(\mathcal{F}) \to c_o$ is given coordinate-wise by a sequence $\{\mu_n\}_{n=1}^{\infty}$ in $B(\mathcal{F})^*$ tending weak* to zero. Putting

$$\eta = \sum_{n=1}^{\infty} 2^{-n} |\mu_n|,$$

η satisfies the assumptions of (vi). Hence

$$\lim_{\eta(A)\to 0} \|\mu(A)\| = 0,$$

which means that $\{\mu_n\}_{n=1}^{\infty}$ is uniformly exhaustive thus weakly tending to zero (prop. 1.2). This is equivalent to the fact that T is a weakly compact operator (c.f. [D 73]).

□

<u>5.3 Proposition.</u> $\mathcal{F}$ has (G) iff

(vii) There is no subspace of $B(\mathcal{F})$ isometric to c_o and complemented in $B(\mathcal{F})$.

<u>Proof:</u> (ii) ⇒ (vii) is trivial.

(vii) ⇒ (i): If $\mathcal{F}$ has not (G) we can find a sequence $\{\mu_n\}_{n=1}^{\infty}$ in $B(\mathcal{F})^*$ converging weak* to zero and a disjoint sequence $\{E_n\}_{n=1}^{\infty}$ in $\mathcal{F}$ such that $|\mu_n(E_n)| > \varepsilon$ for some $\varepsilon > 0$. By multiplying with a bounded sequence of scalars we may assume $\mu_n(E_n) = 1$. Denote by $\mathcal{E}$ the ring generated by $\{E_n\}_{n=1}^{\infty}$ in $\mathcal{F}$. Then $B(\mathcal{E})$ is a subspace of $B(\mathcal{F})$ isometric to c_o. Define an operator $T : B(\mathcal{F}) \to c_o = X$ on the indicator-functions by $T(\chi_A) = \{\mu_n(A)\}_{n=1}^{\infty}$ and extend it by linearity and continuity to

$B(\mathcal{F})$. By [D 76] there is an infinite set $M \subseteq \mathbb{N}$ such that $T_o = T_{|B(\mathcal{F}_M)}$ is an isomorphism onto its image, where $\mathcal{F}_M$ is the ring generated by $\{E_n, n \in M\}$. By a theorem of Sobczyk (see e.g. [J 74], 29.22.) there is a continuous projection P_1 from X onto $T_o(B(\mathcal{F}_M))$. Hence $T_o^{-1} P_1 T$ is a continuous projection from $B(\mathcal{F})$ onto $B(\mathcal{F}_M)$, an isometric copy of c_o. □

Remark: Proposition 5.3 may be rephrased as follows: If $\mathcal{F}$ does not have (G) there is a sequence $\{\mu_n\}_{n=1}^{\infty}$ in $B(\mathcal{F})^*$, tending weak* to zero, and a disjoint sequence $\{E_n\}_{n=1}^{\infty}$ in $\mathcal{F}$ so that

$$\mu_n(E_m) = \delta_{n,m},$$

$\delta_{n,m}$ denoting the Kronecker-symbol. Indeed, the projection $T_o^{-1} P_1 T$ constructed above, viewed as an operator into c_o, defines such a sequence $\{\mu_n\}_{n=1}^{\infty}$.

Note, however, that it follows from 3.11, that $\{\mu_n\}_{n=1}^{\infty}$ may not, in general, be replaced by a sequence $\{\delta_{x_n}\}_{n=1}^{\infty}$ of Dirac measures on Ω.

5.4 We say that a bounded set $\{f_i\}_{i\in I}$ in a Banach space X is equivalent to the l^1-basis, iff the continuous operator $T : l_I^1 \longrightarrow X$, taking the unit-vectors onto the corresponding f_i's, is an isomorphism; equivalently, if there exists a constant $\alpha > 0$, such that for every choice of scalars $\mu_1, \ldots, \mu_n$ and $f_{i_1}, \ldots, f_{i_n}$ we have $\| \sum_{k=1}^{n} \mu_k f_{i_k} \| \geq \alpha \cdot \sum_{k=1}^{n} |\mu_k|$.

We say that $\{f_i\}_{i\in I}$ is equivalent to the l^1-basis and complemented, if in addition the space spanned by $\{f_i\}_{i\in I}$ is complemented in X.

Note also, that it follows from the special nature of the l^1-norm, that if $T : X \to Y$ is a continuous operator and $\{f_i\}_{i\in I}$ in Y is equivalent to an l^1-basis and there exists a bounded sequence $\{e_i\}_{i\in I}$ in X, s.t. $T(e_i) = f_i$, then $\{e_i\}_{i\in I}$ is also equivalent to an l^1-basis and T reduces to an isomorphism on the span of $\{e_i\}_{i\in I}$. Moreover, if $\{f_i\}_{i\in I}$ is assumed to be complemented, then $\{e_i\}_{i\in I}$ is complemented.

5.5 Proposition. F has (G) iff

(viii) Every weak* compact convex subset of $B(F)^*$ which is not weakly compact contains a family $\{f_t\}_{t\in[0,1]}$ equivalent to the l^1-basis and complemented in $B(F)^*$.

Proof: (viii) $\Rightarrow$ (vii): If (vii) does not hold there is a complemented subspace of $B(F)$, isomorphic to c_o. Hence l^1 is isomorhic to a weak*-complemented and therefore weak*-closed subspace of $B(F)^*$. The intersection of this space with the unit ball is weak*-compact and convex but contains no family $\{f_t\}_{t\in[0,1]}$ equivalent to the l^1-basis.

(v) $\Rightarrow$ (viii): Let K in $B(F)^*$ be weak*-compact convex and let $\bar{K}$ be its circled hull, which is weak*-compact again. Consider the Banach space spanned by $\bar{K}$ in $B(F)^*$.

with $\overline{K}$ the unit ball in X. As $\overline{K}$ is weak*-compact, X is a dual Banach space, i.e., there exists a Banach space Y such that X is isometric and weak*-isomorphic to Y^* (see e.g. [C 78]).

Consider the canonical injection from Y^* into $B(F)^*$, $j : Y^* \to B(F)^*$. The injection j is continuous with respect to the weak*-topologies, hence there exists a continuous operator $j_* : B(F) \to Y$ such that $(j_*)^* = j$. If K is not weakly compact, j_* is not a weakly compact operator, so by (v) j_* fixes a copy of $C[0,1]$, i.e. we have the commuting diagramm

$$\begin{array}{ccccc} & & B(F) & \xrightarrow{\ j_*\ } & Y \\ & & \uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle i_2} \\ C[0,1] & \xrightarrow{\ k\ } & V & \xrightarrow{\ l\ } & W \end{array}$$

where i_1, i_2 are isometric injections and k and l are isomorphisms. Transposing we get

$$\begin{array}{ccccc} & & B(F)^* & \xleftarrow{\ j\ } & X \\ & & \downarrow{\scriptstyle i_1^*} & & \downarrow{\scriptstyle i_2^*} \\ M[0,1] & \xleftarrow{\ k^*\ } & V^* & \xleftarrow{\ l^*\ } & W^* \end{array}$$

Note that $k^* i_1^* j$ is an open mapping onto, as $k^* l^* i_2^*$ is. Whence $k^* i_1^* j(\text{ball }(X))$, which is just $k^* i_1^*(\overline{K})$, contains for some $\alpha > 0$ an α-ball around zero in $M[0,1]$. The Dirac measures $\{\delta_t\}_{t\in[0,1]}$ from a complemented l^1-basis in $M[0,1]$.

There exists a sequence $\{f_t\}_{t\in[0,1]}$ in $\overline{K}$ such that $k^* i_1^* (f_t) = \alpha . \delta_{\{t\}}$. It is evident that we may find $\{\tilde{f}_t\}_{t\in[0,1]}$ in K, such that $k^* i_1^* (\tilde{f}_t) = \alpha_t \delta_{\{t\}}$ where $\{\alpha_t\}_{t\in[0,1]}$ is a bounded sequence of scalars such that $|\alpha_t| \geq \alpha$. As the $\{\alpha_t . \delta_{\{t\}}\}_{t\in[0,1]}$ are also equivalent to an l^1-basis and complemented in $M[0,1]$, we conclude by the remark predeeding the proposition that $\{\tilde{f}_t\}_{t\in[0,1]}$ is equivalent to an l^1-basis and complemented in $B(F)^*$.

□

<u>5.6 Corollary:</u> F has (G) iff one of the following two equivalent conditions holds.

(ix) Every weak*-closed subspace Z of $B(F)^*$, which does not contain a complemented subspace isomorphic to $l^1[0,1]$ is reflexive.

(x) Every bounded measure $\mu : F \to X$ is either exhaustive or there is a bounded family $\{x_t^*\}_{t\in[0,1]}$ in X^* such that $\{x_t^* \circ \mu\}_{t\in[0,1]}$ is equivalent to the l^1-basis and complemented on $B(F)^*$.

<u>Proof:</u> We only show (viii) $\Rightarrow$ (ix) : The intersection of Z with the unit ball of $B(F)^*$ is weak*-compact convex. So if it is not weakly compact (i.e., Z is not reflexive), by (viii) it contains a complemented subspace isomorphic to $l^1[0,1]$.

□

To finish this section we remark that prop. 5.5 (as well as corr. 5.6) may be strengthened, if one assumes (R) instead of (G).

5.7 Propositon. If F has (R), then every weak*-compact convex subset of $B(F)^*$ which is not weakly compact contains a family $\{f_i\}_{i\in I}$ equivalent to the l^1-basis and complemented in $B(F)^*$, where I is a set of cardinality $2^{[0,1]}$.

Proof: Just copy the proof of 5.5 replacing $C[0,1]$ by l^∞. The dual $(l^\infty)^*$ may be identified with the Radon measures on the Stone-Cech-compactification of $\mathbb{N}$; the Dirac measures on $\beta\mathbb{N}$ are equivalent to the l^1-basis and complemented in $(l^\infty)^*$, hence the rest of the proof carries over, replacing everywhere $[0,1]$ by $\beta\mathbb{N}$, a set of cardinality $2^{[0,1]}$.

□

§ 6 THE ORLICZ - PETTIS - PROPERTY

In this section we shall show that (G) implies (OP). We first show the result for the special case in which the unit-ball of X^* is weak* sequentially compact (e.g. if X is WCG [D 73]), as in this case we may state a stronger result and the proof is very easy.

6.1 Proposition. Let F satisfy (G), and let X be a Banach space s.t. ball (X^*) is weak*-sequentially compact and let $\Gamma \subseteq X^*$ be a norming subset of X^*.

If a measure $\mu : F \to X$ is such that $x^* \circ \mu$ is σ-additive $\forall x^* \in \Gamma$, then μ is strongly σ-additive.

Proof: Let $\{A_n\}_{n=1}^{\infty}$ be a sequence s.t. $\bigvee_{n=1}^{\infty} A_n \in F$. To show that $\sum_{n=1}^{\infty} \mu(A_n) \to \mu(\bigvee_{n=1}^{\infty} A_n)$ in norm it suffices to show that the set of partial sums is relatively norm-compact, as this,together with $\sum_{n=1}^{\infty}\mu(A_n) \to \mu(\bigvee_{n=1}^{\infty} A_n)$ weakly,will produce the result.

If the set of partial sums is not relatively compact, we may find blocks $B_k = \bigvee_{n=n_k+1}^{n_{k+1}} A_n$ and $\alpha > 0$ s.t. $\|\mu(B_k)\| > \alpha$. As Γ is norming we may find $\{x_k^*\}_{k=1}^{\infty} \subseteq \Gamma$ s.t. $\|x_k^*\| = 1$ and $|x_k^* \circ \mu(B_k)| > \alpha$. By hypothesis, we may find a subsequence $\{x_{k_l}^*\}_{l=1}^{\infty}$ that converges weak*. But then $\{x_{k_l}^* \circ \mu\}_{l=1}^{\infty}$ is converging weak* in $B(F)^*$ and by (G) it is uniformly exhaustive, in contradiction to $|x_{k_l}^* \circ \mu(B_{k_l})| > \alpha$ for every l.

□

The following example will be typical, as we shall reduce the general case of an algebra not having (OP) essentially to the sutuation of this example.

6.2 Example: Let F be the algebra of subsets of $\mathbb{N}$ consisting of the sets $A \subseteq \mathbb{N}$ s.t. for all but finitely many $k \in \mathbb{N}$ the pair $\{2k - 1, 2k\}$ is either in A or in its complement.

Define $\mu : F \to c_0$ coordinate-wise by

$$\mu(A)_k = \begin{cases} 0, & \text{if } \{2k-1, 2k\} \text{ is either in } A \text{ or in } CA, \\ +1, & \text{if } \{2k-1\} \in A, \text{ while } \{2k\} \notin A \\ -1, & \text{if } \{2k\} \in A, \text{ while } \{2k-1\} \notin A. \end{cases}$$

Then clearly μ takes its values in the unit ball of c_o. As on this unit ball the weak topology coincides with the topology of pointwise convergence one checks immediatly that μ is weakly σ-additive. Obviously μ is not strongly σ-additive.

The fact that F does not satisfy (G) is quickly seen by considering the sequence $\{\delta_{\{2k-1\}} - \delta_{\{2k\}}\}_{k=1}^{\infty}$ which converges weak* in $B(F)^*$ but is evidently not uniformly exhaustive .

□

To prove (G) ⇒ (OP) we still need 2 technical lemmas:

<u>6.3 Lemma.</u> Let F be a Boolean algebra, $\mu : F \to X$ a bounded measure and $\{A_n\}_{n=1}^{\infty}$ a disjoint sequence in F s.t. $\|\mu(A_n)\| \geq \alpha > 0$.

Then there exists a continuous linear map $T : X \to l^{\infty}$ s.t. for some subsequence $\{n_k\}_{k=1}^{\infty}$ we have $T \circ \mu(A_{n_k}) = e_k$, where e_k is the k-th unit vector of l^{∞}.

<u>Proof:</u> By ([D-U 77], th.I.4.2.) there exists $\{A_{n_k}\}_{k=1}^{\infty}$ s.t. $\{\mu(A_{n_k})\}_{k=1}^{\infty}$ spans c_o in X. Define $T : X \to l^{\infty}$ on the subspace spanned by $\{\mu(A_{n_k})\}_{k=1}^{\infty}$ putting $T(\mu(A_{n_k})) = e_k$ and extending it to X by the injectivity of l^{∞}.

□

<u>6.4 Lemma.</u> Let $\{x_n\}_{n=1}^{\infty}$ in X be s.t. $\sum_{n=1}^{\infty} x_n$ converges weakly to $x_o \in X$.

Then for $\varepsilon > 0$ and $N \in \mathbb{N}$ there is $M \geq N$ such that for each block of mutually distinct natural numbers $n_1, \ldots, n_k > M$ there are natural numbers $m_1, \ldots, m_l > N$ and scalars $\lambda_1, \ldots, \lambda_l$ with $\lambda_j \in [0,1]$ so that $m_1, \ldots, m_l$, $n_1, \ldots, n_k$ are all mutually distinct and

$$\left\| \sum_{i=1}^{k} x_{n_i} + \sum_{j=1}^{l} \lambda_j \, x_{m_j} \right\| < \varepsilon .$$

<u>Proof:</u> Given $\varepsilon > 0$ and $N \in \mathbb{N}$ the set of points of the form $\sum_{n=1}^{N} x_n + \sum_{n=N+1}^{M} \mu_n \, x_n$, where $M > N$ and $\mu_n \in [0,1]$, is a convex subset of X. As weak and strong closure of a convex set coincide, for some M and $\bar{\mu}_{N+1}, \bar{\mu}_{N+2}, \ldots, \bar{\mu}_M$ we have

$$\left\| \sum_{n=1}^{N} x_n + \sum_{n=N+1}^{M} \bar{\mu}_n \, x_n - x_0 \right\| < \varepsilon/2$$

If $n_1, \ldots, n_k > M$ are fixed, then again the set of points of the form

$$\left[\sum_{n=1}^{N} x_n + \sum_{n=N+1}^{M} \bar{\mu}_n \, x_n + \sum_{i=1}^{k} x_{n_i} \right] + \sum_{j=1}^{l} \lambda_j \, x_{m_j},$$

where for each $j, m_j \notin \{1, \ldots, N, \; n_1, \ldots, n_k\}$, and either $\lambda_j \in [0,1]$ or $\lambda_j \in [0, 1-\bar{\mu}_{m_j}]$ if $m_j \in \{N+1, \ldots, M\}$, and l ranges in $\mathbb{N}$, is a convex set whose strong closure contains x_0. So again we may find $\bar{\lambda}_1, \ldots, \bar{\lambda}_j$ and $\bar{m}_i, \ldots, \bar{m}_j$ s.t.

$$\left\| \sum_{n=1}^{N} x_n + \sum_{n=N+1}^{M} \bar{\mu}_n \, x_n + \sum_{i=1}^{k} x_{n_i} + \sum_{j=1}^{l} \bar{\lambda}_j \, x_{\bar{m}_j} - x_0 \right\| < \varepsilon/2 .$$

Whence $\left\| \sum_{i=1}^{k} x_{n_i} + \sum_{j=1}^{l} \bar{\lambda}_j \, x_{\bar{m}_j} \right\| < \varepsilon .$

□

6.5 Theorem. (G) $\Rightarrow$ (OP), i.e., an algebra F having (G) has (OP).

Proof: Let $\mu : F \to X$ be a bounded measure that is weakly σ-additive but not strongly. Then there is a disjoint sequence $\{A_n\}_{n=1}^{\infty}$, such that $\bigvee_{n=1}^{\infty} A_n = A_\infty$ exists in F, and such that $\sum_{n=1}^{\infty} \mu(A_n)$ converges weakly but not strongly to $\mu(A_\infty)$. By may be forming blocks of the form $A_{n_m+1} \vee \ldots \vee A_{n_{m+1}}$ we may suppose that there is $\alpha > 0$, such that $\|\mu(A_n)\| \geq \alpha$, for all n.

By the lemma 6.3 there is a continuous operator $T : X \to l^\infty$, such that, for some $\{n_k\}_{k=1}^{\infty}$,

$$T(\mu(A_{n_k})) = e_k .$$

Write ν for $T \circ \mu$, which clearly is a weakly σ-additive bounded l^∞-valued measure, and denote $\nu_k = e_k \circ \nu$ (this time e_k denoting the k-th unit vector in l^1).

Now we form inductively a sequence $\{B_l\}_{l=1}^{\infty}$ such that each B_l is a finite union of elements of the sequence $\{A_n\}_{n=1}^{\infty}$ and such that the B_l's mimic the pairs $\{2l - 1, 2l\}$ in example 6.3.

At the first step let $k(1) = 1$ and add to $A_{n_{k(1)}}$ finitely many $A_1^{(1)}, \ldots, A_{p(1)}^{(1)}$ (taken from the sequence $\{A_n\}_{n=1}^{\infty}$), such that for

$$B_1 = A_{n_{k(1)}} \vee A_1^{(1)} \vee \ldots \vee A_{p(1)}^{(1)} ,$$

$$|\nu_1| (A_\infty \setminus B_1) < 1.$$

$|\nu_1|$ denoting the variation measure of ν_1. Define

$$\alpha(1) = \liminf_{k \to \infty} |\nu_k| (B_1)$$

and let S_1 be an infinite subset of $\mathbb{N}$, such that, for $k \in S_1$,

$$| |\nu_k| (B_1) - \alpha_1| < \frac{1}{2} .$$

At the l-th step suppose $k(1),\ldots,k(l-1)$, $B_1,\ldots,B_{l-1}$, $\alpha(1),\ldots,\alpha(l-1)$ and S_{l-1} have been defined. Choose $k(l)$ from S_{l-1}, such that $A_{n_{k(l)}}$ has not been used to build one of the blocks $B_1,\ldots,B_{l-1}$. Choose elements $A_1^{(l)}, \ldots, A_{p(l)}^{(l)}$ from the sequence $\{A_n\}_{n=1}^{\infty}$, such that neither of them has been built into one of the blocks $B_1,\ldots,B_{l-1}$ and such that for

$$B_l = A_{n_{k(l)}} \vee A_1^{(l)} \vee \ldots \vee A_{p(l)}^{(l)}$$

we have

$$|\nu_{k(l)}| (A_\infty \setminus (B_1 \vee \ldots \vee B_l)) < \frac{1}{l} . \qquad (*)$$

To make sure that the B_l's finally exhaust all of the sequence $\{A_n\}_{n=1}^{\infty}$ we may and do assume that $A_1,\ldots,A_{l-1}$ have all been built into one of the blocks $B_1,\ldots,B_l$. Define

$$\alpha(l) = \liminf_{k \in S_{l-1}} |\nu_k| (B_l)$$

and let S_l be an infinite subset of S_{l-1}, such that, for $k \in S_l$

$$||\nu_k| (B_l) - \alpha(l)| < 2^{-l} \qquad (**)$$

This completes the induction step.

We shall show that for each C in F the sequence $\{\nu_{k(l)}(C \wedge B_l)\}_{l=1}^{\infty}$ tends to zero. This will imply that F does not satisfy (G). Indeed the sequence of measures $\{\nu_{k(l)}|B_l\}_{l=1}^{\infty}$ (i.e., the restrictions of the $\nu_{k(l)}$'s to the B_l's) tends weak* to zero. But the $\nu_{k(l)}|B_l$ are disjointly supported and

$$\|\nu_{k(l)}|B_l\| \geq |\nu_{k(l)} (A_{n_{k(l)}})| = 1 ,$$

whence $\{\nu_{k(l)}|B_l\}_{l=1}^{\infty}$ cannot be uniformly exhaustive thus does not tend to zero weakly.

First observe that $\sum_{l=1}^{\infty} \alpha(l) < \infty$. Indeed, $\nu : F \to l^{\infty}$ is a bounded measure, so there exists some constant $K > 0$ such that for each $k \in \mathbb{N}$ the variation norm $\|\nu_k\|$ is majorized by K. If $\sum_{l=1}^{\infty} \alpha(l) = \infty$, we could find $L \in \mathbb{N}$, such that

$$\sum_{l=1}^{L} (l) > K + 1 .$$

For any $k \in S_L$,
$$\begin{aligned} \|\nu_k\| &\geq |\nu_k| (B_1) + \ldots + |\nu_k| (B_L) \\ &\geq \alpha(1) - \tfrac{1}{2} + \ldots + \alpha(L) - 2^{-L} \qquad (\text{by}\,(**)) \\ &> \sum_{l=1}^{L} \alpha(l) - 1 > K, \end{aligned}$$

a contradiction.

Now fix $C \in \mathcal{F}$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} (\alpha(i) + 2^{-i}) + N^{-1} < \frac{\varepsilon}{2} .$$

By the lemma 6.4 there is $M \geq N$ such that for every l greater than M we may find an element

$$\lambda_1 \cdot \nu(C \wedge B_{i_1}) + \ldots + \lambda_r \cdot \nu(C \wedge B_{i_r})$$

s.t.

$$\left\| \lambda_1 \cdot \nu(C \wedge B_{i_1}) + \ldots + \lambda_r \cdot \nu(C \wedge B_{i_r}) + \nu(C \wedge B_l) \right\| < \frac{\varepsilon}{2}$$

where $\lambda_j \in [0,1]$ and the indices $\{i_j\}_{j=1}^{r}$ are greater than N and different from l. In particular we have

$$|\lambda_1 \cdot \nu_{k(l)}(C \wedge B_{i_1}) + \ldots + \lambda_r \cdot \nu_{k(l)}(C \wedge B_{i_r}) + \nu_{k(l)}(C \wedge B_l)| < \frac{\varepsilon}{2} .$$

On the other hand

$$|\lambda_1 \cdot \nu_{k(l)}(C \wedge B_{i_1}) + \ldots + \lambda_r \cdot \nu_{k(l)}(C \wedge B_{i_r})| \leq$$
$$|\nu_{k(l)}| \, ((C \wedge B_{i_1}) \vee \ldots \vee (C \wedge B_{i_r})) \leq$$
$$|\nu_{k(l)}| \, (\bigvee_{i=N+1}^{l-1} B_i \vee \bigvee_{i=l+1}^{\infty} B_i) \leq$$
$$\sum_{i=N+1}^{l-1} (\alpha(i) + 2^{-i}) + l^{-1} < \frac{\varepsilon}{2} ,$$

where the last line follows from (*) and (**).

So

$$|\nu_{k(1)}(C \wedge B_1)| < \varepsilon,$$

which completes the proof.

□

Remark: From 6.5 and 2.11 it actually follows that if $\mathcal{F}$ has (G), every quotient algebra of $\mathcal{F}$ has (OP). I do not know if the converse holds true.

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