SAKS SPACES AND VECTOR VALUED MEASURES

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CONTENTS

INTRODUCTION.

The purpose of this note is to provide a systematic approach to some aspects of vector measure theory using the concept of a Saks space. A Saks space is a vector space with two structures, a norm and a locally convex topology, which are in some sense compatible. At first sight this may seem a rather strange object and its relevance to measure theory is not at all obvious. However, we hope that this paper demonstrates the thesis that they are a suitable tool for some aspects of the theory. Here we would like to mention one argument which may make this claim more plausible. One of the features of a σ-additive measure with values in a Banach space and defined on a σ-field is that it takes its values in a weakly compact set. This means exactly that it takes its values in a Saks space and indeed of a very special kind - one with compact unit ball. This trivial remark is useful for two reasons. Firstly, such Saks spaces are precisely those which are expressible as projective limits (in a suitable sense) of finite dimensional spaces and this allows us to prove several results very simply by reducing to the finite dimensional case (i.e. essentially to the scalar case). Also the theory of Saks spaces is taylor-made to handle just such situations of a ball and an auxiliary topology (in this case the closed absolutely convex hull of the range of the measure with the weak topology) so that it is not surprising that it gives a more precise and efficient approach.

In fact, we use only some very basic and simple results on Saks spaces and we recall these, together with the definitions in § 1 which also contains some new results on operators between Saks spaces. In §§ 2 and 3 we develop the basic theory of measures and measurable functions with values in Saks spaces. Using the representation of a Saks space as a projective limit of Banach spaces, there is no difficulty in carrying over the definitions and results we need. However, this gain in generality, although easily won, allows us to unify several distinct concepts of measurability and integrability. In § 4 we consider L^p-spaces of functions with values in a Saks space and show how their duality theory can be deduced very easily by formal manipulation with inductive and projective limits.
In § 5 we consider Riesz representation theory for operators from $C^b(S)$ (S a completely regular space) into a Saks space E. We prove one result for the special case where E is a Saks space with compact unit ball and show how some known results (and generalisations thereof) follow easily and naturally from it.

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§ 1. PRELIMINARIES ON SAKS SPACES.

1.1. Definition: A Saks space is a triple $(E,\|\ \|,\tau)$ where $(E,\|\ \|)$ is a normed space and τ is a locally convex topology on E so that $B_{\|\ \|}$, the unit ball of $(E,\|\ \|)$, is τ-closed and bounded. We then write $\gamma[\|\ \|,\tau]$ or simply γ for the finest locally convex topology on E which coincides with τ on $B_{\|\ \|}$. We resume the most important elementary properties of γ in the following Proposition (cf. COOPER [2]):

1.2. Proposition: 1) $\tau \subseteq \gamma \subseteq \tau_{\|\ \|}$;

2) the γ-bounded subsets of E coincide with the norm-bounded sets;

3) a sequence (x_n) in E converges to zero with respect to γ if and only if it is norm bounded and τ-convergent to zero;

4) a subset of E is γ-compact if and only if it is norm-bounded and τ-compact;

5) (E,γ) is complete if and only if $B_{\|\ \|}$ is τ-complete;

6) the dual E'_γ of (E,γ) is the norm-closure of $(E,\tau)'$ in the dual of $(E,\|\ \|)$.

1.3. Examples: I. If E is a Banach space, then the following triples are Saks spaces:

$$(E,\|\ \|,\tau_{\|\ \|}),\ (E,\|\ \|,\sigma(E,E')),\ (E',\|\ \|,\sigma(E',E)).$$

II. If S is a completely regular space and $C^b(S)$ denotes the space of bounded, continuous complex-valued functions on S then $(C^b(S),\|\ \|,\tau_K)$ is a Saks space where τ_K is the topology of compact convergence.

III. If H is a Hilbert space and L(H) is the algebra of continuous linear operators on H we denote by τ_w and τ_s the weak resp. strong operator topology on L(H). Hence τ_w is defined by the seminorms $T \longmapsto |f(Tx)|$ $(x \in H,\ f \in H')$ and τ_s is defined by the seminorms $T \longmapsto \|Tx\|$ $(x \in H)$. Then

$$(L(H), \|\ \|, \tau_w) \quad \text{and} \quad (L(H), \|\ \|, \tau_s)$$

are Saks spaces.

IV. Let μ be a positive, finite, σ-additive measure on the space (Ω, Σ) and denote by $L^\infty(\mu)$ the corresponding L^∞-space. Then $(L^\infty(\mu), \|\ \|, \tau_1)$ is a Saks space where τ_1 is the topology induced by the L^1-norm and its dual is L^1.

1.4. <u>Completions</u>: If $(E, \|\ \|, \tau)$ is a Saks space its completion is defined as follows: we let $\hat{B}$ denote the τ-completion of $B_{\|\ \|}$ i.e. the closure of $B_{\|\ \|}$ in the completion $\hat{E}_\tau$ of (E,τ). Then if $\hat{E}$ is the span of $\hat{B}$ and $\|\ \|^\wedge$ denotes the Minkowski functional of $\hat{B}$ $(\hat{E}, \|\ \|^\wedge, \tau)$ is the required completion. As an example, if E is a normed space then the completion of the Saks space $(E, \|\ \|, \sigma(E,E'))$ is the Saks space $(E'', \|\ \|, \sigma(E'',E'))$.

1.5. <u>Saks space products and projective limits</u>: If $\{(E_\alpha, \|\ \|_\alpha, \tau_\alpha)\}_{\alpha \in A}$ is a family of Saks spaces we form their product as follows: if E denotes the Cartesian product $\prod_{\alpha \in A} E_\alpha$ we put

$$E_o := \{x = (x_\alpha) \in E : \|x\| := \sup \|x_\alpha\|_\alpha < \infty\}$$

Then $(E_o, \|\ \|, \tau)$ is a Saks space where τ is the Cartesian product of the topologies $\{\tau_\alpha\}$. $(E_o, \| \|, \tau)$ is called the Saks space product of $\{E_\alpha\}$ and is denoted by $S \Pi E_\alpha$.

Now let $\{\pi_{\beta\alpha} : E_\beta \longrightarrow E_\alpha, \alpha, \beta \in A, \alpha \leq \beta\}$ be a projective spectrum of Saks spaces. As usual, we define the projective limit of this spectrum as the subspace of the product formed by the threads i.e. as

$$E_1 := \{(x_\alpha) \in S \Pi E_\alpha : \pi_{\beta\alpha}(x_\beta) = x_\alpha \quad \text{for } \alpha \leq \beta\}$$

E_1 is denoted by $S \varprojlim\{E_\alpha\}$. As an example, if S is a locally compact space and $K(S)$ denotes the family of compact subsets of S then

$$\{\rho_{K_1,K} : C(K_1) \longrightarrow C(K), K \subseteq K_1\}$$

forms a projective spectrum of Banach spaces (where C(K) denotes the Banach space of continuous, complex-valued functions on K and $\rho_{K_1,K}$ is the restriction operator) and its Saks space projective limit is naturally identifiable with $(C^b(S), \| \|, \tau_K)$.

Now if $(E, \| \|, \tau)$ is a Saks space, we say that a family of seminorms S which generates τ is a suitable family if it satisfies the condition:

1) if $p, q \in S$ then $\max\{p, q\} \in S$;
2) $\| \| = \sup S$.

If $p \in S$, E_p denotes the Banach space generated by p (i.e. the completion of the normed space E/N_p where N_p is the kernel of p) and if $p \leq q$ then ω_{qp} denoted the natural mapping from E_q to E_p. Then $\{\omega_{qp} : E_q \longrightarrow E_p\}$ forms a projective spectrum of Banach spaces.

1.6. Proposition: If $(E, \| \|, \tau)$ is a complete Saks space, then E is naturally identifiable with $S\varprojlim_{p \in S} E_p$.

1.7. Proposition: Let $(E, \| \|, \tau)$ be a Saks space. Then the following are equivalent:

1) $B_{\| \|}$ is τ-compact;

2) E is the Saks space projective limit of finite dimensional Banach spaces;

3) E has the form $(F', \| \|, \sigma(F',F))$ for some Banach space F.

Then $\gamma = \tau_c(F',F)$, the topology of uniform convergence on the compact sets of E, is the finest topology on E which agrees with τ on $B_{\| \|}$.

In fact, if 1) is fullfilled, then E is naturally identifiable with $S\varprojlim_{F \in \mathcal{F}(E'_\gamma)} \{F'\}$ where $\mathcal{F}(E'_\gamma)$ denotes the family of finite dimensional subspaces of E'_γ.

Further, the following are equivalent:

1) B is τ-compact and metrisable;

2) E is the Saks space projective limit of a sequence of finite dimensional Banach spaces;

3) E has the form $(F', \| \|, \sigma(F',F))$ for a separable Banach space F;

4) $B_{\| \|}$ is τ-compact and normable (i.e. there is a norm $\| \|_1$ on E so that $\tau = \tau_{\| \|_1}$ on $B_{\| \|}$).

1.8. The Hom functor: If E is a Banach space, $(F, \| \|, \tau)$ a Saks space, then Hom(E,F) denotes the set of $\| \|$-γ continuous linear operators from E into F. Note that as a vector space, this coincides with the space of norm-bounded linear operators from E into F. We regard Hom(E,F) as a Saks space with the supremum norm and τ_p, the topology of pointwise convergence, with respect to τ, Note that on the unit ball of Hom(E,F), this topology coincides with that of compact convergence, resp. with that of pointwise convergence on a dense subspace of E.

1.9. Proposition: 1) If $\{E_\alpha\}$ is an inductive system in BAN_1 and F is a complete Saks space then there is a natural isomorphism between the Saks spaces

$$\text{Hom}\ (B \varinjlim_{\alpha} E_\alpha, F) \quad \text{and} \quad S \varprojlim_{\alpha} \text{Hom}\ (E_\alpha, F).$$

In particular, if E is a Banach space, we have

$$\text{Hom}\ (E,F) = \text{Hom}\ (B \varinjlim_{G \varepsilon \mathcal{F}(E)} G, F) = S \varprojlim_{G \varepsilon \mathcal{F}(E)} \text{Hom}\ (G, F).$$

2) if $\{F_\alpha\}$ is a projective system of Banach spaces, E a Banach space, then there is a natural isomorphism between the Saks spaces

$$\text{Hom}\ (E, S \varprojlim_{\alpha} F_\alpha) \quad \text{and} \quad S \varprojlim_{\alpha} \text{Hom}\ (E, F_\alpha).$$

In particular, if F is a Saks space with $B_{\|\ \|}$ τ-compact, we have

$$\text{Hom}\ (E,F) = \text{Hom}\ (E, S \varprojlim_{G \varepsilon \mathcal{F}(F'_\gamma)} G') = S \varprojlim_{G \varepsilon \mathcal{F}(F'_\gamma)} \text{Hom}\ (E, G').$$

In the following, we bring some new results on operators between Saks spaces. A key role is played by the following simple Lemma:

1.10. Lemma: Let E be a vector space, τ and γ locally convex topologies on E so that $\tau \subseteq \gamma$ and γ has a basis of τ-closed neighbourhoods. Let K be a τ-closed, γ-complete subset of E. Then K is τ-compact if and only if the following condition is satisfied:

for each γ-neighbourhood U of zero, there is a τ-compact set K_U so that $K \subseteq K_U + U$.

Proof: $\Longrightarrow$ is trivial.

$\Longleftarrow$: K is then clearly precompact. We show that it is complete. Let (x_α) be a τ-Cauchy net in K. If U is γ-neighbourhood of zero, we can write

$$x_\alpha = y_\alpha^U + z_\alpha^U$$

where $y_\alpha^U \in K_U$, $z_\alpha^U \in U$. Let y^U be a τ-cluster point of $\{y_\alpha^U\}_{\alpha \in A}$ in K_U. Now for any U_1, U_2 and $\alpha \in A$, $\beta \in A$, we have

$$y_\alpha^{U_1} - y_\beta^{U_2} = (x_\alpha - x_\beta) - (z_\alpha^{U_1} - z_\beta^{U_2})$$

and so $y^{U_1} - y^{U_2} \in \overline{U_1 + U_2}^\tau$. Hence $\{y^U\}$ is a γ-Cauchy net. Let $x = \lim y^U$. Then x_α is τ-convergent to x. To prove this it is sufficient to verify that x is a limit point of $\{x_\alpha\}$. Let V be a τ-neighbourhood of zero and choose a τ-neighbourhood V_1 so that $V_1 + V_1 + V_1 \subseteq V$. There is a $U_2 \subseteq V_1$ so that $x - y^{U_2} \in V_1$. For a cofinal family of α's, we have

$$y_\alpha^{U_2} - y^{U_2} \in V_1.$$

Hence for those α, we get

$$x - x_\alpha = (x - y^{U_2}) + (y^{U_2} - y_\alpha^{U_2}) - z_\alpha^{U_2} \in V_1 + V_1 + V_1 \subseteq V.$$

1.11. Corollary: Let K be a weakly closed subset of a Banach space E. Then K is weakly compact if and only if the following condition is satisfied:

for each $\varepsilon > 0$ there is a weakly compact subset K_ε of E so that $K \subseteq K_\varepsilon + \varepsilon B_{\| \|}$.

1.12. Remark: I. The above Corollary is known (cf. GROTHENDIECK [8], p. 221). In fact, the Lemma can be deduced from this Corollary. For, using the BOURBAKI completeness theorem, we can reduce to the case where τ is the weak topology and then it is easy to reduce to the case where E is a Banach space. We have preferred the above proof since, although rather less elegant, it is more elementary (in particular, it avoids any use of the bidual). It can also easily be adapted to prove a suitable modification of the Proposition which is valid for uniform spaces.

The following is an abstract version of a result of SENTILLES [15] on operators between $C^b(S)$ spaces. As so often happens, the proof of the abstract version is considerably shorter and clearer than the original one.

1.13. Proposition: Let $(E, \| \|, \tau)$ and $(F, \| \|_1, \tau_1)$ be Saks spaces with $(F, \| \|_1)$ a Banach space. Then a linear operator $T : E \longrightarrow F$ maps a γ-neighbourhood of zero in E into a relatively (weakly) compact subset of (F, γ) if and only if T maps bounded sets into relatively (weakly) compact sets and is γ-$\| \|$ continuous.

Proof: $\Longrightarrow$ is trivial.

$\Longleftarrow$: Let U be an absolutely convex γ-neighbourhood of zero with $T(U) \subseteq B_1(F)$ ($B_\lambda(F)$ is the λ-ball in $(F, \| \|_1)$). Then $T(\lambda U) \subseteq B_\lambda(F)$ and $T(U \cap B_n(E))$ is relatively (weakly) compact for each n. Let

$$\widetilde{U} := \bigcup_{n=1}^{\infty} \sum_{k=1}^{n} (2^{-k}U \cap B_k(E)).$$

Then $\widehat{U}$ is a γ-neighbourhood of zero and we claim that its image is relatively (weakly) compact. For

$$\begin{aligned} T(\widetilde{U}) &= \bigcup_{n=1}^{\infty} \sum_{k=1}^{n} T(2^{-k}U \cap B_k(E)) \\ &\subseteq \sum_{k=1}^{n} (T(2^{-k}U \cap B_k(E)) + \bigcup_{l=n+1}^{\infty} \sum_{k=n+1}^{l} T(2^{-k}U) \\ &\subseteq \sum_{k=1}^{n} T(2^{-k}U \cap B_k(E)) + B_{2^{-n}}(F) \end{aligned}$$

for each n and so the result follows from the Lemma.

We now give some useful factorisation theorems for operators between Saks spaces. They are consequences of the following recent factorisation theorems (due to the combined efforts of DAVIS, FIGIEL, JOHNSON, PEŁCZYŃSKI cf. [3],[6],[10]).

1.14. Theorem: Let E and F be Banach spaces, $T : E \longrightarrow F$ a weakly compact linear operator. Then there is a reflexive Banach space G and continuous linear operators $S : G \longrightarrow F$ and $R : E \longrightarrow G$ so that $T = SR$.

If T is compact, we can find G,R,S with the additional properties that G be separable and R and S be compact.

This theorem follows immediately from the following Lemma. Since we are able to give a simple and short proof using the Lemma 1.10 above, we do so for completeness.

1.15. <u>Lemma</u>: Let E be a Banach space and $W \subseteq E$ be a weakly compact, absolutely convex subset. Then there exists a weakly compact, absolutely convex subset C of E such that $W \subseteq C$ and $(E_C, \| \|_C)$ is reflexive ($(E_C, \| \|_C)$ denotes the normed space with C as unit ball).

If W is norm compact, then one may construct C so that W is compact in $(E_C, \| \|_C)$, C is compact in E and $(E_C, \| \|_C)$ is a separable, reflexive Banach space.

<u>Proof</u>: Denote by B the unit ball of E and put $W_n := 2^n K + n^{-1} B$. Then W_n is a closed, absolutely convex subset of E. Let $\| \|_n$ be the Minkowski functional of W_n and write E_n for the normed space $(E, \| \|_n)$. Then $\| \|_n$ is equivalent to $\| \|$ (and so E_n is a Banach space). Now let

$$C := \{x \in E : \sum_{n=1}^{\infty} \|x\|_n^2 \leq 1\}.$$

Then C is a closed absolutely convex subset of E and a simple calculation shows that $K \subseteq C$. Also $C \subseteq 2^n K + n^{-1} B(E)$ for each n and so C is weakly compact by 1.11. We show that $\sigma(E,E')$ and $\sigma(E_C, E_C')$ coincide on C and this will conclude the first part. First note that the diagonal mapping

$$x \longmapsto (x, x, \ldots .)$$

is an isometric embedding from E_C onto a closed subspace of $\ell^2(E_n)$. Hence, since the dual of $\ell^2(E_n)$ is $\ell^2(E_n')$, it suffices to show that $\sigma(\ell^2(E_n),\ell^2(E_n'))$ coincides with $\sigma(E,E')$ on C. Now if we denote by $\phi(E_n')$ the subspace of $\ell^2(E_n')$ consisting of those sequences which have at most finitely many non-zero elements, $\phi(E_n')$ is norm-dense in $\ell^2(E_n')$ and so $\sigma(\ell^2(E_n),\ell^2(E_n'))$ agrees with $\sigma(\ell^2(E_n),\phi(E_n'))$ on the bounded set C. But the latter topology induces $\sigma(E,E')$ on C (since the restriction of a form in $\phi(E_n')$ to C is in E').

Now we turn to the second part. If K is norm-compact, then it follows from 1.10 that C is compact in E.

We now show that K is compact in E_C. It will suffice to show that it is precompact. Let $\varepsilon > 0$ - we shall find an ε-net for K with respect to $\| \ \|_C$. First note that if $x \in K$ then $\|x\|_n \leq 2^{-n}$. Hence there is an $N > 0$ so that $\left(\sum_{n=N+1}^{\infty} \|x\|_n^2\right) \leq \varepsilon^2/8$ for each $x \in K$. Now the norm $\left(\sum_{n=1}^{N} \| \cdot \|_n^2\right)^{1/2}$ is equivalent to $\| \ \|$ and so there is a finite set $\{x_1,\ldots,x_k\}$ in K so that for each $x \in K$ there is an i with $\left(\sum_{n=1}^{N} \|x - x_i\|_n^2\right) \leq \varepsilon^2/2$. Then we have $\|x - x_i\|_C \leq \varepsilon$.

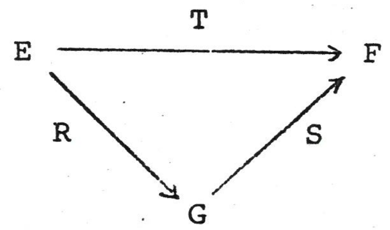
To show that $(E_C, \| \ \|_C)$ is separable, note that since C is norm compact, the norm topology agrees with $\sigma(E,E')$ on C and so the latter is metrisable. However, as we know, $\sigma(E,E')$ agrees with $\sigma(E_C,E_C')$ on C and so the latter is also metrisable.

From this it follows that $(E_C, \| \|_C)$ is separable.

The following Proposition is perhaps of some independent interest although we shall not actually use it in this form (so that we do not require the factorisation theorem 1.14):

1.16. <u>Proposition</u>: Let $(E, \| \|, \tau)$ be a Saks space, $(F, \tilde{\tau})$ a locally convex space, $T : E \longrightarrow F$ γ-continuous and linear. Then

1) T takes bounded sets into relatively compact sets if and only if T factorises as follows



where $(G, \| \|_1, \tau_1)$ is a Saks space with $B_{\| \|_1}$ τ_1-compact $R : E \longrightarrow G$ is γ-γ-continuous (and so takes bounded sets into relatively compact sets) and $S : G \longrightarrow F$ is γ-$\tilde{\tau}$-continuous.

If F is a Banach space with the weak topology, we can assume that $(G, \| \|_1)$ is reflexive and $\tau_1 = \sigma(G,G')$ and if F is a Banach space with the norm topology, we can assume that $(G, \| \|_1)$ is a separable reflexive and $\tau_1 = \sigma(G,G')$ (so that $B_{\| \|_1}$ is τ_1-compact and metrisable).

2) T is γ-compact if and only if T has a factorisation as in the diagram with R γ-$\| \|_1$ continuous.

Proof: 1) $\Longleftarrow$ is clear.

$\Longrightarrow$: Let $C := \overline{T(B_{\| \|})}$. Then $(E_C, \| \|_C, \tilde{\tau}|_{E_C})$ is the required space where R is the corestriction of T to E_C and S is the injection $E_C \longrightarrow F$.

Now if F is a Banach space with the weak topology then T is weakly compact from $(E, \| \|)$ into $(F, \| \|)$. Hence by the factorisation theorem, there is a ball B_1 with $T(B) \subseteq B_1 \subseteq F$ so that $(E_{B_1}, \| \|_{B_1})$ is reflexive. Then if $G := E_{B_1}$, $\tau := \sigma(G,G')$ on B_1 by compactness so we can complete the proof as above.

If F is a Banach space with the norm topology then we can proceed as above except, using the factorisation theorem for compact operators, we can assume that B_1 is norm-compact.

2) $\Longleftarrow$ is clear for then R is γ-compact by 1.13 and hence so is T.

$\Longrightarrow$: Suppose that T is γ-compact. Then there is an absolutely convex γ-neighbourhood V of zero so that $C_1 := \overline{T(V)}$ is $\tilde{\tau}$-compact. We can construct G as above, using C_1 as the unit ball. Then R is γ-compact as it sends V into a compact set in (G,γ).

In this order of ideas, we recall the following result of GROTHENDIECK (cf. [7], Lemma 1, p.131):

1.17. Lemma: Let $T : E \longrightarrow F$ be a continuous linear operator from one locally convex space into another. Consider the following three properties:

1) T takes bounded sets of E into relatively weakly compact sets;

2) $T''(E'') \subseteq F$;

3) T' transforms equicontinuous sets in F' into relatively $\sigma(E',E'')$-compact sets.

Then 1) $\Longleftrightarrow$ 2) $\Longrightarrow$ 3) and 3) $\Longrightarrow$ 2) if F is quasi-complete.

§ 2. MEASURES WITH VALUES IN A SAKS SPACE.

2.1. Definition: Let $(E, \| \|, \tau)$ be a Saks space, S a suitable family of τ-seminorms on E and (Ω, F) a measure space (i.e. F is a field of subsets of Ω). A function $\mu : F \longrightarrow E$ is a (finitely additive) measure if $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$ if $A_1, A_2 \in F$ are disjoint. A measure is

bounded if its range is norm-bounded i.e. if its semi-variation $\|\mu\| : A \longmapsto \sup \{ |f \circ \mu| (A) : f \in B(E'_{\| \|}) \}$ with respect to the norm is finite-valued;

From now on we shall only consider bounded measures. μ is

countably additive (or σ-additive) if $\mu(\bigcup A_n) = \Sigma \mu(A_n)$ (convergence with respect to τ) whenever (A_n) is a disjoint sequence in F with $\bigcup A_n \in F$;

strongly additive if for each disjoint sequence (A_n) in F, $\Sigma \mu(A_n)$ converges (again with respect to τ).

2.2. Remarks: If (E, τ) is a locally convex space and $\mu : \Sigma \longrightarrow E$ is a bounded finitely additive measure, then μ factorises over a Saks space and the above concepts coincide with the classical ones. For if B is the closed, absolutely convex hull of the range of μ and $\| \|$ is the Minkowski functional of B then $(E_B, \| \|, \tau)$ is a suitable Saks space. If we consider the Saks spaces of examples I and III of 1.3, we obtain the most important regularity concepts for measures with values in Banach spaces or spaces of operators.

2.3. Remark: Using classical results on Banach space valued measures, it is easy to see that the following results hold:

1) if $\mathcal{F}$ is a σ-field, then σ-additivity implies strong additivity

2) we can drop the assumption that be bounded in the definition of strongly additive;

3) if E is complete, then μ is strongly additive if for each decreasing sequence (A_n) in $\mathcal{F}$, $\mu(A_n)$ exists in $\sigma(E,E'_\tau)$;

4) if $\mathcal{F}$ is a σ-field and $f \circ \mu$ is bounded for each $f \in E'_\tau$, then μ is bounded (DIEUDONNÉ-GROTHENDIECK theorem). For a reference see, for example, |4|.

Now if X is a completely regular space we denote by Bo(X) the σ-field of Borel subsets of X. If $(E,\|\ \|,\tau)$ is a Saks space an E-valued Radon measure on X is a bounded measure $:Bo(X) \quad E$ which is inner regular with respect to τ i.e. satisfies the condition that

$$\lim_{\substack{K \in \mathcal{K}(S) \\ K \subseteq A}} \omega_p \quad \mu(K) = \omega_p \quad \mu(A) \quad \text{in } E_p$$

for each Borel set A in X and each $p \in S$.

We now introduce the following notation:

$M_b(\mathcal{F};E) = \{\mu : \mathcal{F} \longrightarrow E : \mu \text{ is a bounded measure}\}$

$M\sigma(\mathcal{F};E) = \{\mu : \mathcal{F} \longrightarrow E : \mu \text{ is } \sigma\text{-additive}\}$

$M_R(X;E) = \{\mu : Bo(X) \longrightarrow E : \mu \text{ is Radon}\}$.

The semivariation induces a norm on the above space, under which they are all complete.

We regard then as Saks spaces with the following auxiliary topologies; in the first two cases that of pointwise convergence on sets of $\mathcal{F}$ (with respect to τ) and in the lust case pointwise convergence on the functions of $C^b(X)$ via integration (see 2.5 below).

2.4. <u>Proposition</u>: If $(E, \| \|, \tau)$ is a complete Saks space, then there are natural identifications:

$$M_b(F;E) = S \varprojlim_{p \in S} M_b(F;E_p)$$
$$M_\sigma(F;E) = S \varprojlim_{p \in S} M_\sigma(F;E_p)$$
$$M_R(S;E) = S \varprojlim_{p \in S} M_R(S;E_p)$$

In particular, the spaces on the left hand side are complete.

<u>Proof</u>: The proof in each case is the same. If μ is in the left-hand side, then the elements $(\omega_p \circ \mu)$ form a thread which defines an element on the right-hand side.
On the other hand, a thread (μ_p) on the right hand side pieces together to form a bounded E-valued measure and the correspondence between the regularity conditions follows from the definition.

2.5. <u>Integration</u>: If F is a field of subsets of Ω we denote by $S(F)$ the vector space of simple (complex-valued) functions F-measurable functions. $L^\infty(F)$ denotes the sup-norm completion of this space. Hence when F is a σ-field, then $L^\infty(F)$ is just the space of bounded, measurable functions.

If $x = \Sigma \alpha_i \chi_{A_i}$ is a simple function and μ is an E-valued bounded measure (E a Saks space) we define $\int x \, d\mu$ to be $\Sigma \alpha_i \mu(A_i)$.

Then $T_\mu : x \longmapsto \int x \, d\mu$

is a bounded linear operator from $S(F)$ (with the supremum norm) into E and $\|T_\mu\| = \|\mu\|$ (semi-variation of μ with respect to the norm). Hence we can extend T_μ to a continuous linear mapping from $L^\infty(F)$ into $\hat{E}$ (the norm completion of E).
Hence we have:

2.6. Proposition: Let E be a complete Saks space. Then integration induces a natural norm isometry between the spaces

$$\text{Hom}\,(L^\infty(F);\, E) \quad \text{and} \quad M_b(F;E).$$

This is, in addition, a Saks space isomorphism if the left-hand side has its natural Saks space structure.

Now let μ be a σ-additive E-valued measure on a σ-field, ν a non-negative (finite, σ-additive) measure on Σ. Then the following are equivalent:

1) $\lim\limits_{\nu(A)\to 0} \mu(A) = 0$ (limit w.r.t. τ);

2) $\nu(A) = 0 \Longrightarrow \mu(A) = 0 \quad (A \in \Sigma)$.

Of course 1) $\Longrightarrow$ 2) is trivial and 2) $\Longrightarrow$ 1) follows from the corresponding result for Banach spaces since if 2) holds then $\nu(A) = 0 \Longrightarrow \omega_p \circ \mu(A) = 0$ and so $\nu(A) \longrightarrow 0 \Longrightarrow \omega_p \circ \mu(A) \longrightarrow 0$ for each $p \in S$ i.e. $\mu(A) \longrightarrow 0$ in τ (see DIESTEL, UHL [4], Th. I.2.1).

We denote by $M_b^\nu(\Sigma;E)$ the family of $\mu \in M_b(\Sigma;E)$ such that 2) holds. We regard $M_b^\nu(\Sigma;E)$ as a Saks space as a subspace of $M_b(\Sigma;E)$.

2.7. <u>Proposition</u>: Let E be a complete Saks space, Σ a σ-field. Then the isomorphism

$$\mathrm{Hom}\ (L^{\infty}(\Sigma);E) \cong M_b(\Sigma;E)$$

induces an isomorphism

$$\mathrm{Hom}\ (L^{\infty}(\nu);E) = M_b^{\nu}(\Sigma;E).$$

<u>Proof</u>: Clear since $T : L^{\infty}(\Sigma) \longrightarrow E$ lifts to an operator on $L^{\infty}(\nu)$ exactly when condition 2) above is fulfilled.

In the following Proposition, we characterise σ-additivity of the measure in terms of continuity properties of the induced linear mapping. Recall that if Σ and ν are as above, then the vector space $L^{\infty}(\nu)$ has a natural Saks structure $(\|\ \|,\tau_1)$ where τ_1 is the topology induced by the L^1-norm. The corresponding mixed topology is denoted by β_1 (see [2], Ch. III for some results on β_1):

2.8. <u>Proposition</u>: Let $\mu \in M_b^{\nu}(\Sigma;E)$. Then μ is σ-additive if and only if $T_{\mu} : L^{\infty}(\nu) \longrightarrow E$ is β_1-γ continuous.

<u>Proof</u>: We show that if T_{μ} is β_1-continuous, then μ is σ-additive. For if (A_n) is a disjoint sequence in Σ then $\Sigma\, \chi_{A_n} \longrightarrow \chi_{\cup A_n}$ in $L^1(\nu)$ and so with respect to the topology β_1. Hence

$$\mu(\bigcup A_n) = T_{\mu}(\Sigma\, \chi_{A_n}) = \Sigma\, T_{\mu}(\chi_{A_n}) = \Sigma\, \mu(A_n).$$

On the other hand, if μ is σ-additive, we must show that T_{μ} is β_1-continuous. Since β_1 is the Mackey topology, it is enough to show that $f \circ T_\mu$ is β_1-continuous for each $f \in E'_\gamma$ i.e. is given by a function in $L^1(\nu)$. But $f \circ T_\mu$ is σ-additive and ν-absolutely continuous and so the result follows from the Radon-Nikodym theorem (see [1], Th. 2.2.4.).

2.9. <u>Corollary</u>: Let $\mu \in M(\Sigma;E)$ where E is a Banach space. Then μ is σ-additive if and only if there is a ν so that μ induces a β_1-continuous linear operator on $L^\infty(\nu)$.

This is extentially a reformulation of the BARTLE-DUNFORD-SCHWARTZ theorem (see [4], p.14).

In the next Proposition and its Corollary we reproduce an important theorem on Banach-space valued measures. The extention to measures with values in a Saks space is trivial.

2.10. <u>Proposition</u>: Let F be a field of subsets of Ω, Σ the σ-field generated by it, E a complete Saks space and μ a bounded E-valued measure on F with the property that $f \circ \mu$ is σ-additive for each $f \in E'_\tau$. Then the following are equivalent:

1) μ has a σ-additive extension to a measure on Σ;

2) for each $p \in S$, $\omega_p \circ \mu$ has a positive σ-additive control measure;

3) μ is strongly additive;

4) $\mu(F)$ is relatively weakly compact in E.

<u>Proof</u>: 1) $\Longrightarrow$ 2): if μ has a σ-additive extension, so does $\omega_p \circ \mu$ and so the latter has a control measure.

2) $\Longrightarrow$ 3): it follows from 2) that each $\omega_p \circ \mu$ is strongly additive and hence so is μ.

3) $\Longrightarrow$ 4): if μ is strongly additive, so is $\omega_p \circ \mu$ and so $(\omega_p \circ \mu)(F)$ is relatively weakly compact in E_p. But $(\omega_p \circ \mu)(F) = \omega_p(\mu(F))$ and so the result follows from the characterisation of weak compactness in a Saks space (see COOPER [2], I.1.20).

4) $\Longrightarrow$ 1): if 4) holds, then $\omega_p \circ \mu(F)$ is relatively weakly compact in E and so $\omega_p \circ \mu$ has a unique extension to a σ-additive measure μ_p on Σ. But (μ_p) is then compatible and so defines an E-valued measure on Σ (recall that $M_\sigma(\Sigma;E) = S \varprojlim_{p \in S} M_\sigma(\Sigma;E_p)$).

2.11. Corollary: Let μ be a bounded E-valued measure on a field F (E a complete Saks space). Then the following are equivalent:

1) for each $p \in S$, $\omega_p \circ \mu$ has a finitely additive positive control measure ν_p;

2) μ is strongly additive;

3) $\mu(F)$ is relatively weakly compact.

This follows from the Stone Representation theorem (cf. DIESTEL and UHL [4], Th. I.5.3.).

§ 3. MEASURABLE FUNCTIONS WITH VALUES IN A SAKS SPACE

3.1. Definition: Let (Ω,ν) be a finite measure space i.e. ν is a positive, σ-additive measure on a σ-field Σ of subsets of Ω. If E is a Saks space, once again, S denotes a suitable family of seminorms which define τ and are dominated by $\| \|$. If x is a function from Ω into E, x is said to be measurable if for each $p \in S$, $\omega_p \circ x$ is Pettis-measurable as a function from Ω into the Banach space $(E_p, \| \|_p)$. Hence x is measurable if and only if for each $p \in S$, there is a sequence (x_n) of simple functions so that $\omega_p(x_n - x) \longrightarrow 0 \quad \nu$ a.e.
Note that at this stage, the definition of measurability has nothing to do with Saks spaces since the above definition is essentially that of measurability as a function into (E,τ) - in particular, it depends on the global form of τ and not just its character on the unit ball.

From the classical theorem of PETTIS we can immediately deduce:

3.2. Proposition: x is measurable if and only if

1) for each $f \in E'_\tau$, $f \circ x$ is measurable;

2) for each $p \in S$ there is a ν-negligible set A so that $x(\Omega \setminus A)$ is separable in E_p.

In particular, if $x(M)$ is separable in (E,τ) and $f \circ x$ is measurable for each $f \in E'_\tau$ then x is measurable.

3.3. <u>Examples</u>: I. If E is a Banach space with the Saks structure $(\|\ \|, \tau_{\|\ \|})$, then we obtain the standard notion of measurability.

II. If E is a Banach space and we consider E as a Saks space with the structure $(\|\ \|, \sigma(E,E'))$ we obtain the notion of scalar measurability.

III. If E = F' the dual of a Banach space F and we use the Saks space structure $(\|\ \|, \sigma(F',F))$ we obtain the notion of scalar $*$-measurability.

IV. If E = L(F,G), the space of continuous linear mappings between the Banach spaces F,G we obtain interesting notions of measurability by considering the Saks structures $(\|\ \|, \tau_W)$ and $(\|\ \|, \tau_S)$ (cf. 1.3.III).

3.4. <u>The space $L^\infty(\nu;E)$</u>: Recall that $\mathcal{L}^\infty(\Sigma;E)$ denotes the space of bounded, measurable functions into the Saks space E. On $\mathcal{L}^\infty(\Sigma;E)$ we consider the norm

$$\|\ \|_E : x \longmapsto \sup\{\|x(t)\| : t \in \Omega\}$$

and the seminorms

$$p : x \longmapsto \int_\Omega \|\omega_p \circ x\| d\nu \qquad (p \in S).$$

$L^\infty(\nu;E)$ denotes the quotient space of $\mathcal{L}^\infty(\Sigma;E)$ with respect to N where

$$N := \{x \in \mathcal{L}^\infty(\Sigma;E) : \omega_p \circ x = 0 \quad \nu \text{ a.e. for each } p \in S\}.$$

Then $\|\ \|_E$ and p induce a norm resp. seminorms on $L^\infty(\nu;E)$ and these induce a Saks space structure $(\|\ \|_E, \tau_E)$ on $L^\infty(\nu;E)$.

We recall that the measure space $(\Omega;\nu)$ possesses a <u>multiplicative lifting</u> (cf. [9]). i.e. a linear mapping $\rho : L^{\infty}(\nu) \longrightarrow \mathcal{L}^{\infty}(\Sigma)$ of norm one so that ρ is multiplicative and a right inverse of the projection $\mathcal{L}^{\infty}(\Sigma) \longrightarrow L^{\infty}(\nu)$.

3.5. <u>Proposition</u>: Let E be a Banach space. Then there exists a vector valued lifting $\rho_E : L^{\infty}(\nu;E) \longrightarrow \mathcal{L}^{\infty}(\Sigma;E)$ i.e. a linear mapping of norm one which is a right inverse of the projection $\mathcal{L}^{\infty}(\Sigma;E) \longrightarrow L^{\infty}(\nu;E)$ so that $\rho_E(x \otimes \chi_A) = x \otimes \rho(\chi_A)$ for each $x \in E$, $A \in \Sigma$.

<u>Proof</u>: ρ induces a mapping $\tilde{\rho}$

$$\Sigma / N(\nu) \longrightarrow \Sigma$$

(where $N(\nu)$ is the family of ν-negligible sets)

which associates to each equivalence class [A] of elements of Σ an element $\tilde{A}$ of so that if $[A] \cap [B] = \emptyset$ (i.e. $A_1 \in [A]$, $B_1 \in [B] \Longrightarrow \mu(A_1 \cap B_1) = \emptyset$), then $\tilde{A} \cap \tilde{B} = \emptyset$. For if $A \in \Sigma$, then the equivalence class of χ_A in $L^{\infty}(\nu)$ is an idempotent. Hence so is its image under ρ and hence the letter is the characteristic function of some subset of Ω.

Now if $\Sigma x_i \otimes \chi_{A_i}$ is a countably valued function in $L^{\infty}(\nu;E)$ with $\{A_i\}$ mutually disjoint in the above sense, we can define

$$\rho_E(\Sigma x_i \otimes \chi_{A_i}) := \Sigma x_i \otimes \chi_{\tilde{\rho}(A_i)}$$

then ρ_E is well-defined on $L^{\infty}_{\sigma}(\nu;E)$, the norm-dense subspace of $L^\infty(\nu;E)$ consisting of the countably-valued functions. Its continuous extension to an operator on $L^\infty(\nu;E)$ has the required properties.

Note that the above construction is functorial in the following sense. Suppose that E and F are Banach spaces and $T : E \longrightarrow F$ is continuous and linear. Then T induces in a natural way operators from $L^\infty(\nu;E)$ into $L^\infty(\nu;F)$ resp. from $\mathcal{L}^\infty(\Sigma;E)$ into $\mathcal{L}^\infty(\Sigma;E)$ and the following diagram commutes:

$$\begin{array}{ccc} L^\infty(\nu;E) & \xrightarrow{\rho_E} & \mathcal{L}^\infty(\Sigma;E) \\ \downarrow & & \downarrow \\ L^\infty(\nu;F) & \xrightarrow[\rho_F]{} & \mathcal{L}^\infty(\Sigma;F) \end{array}$$

3.6. Proposition: Let E be a complete Saks space with canonical representation $E = S \varprojlim \{E_p : p \in S\}$. Then

$$L^\infty(\nu;E) = S \varprojlim_{p \in S} L^\infty(\nu;E_p).$$

Proof: There is a natural mapping $x \longmapsto \{\omega_p \circ x\}$ from the left-hand side into the right side. We need only show that it is onto (that this vector space isomorphism is a Saks space isomorphism follows immediately from the definitions of the appropriate structures).

Let $\{x_p\}$ be a thread in the spectrum defining the right-hand side. Put $\tilde{x}_p = \rho_{E_p}(x_p)$. Then by the functoriality of the construction of ρ_E $\{\tilde{x}_p\}$ is a thread in the spectrum $\{L^\infty(\Sigma;E_p)\}$. Hence it defines a bounded measurable function $\tilde{x} : \Omega \longrightarrow E$. Then x, the projection of $\tilde{x}$ in $L^\infty(\nu;E)$ is the required limit of the given thread.

3.7. <u>Corollary</u>: If $(E,\|\ \|,\tau)$ is complete, then so is $L^\infty(\nu;E)$.

3.8. <u>Corollary</u>: If E is a Saks space with $B_{\|\ \|}$ τ-compact so that $E = S\varprojlim\{F' : F \in F(G)\}$ $(G := E'_\gamma)$ then

$$L^\infty(\nu;E) = S\varprojlim\{L^\infty(\nu;F') : F \in F(G)\}.$$

We recall that if F is a Banach space, there are natural isometric injections:

1) $L^\infty(\nu;F') \subseteq L^1(\nu;F)'$;

2) $L^\infty(\nu;F') \subseteq \mathrm{Hom}(F,L^\infty(\nu))$;

3) $\mathrm{Hom}(L^1(\nu);F') \subseteq \mathrm{Hom}(F;L^\infty(\nu))$

which are defined as follows:

1) if $f \in L^\infty(\nu;F')$, then f defines a linear form T_f on $L^1(\nu;F)$ as follows:

$$T_f : x \longmapsto \int_\Omega \langle x,f\rangle\, d\nu \qquad (x \in L^1(\nu;F))$$

(Note that the scalar-valued function $\langle x,f\rangle$ is integrable since x is integrable and f is bounded).

2) if $f \in L^\infty(\nu;F')$, then f defines an operator

$$S_f : y \longmapsto (t \longmapsto \langle y,f(t)\rangle) \qquad (y \in F)$$

from F into $L^\infty(\nu)$.

3) is defined by transposition.

Simple estimates show that these injections are isometric. Now in the case where F is finite dimensional, it is easy to see that these are all surjective (for example, by working componentwise, one can reduce to the case where F is one-dimensional). Hence we have the following natural isometric equalities in this case:

$$\begin{aligned}\mathrm{Hom}(F,L^\infty(\nu)) &\cong L^\infty(\nu;F') \cong \mathrm{Hom}(L^1(\nu);F') \\ &\cong L^1(\nu;F)'.\end{aligned}$$

Note that all of these expressions represent functors from the category BAN_1 into itself and we are claiming that these functors are equivalent with respect to suitable natural transformation, a fact which we shall use implicitly in the following proof:

3.9. <u>Proposition</u>: Let E be a Banach space and denote by E'_σ the Saks space $(E',\|\ \|,\sigma(E',E))$. Then there are natural Saks space isomorphisms

$$\begin{aligned}\mathrm{Hom}(E,L^\infty(\nu)) &= L^\infty(\nu;E'_\sigma) = \mathrm{Hom}(L^1(\nu),E'_\sigma) \\ &= L^1(\nu;E)'.\end{aligned}$$

<u>Proof</u>: The above preparatory remarks and results allow us to reduce the proofs to formal manipulations with the various functors:

$$\begin{aligned}
\mathrm{Hom}(E,L^{\infty}(\nu)) &= \mathrm{Hom}(B\varinjlim_{F\in\mathcal{F}(E)} F;L^{\infty}(\nu)) \\
&= S\varprojlim_{F\in\mathcal{F}(E)} \mathrm{Hom}(F,L^{\infty}(\nu)) \\
&= S\varprojlim_{F\in\mathcal{F}(E)} L^{\infty}(\nu;F') \\
&= L^{\infty}(\nu;E'_{\sigma}).
\end{aligned}$$

$$\begin{aligned}
L^{\infty}(\nu;E'_{\sigma}) &= S\varprojlim_{F} L^{\infty}(\nu;F') \\
&= S\varprojlim_{F} \mathrm{Hom}(L^{1}(\nu),F') \\
&= \mathrm{Hom}(L^{1}(\nu),S\varprojlim_{F} F') \\
&= \mathrm{Hom}(L^{1}(\nu),E'_{\sigma}).
\end{aligned}$$

$$\begin{aligned}
\mathrm{Hom}(L^{1}(\nu),E'_{\sigma}) &= \mathrm{Hom}(L^{1}(\nu),\ S\varprojlim_{F} F') \\
&= S\varprojlim_{F} \mathrm{Hom}(L^{1}(\nu),F') \\
&= S\varprojlim_{F} L^{1}(\nu;F)' \\
&= (B\varinjlim_{F} L^{1}(\nu;F))' \\
&= L^{1}(\nu;E)'.
\end{aligned}$$

In the last step we are using the fact that

$$L^{1}(\nu;E) = B\varinjlim_{F} L^{1}(\nu;F)$$

This follows from the fact that $L^1(\nu;F)$ is (isometrically) a subspace of $L^1(\nu;E)$ and $\bigcup_{F \in \mathcal{F}(E)} L^1(\nu;F)$ is clearly dense in $L^1(\nu;E)$.

§ 4. INTEGRATION OF SAKS SPACE VALUED FUNCTIONS.

4.1. Let $(E, \|\ \|, \tau)$ be a Saks space, (Ω, ν) a measure space as above. If $x : \Omega \longrightarrow E$ is a measurable function such that

1) for each $p \in S$, $\omega_p \circ x$ is Bochner integrable;
2) $\sup\limits_{A \in \Sigma} \sup\limits_{p \in S} \{ \| \int_A \omega_p \circ x \, d\nu \|_p \} < \infty$

then for each $A \in \Sigma$, $\{ \int_A \omega_p \circ x \, d\nu \}$ is an element of $\hat{E}_\gamma$, the Saks space completion of E. We then say that x is integrable and write $\int_\Omega x \, d\nu$ $(\in \hat{E}_\gamma)$ for its integral.

We can also define $\int_A x \, d\nu$ $(A \in \Sigma)$. If $\int_A x \, d\nu \in E$ for each $A \in \Sigma$, we say that x is E-integrable.

4.2. Examples: If E is a Banach space, then we obtain the notion of Bochner integrability. If E is a Banach space with structure $(\|\ \|, \sigma(E,E'))$ then integrability in the above sense is Dunford integrability (or Gelfand integrability) - cf. DIESTEL-UHL [4], Ch. 2. E-integrability is, in this case, Pettis-integrability.

Now if E is a Banach space, we denote by $L^\infty_E(\nu; E''_\sigma)$ the space of functions in $L^\infty(\nu; E''_\sigma)$ (E''_σ is the Saks space $(E'', \|\ \|, \sigma(E'',E'))$) which are E-integrable (i.e. $\int_A x d\nu \in E$ for each $A \in \Sigma$).

4.3. Proposition: There is a natural isomorphism

$$\mathrm{Hom}(L^1(\nu), E) = L^\infty_E(\nu; E'')$$

Proof: By Proposition 3.9, we have

$$\mathrm{Hom}(L^1(\nu), E''_\sigma) = L^\infty(\nu; E''_\sigma)$$

and it is clear that a function on the right-hand side induces an operator with values in E if and only if it is E-integrable.

The following Lemma is a version of the Pettis theorem on weak and strong measurability:

4.4. Lemma: Let E be a Saks space where $(E, \| \|)$ is a separable Banach space and $\tau = \sigma(E, E')$. Then there is a natural isometric isomorphism between the Banach spaces

$$L^\infty(\nu; E_{\| \|}) \quad \text{and} \quad L^\infty(\nu; E).$$

If F and G are Banach spaces , we write $L_C(F,G)$ for the (Banach) space of compact operators in $L(F,G)$. We also write $L_C^\infty(\nu;F)$ for the space of functions in $L^\infty(\nu;F)$ which have essentially relatively compact range (so that $L_C^\infty(\nu;F) = L^\infty(\nu) \hat{\otimes} F$ is the norm closure of $L_S^\infty(\nu;F)$, the space of measurable simple functions).

4.5. Proposition: If E is a Banach space, the isomorphism

$$\mathrm{Hom}(L^1(\nu); E) \cong L_E^\infty(\nu; E''_\sigma)$$

induces an isometric isomorphism between the Banach spaces

$$L_C(L^1(\nu), E) \quad \text{and} \quad L_C^\infty(\nu; E).$$

Proof: We need only show that each $T \in L_C(L^1(\nu), E)$ is representable by a function in $L_C^\infty(\nu;E)$. But by 1.16 we know that T factors over a mapping $R : L^1(\nu) \longrightarrow G$ where G is a Saks space of the form $(H, \| \|, \sigma(H,H'))$ where the Banach space H is separable. Now R is representable by a function in $L^\infty(\nu; H_\sigma)$ and so by a function in $L^\infty(\nu; H_{\| \|})$ by the Lemma.

Similarly, we denote by $L_{WC}(F,G)$ the space of weakly compact operators between the Banach spaces F and G and by $L^\infty_{WC}(\nu;F)$ the space of functions in $L^\infty(\nu;F)$ which have essentially weakly compact range.

4.6. <u>Proposition</u>: The isomorphism

$$\text{Hom}(L^1(\nu);E) = L^\infty_E(\nu;E''_\sigma)$$

induces an isometric isomorphism between the Banach spaces

$$L_{WC}(L^1(\nu);E) \quad \text{and} \quad L^\infty_{WC}(\nu;E).$$

We remark that in the special case where E is separable this can be proved exactly as 4.5. The general case follows from the fact that a weakly compact operator from $L^1(\nu)$ into E has separable range (sketch of proof: since $L^1(\nu)$ has the DUNFORD-PETTIS property, such an operator takes relatively weakly compact sets into compact sets. Hence it takes the set $\{\chi_A : A \in \Sigma\}$ into a relatively compact set. But the former generates $L^1(\nu)$ - so the range of the operator is compactly generated and so separable) - for details see DIESTEL and UHL [4], pp. 75,76).

4.7. <u>The spaces $L^q(\nu;E)$</u>: Let E be a Saks space. We denote by $\mathcal{L}^q(\nu;E)$ $(1 \leq q < \infty)$ the space of measurable functions

$$x : \Omega \longrightarrow E$$

for which $\int' \|x\|^q d\nu < \infty$ where $\int'$ is the upper integral ($\|x\|^q$ need not be integrable). Then

$$\| \ \| : x \longmapsto (\int' \|x\|^q d\nu)^{1/q}$$

is a seminorm on $\mathcal{L}^q(\nu;E)$ and induces a norm (also denoted by $\| \ \|_q$) on $L^q(\nu;E) := \mathcal{L}^q(\nu;E)/N_q$ where

$$N_q := \{x \in \mathcal{L}^q : \omega_p \circ x = 0 \quad \nu \text{ a.e. for each } p \in S\}.$$

We regard $L^q(\nu;E)$ as a Saks space with the norm and the locally convex topology defined by the seminorms

$$x \longmapsto (\int \|\omega_p \circ x\|^q d\nu)^{1/q}$$

$(p \in S)$.

4.8. <u>Examples</u>: If E is a Banach space, then this space coincides with the classical notion of Banach space valued L^q-spaces. If E has the form $(F', \| \ \|, \sigma(F',F))$ (F a Banach space) then we obtain the lower-star space $L^q_*(\nu;E')$ studied by SCHWARTZ in [14].

4.9. <u>Proposition</u>: If E is a complete Saks space with canonical representation $E = S \varprojlim_p \{E_p : p \in S\}$ then

$$L^q(\nu;E) = S \varprojlim_p L^q(\nu;E_p).$$

In particular, $L^q(\nu;E)$ is complete.

$S \varprojlim_p L^q(\nu;E_p)$ is surjective which is the only non-trivial part of the Proposition.

Proof: If $x \in L^{q}(\nu;E)$ then $(\omega_p \circ x)_{p\in S}$ is a thread which defines an element of the right hand side. On the other hand let (x_p) be a thread in $S \varprojlim L^q(\nu;E_p)$. Note that $(\|x_p(.)\|_p)$ is an increasing family of elements of $L^q(\nu)$ and that by the definition of a Saks-space projective limit $\{\int (\|x_p\|_p)^q d\nu\}$ is uniformly bounded. Let $\{p_n\}_{n=1}^{\infty}$ be an increasing sequence in S such that

$$\sup_{n\in\mathbb{N}} \int (\|x_{p_n}\|_{p_n})^q d\nu = \sup_{p\in S} \int (\|x_p\|_p)^q \, d\nu$$

and let

$$b : \Omega \longmapsto \mathbb{R}_+ \cup \{\infty\}$$

$$\omega \longmapsto \sup_{n\in\mathbb{N}} \|x_{p_n}(\omega)\|_{p_n} .$$

b is ν a.e. defined and finite valued and by the monotone convergenc theorem it belongs to $L^q(\nu)$. Also for $p \in S$, we have

$$\|x_p(\omega)\|_p \le b(\omega) \qquad \nu \text{ a.e.}$$

Indeed for fixed p let p_n' be an increasing sequence in S such that $p_n' \ge p$ and $p_n' \ge p_n$.

Let

$$b' : \omega \longmapsto \sup_{n\in\mathbb{N}} \|x_{p_n'}\|_{p_n'} .$$

Then $b' \ge b$ and $\|b'\|_{L^q} = \|b\|_{L^q}$, whence they are equal ν a.e. and so b dominates $\|x_p(.)\|_p$ ν a.e. .

Define

$$y_p = \frac{x_p}{b} ,$$

with the convention $\frac{0}{0} = 0$. Then (y_p) defines a thread in $S \varprojlim L^{\infty}(\nu;E$ and by the vector - valued lifting theorem 3.5 we can lift it to a thread $(\tilde{y}_p)$ in $S \varprojlim \mathcal{L}^{\infty}(\nu;E_p)$. This thread pieces together to an element $\tilde{y}$ of $\mathcal{L}^{\infty}(\nu;E)$, Let y be the corresponding equivalence class in $L^{\infty}(\nu;E)$ and finally put

$$x := b.y ,$$

which is the element of $L^q(\nu;E)$ corresponding to (x_p). This shows tha the above mapping from $L^q(\nu;E)$ into $S \varprojlim L^q(\nu;E_p)$ is surjective whic is the only non-trivial part of the proposition.

4.10. <u>Proposition</u>: Let E be a Banach space. Then there is a natural isomorphism

$$L^q(\nu;E)' = L^{q'}(\nu;E'_\sigma) \qquad (1 \leq q < \infty)$$

where q' is conjugate to q and E'_σ is the Saks space $(E', \|\ \|, \sigma(E',E))$.

<u>Proof</u>:
$$\begin{aligned} L^q(\nu;E)' &= L^q(\nu; B\varinjlim_{F\in\mathcal{F}(E)} F)' \\ &= (B\varinjlim_{F\in\mathcal{F}(E)} L^q(\nu;F))' \\ &= S\varprojlim_{F} (L^q(\nu;F)') \\ &= S\varprojlim_{F} L^{q'}(\nu;F')) \\ &= L^{q'}(\nu;E'_\sigma). \end{aligned}$$

We note that once again the duality is generated by the bilinear form of integration of the bilinear mapping of pointwise evaluation.

§ 5. REPRESENTATION OF OPERATORS ON $C^b(X)$.

In this paragraph we consider Riesz representation theorems for continuous linear operators from the Saks space $C^b(X)$ (cf. I.3.II) into a Saks space. We prove one main theorem for operators with values in a Saks space with compact unit ball. Using the machinery developed above, this can be proved in a few lines. We then show how the classical representation theorems for bounded, weakly compact and compact operators from $C(K)$ into a Banach space follows immediately from this.

First we recall that if F is a Banach space, X a completely regular space, then if $\mu : Bo(X) \longrightarrow F$ is a Radon measure.

$$T_\mu : x \longmapsto \int x \, d\mu$$

is a β-continuous linear operator from $C^b(X)$ into F. $\mu \longmapsto T_\mu$. is an isometry from $M_R(X;F)$ into $Hom(C^b(X),F)$. In general it is not onto as the example of the identity operator on $C[0,1]$ shows. However, if F is finite dimensional it clearly is (once again, we can easily reduce to the case where F is one-dimensional and then the result can be found, for example, in COOPER [2], II.3.3).

Hence we have, for finite dimensional Banach spaces F,

$$M_R(X;F) \cong Hom(C^b(X),F).$$

Once again, this is a natural isomorphism of functors.

The above isomorphism is a Saks space isomorphism if we give the left-hand side the topology of pointwise convergence on $C^b(X)$ as auxiliary topology.

5.1. <u>Proposition</u>: Let $(E, \| \|, \tau)$ be a Saks space with $B_{\| \|}$ τ-compact, X a completely regular space. Then if $T : C^b(X) \longrightarrow E$ is a β-γ continuous linear operator then there exists a Radon measure $\mu : Bo(X) \longrightarrow E$ representing T i.e. T is the operator

$$T_\mu : x \longmapsto \int x \, d\mu$$

Conversely, every Radon measure μ defines a β-γ continuous linear operator T_μ in the above manner.
Hence integration establishes a Saks space isomorphism

$$M_R(X;E) = Hom(C^b(X), E).$$

<u>Proof</u>: We put $G := E'_\gamma$ and calculate:

$$\begin{aligned} Hom(C^b(X), E) &= Hom(C^b(X), S\varprojlim_{F \in \mathfrak{F}(G)} F') \\ &= S\varprojlim_{F} Hom(C^b(X), F') \\ &= S\varprojlim_{F} M_R(X, F') \\ &= M_R(X, S\varprojlim_{F} F') \\ &= M_R(X, E). \end{aligned}$$

5.2. <u>Remark</u>: A less formal demonstration of the above result goes as follows: since T maps bounded sets in $C^b(X)$ into relatively compact subsets of (E,γ), then

$$T'' : C^b(X)'' \longrightarrow E''$$

actually takes its values in E (cf. 1.17).

Noting that if A is a Borel set in X then its characteristic function χ_A defines, by integration, an element of $C^b(X)'' = (M_R(X), \|\ \|)'$ we may define

$$\mu(A) := T''(\chi_A)$$

which is an E-valued measure.

By the continuity of T" (with respect to the norm in $C^b(X)''$) we can deduce that

$$T''(x) = \int x \, d\mu$$

for every bounded, Borel-measurable function on S and so in particular, for $x \in C^b(X)$. The converse fact is easy.

5.3. <u>Corollary</u>: If T and X and E are as above, then the following are equivalent:

1) T is β-$\|\ \|$ continuous;

2) T is compact i.e. takes some β-neighbourhood of zero to a relatively compact set in E;

3) the semi-variation $\|\mu\|$ of μ (with respect to the norm in E) is tight i.e. for each $\varepsilon > 0$ there exists a $K \in \mathcal{K}(S)$

so that for each $A \in Bo(X)$ with $A \subseteq X \setminus K$, $\|\mu\|(A) < \varepsilon$.

Proof: The equivalence of 1) and 2) follows from Proposition 1.13 (which is trivial in this case since $B_{\| \|}$ is compact). The equivalence of 2) and 3) is clear.

Using the above result, we can now easily obtain a result for general operators with values in a locally convex space.

5.4. Proposition: Let E be a locally convex space, X a completely regular space. Then any continuous, linear operator $T : C^b(X) \longrightarrow E$ may be represented by integration with respect to a Radon measure μ from $Bo(X)$ into $(E'',\sigma(E'',E'))$. In fact, μ takes its values in the $\sigma(E'',E')$-closure of $T(B(C^b(X)))$.

If T maps the unit ball of $C^b(X)$ into a relatively weakly compact subset of E, then μ takes its values in E (actually in $\overline{T(B(C^b(X)))}$) and is a Radon measure with respect to the original topology in E.

Proof: For the first assertion, let B be the $\sigma(E'',E')$-closure of $T(B(C^b(X)))$ in E" And let F be the Saks space spanned by B in E" with $\| \|_B$ as norm and $\sigma(E'',E')$ as auxiliary topology. Then this is a Saks space with compact unit ball and the result follows immediately from 5.1.

In the second case, take $B := \overline{T(B(C^b(X)))}$, the closure now being taken in E, and define the Saks space F to be $(E_B, \| \|_B, \sigma(E,E'))$. Then T is represented by an F-valued Radon measure μ (i.e. Radon with respect to $\sigma(E,E')$). Now the Orlicz-Pettis theorem for Radon measures implies that μ is a Radon measure with respect to the finest topology on F compatible with the duality between (F,F'_γ) and in particular with respect to the topology of E.

5.5. Remark: We note that if $\mu \in M_R(X;E)$ is such that its range is contained in a weakly compact, absolutely convex set, then the associated integration operator $T : C^b(X) \longrightarrow E$ sends the $C^b(X)$-ball into this same set and T is weakly continuous and so continuous if $(C^b(X),\beta)$ is a Mackey space (e.g. if S is locally compact, paracompact).

5.6. Proposition: Let (E,τ) be a quasi-complete locally convex space, $T : C^b(X) \longrightarrow E$ a β-continuous linear operator. If T does not map the unit ball of $C^b(X)$ into a relatively weakly compact subset of E there is a sequence (x_n) of functions in $C^b(X)$ with mutually disjoint supports so that if j is the mapping $(\lambda_n) \longrightarrow \Sigma \lambda_n x_n$ from c_o into $C^b(X)$ and A denotes the β-closed span of $\{x_n\}$ in $C^b(S)$ then in the following diagram

$$\begin{array}{ccccc} & & C^b(X) & \xrightarrow{T} & E \\ & & \uparrow\cup & & \uparrow\cup \\ c_o & \xrightarrow{j} & A & \xrightarrow{T|A} & T(A) \end{array}$$

j and $T_{|A}$ are isomorphisms. More informally, T' fixes a subspace of $(C^b(X),\beta)$ which is isomorphic to c_o.

Consequently, if E fails to contain a copy of c_o then every continuous linear operator $T : C^b(X) \longrightarrow E$ takes the unit ball into a relatively weakly compact subset of E.

<u>Proof</u>: If T fails to satisfy the given condition, then by Lemma 1.17, $T' : E' \longrightarrow M_R(X)$ takes some equicontinuous set H in E' to a subset of $M_R(X)$ which is bounded but not relatively $\sigma(M(X),M(X)')$-compact. Then by GROTHENDIECK's characterisation of weakly compact sets in $M_R(X)$ (cf. BUCHWALTER and BUCCHIONI [1], p.76) there exists a sequence (f_n) in H and a sequence (U_n) of disjoint open sets in S and an $\varepsilon > 0$ so that $|T'(f_n)(U_n)| > \varepsilon$ $(n \in \mathbb{N})$. i.e. $|f_n \circ \mu(U_n)| > \varepsilon$ (where μ represents T). By ROSENTHAL's Lemma (sf. DIESTEL-UHL [4], I.4.1) we may suppose that

$$|f_n \circ \mu| \; (\bigcup_{m \neq n} U_m) < \varepsilon/2 \qquad (n \in \mathbb{N}).$$

Now choose a sequence (x_n) in $C^b(S)$ so that $|x_n| \leq \chi U_n$ and

$$|f_n \circ T(x_n)| = |\int_S x_n d(f_n \circ \mu)| > \varepsilon$$

which is possible since $f_n \circ \mu$ is a Radon-measure. Then j, as defined in the statement of the theorem, is clearly a well-defined, continuous injection. We claim that

$$||T \circ j((\lambda_n))||_H \geq \varepsilon/2 \; ||(\lambda_n)||_{c_o}$$

for each $(\lambda_n) \in c_o$, where $\| \|_H$ denotes the seminorm of uniform convergence on the equicontinuous set H. Indeed for any $(\lambda_n) \in c_o$ and any $k \in \mathbb{N}$,

$$\|T \circ j((\lambda_n))\|_H \geq |\langle T \circ j((\lambda_n)), f_k \rangle| = \left| \int_S \sum_{n \in \mathbb{N}} \lambda_n x_n d(f_k \circ \mu) \right|$$

$$\geq \left| \int_S \lambda_k x_k d(f_k \circ \mu) \right| - \|(\lambda_n)\|_{c_o} |f_k \circ \mu| \left(\bigcup_{l \neq k} U_l \right)$$

$$\geq |\lambda_k| \varepsilon - \|(\lambda_n)\|_{c_o} \cdot \varepsilon/2$$

Taking the supremum over k on the right-hand side we get the required estimate.

This shows that $(T \circ j)^{-1}$ is well-defined and continuous on $T \circ j(c_o)$, from which it follows that j, as an operator from c_o to $j(c_o)$ and T, as an operator from $j(c_o)$ to $T \circ j(c_o)$ are isomorphisms (by the following trivial Lemma).

5.7. <u>Lemma</u>: Let X,Y,Z be topological spaces, $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ continuous, surjective mappings such that $g \circ f$ is an isomorphism. Then f and g are also isomorphisms.

<u>Proof</u>: The injectivity of $g \circ f$ implies that of g and f so that f^{-1} and g^{-1} are well-defined. But as $g^{-1} = f \circ (g \circ f)^{-1}$ and $f^{-1} = (g \circ f)^{-1} \circ g$ it is clear that f and g are continuous.

With these results it is now easy to prove the following Proposition:

5.8. Proposition: Let $(E,\|\ \|,\tau)$ be a complete Saks space, $T : C^b(S) \longrightarrow E$ an β-γ continuous linear operator with representing measure $\mu \in M_R(S;E'')$. Then the following are equivalent:

1) T does not fix a copy of c_o;

2) T maps the unit ball of $C^b(S)$ into a relatively weakly compact set;

3) T maps weakly summable sequences to summable sequences;

4) T maps weakly Cauchy sequences into convergent sequences;

5) T maps sequences which tend weakly to zero to convergent sequences;

6) if (x_n) is a bounded sequence of functions in $C^b(S)$ with mutually disjoint supports, then $Tx_n \longrightarrow 0$ in E;

7) T maps weakly compact sets in $C^b(S)$ into compact sets;

8) μ takes its values in E;

9) μ is a Radon measure with values in E;

10) μ is a strongly additive measure with values in E.

Proof: 1) $\Longleftrightarrow$ 2) is Proposition 5.6.

2) $\Longrightarrow$ 9) is Proposition 5.4.

9) $\Longrightarrow$ 2) - 6) are all simple applications of the Lebesgue-dominated convergence theorem. The reverse implications all follow from the fact that if μ is not Radon then the (x_n) constructed in the proof of Proposition 5.6 supply counter-examples.

9) $\Longrightarrow$ 8) is clear and 8) $\Longrightarrow$ 10) follows from the weak σ-additivity of μ and ORLICZ-PETTIS. 10) implies 2) is embedded in the proof of Proposition 5.6.

9) $\Longrightarrow$ 7): Note that $(C^b(S),\beta)$ is not necessarily complete so that we cannot use EBERLEIN-SMULIAN. Firstly, we may assume that E is a Banach space (using the characterisation of compactness in Saks spaces (cf. COOPER [2], I.1.12). Then $\mu : Bo(S) \longrightarrow E$ has tight semi-variation norm so that for $\varepsilon > 0$ we may find $K \in \mathcal{K}(S)$ so that if T_K denotes the operator associated to $\mu|_K$ then $\|T - T_K\| < \varepsilon$. Now if B is weakly compact in $C^b(S)$, then $T_K(B)$ is compact in $(E,\|\ \|)$ (using factorisation through C(K) and EBERLEIN-SMULIAN) and by Lemma 1.10 we may conclude that T(B) is compact in $(E,\|\ \|)$.

In the following we give two applications of the theory developed here - to ORLICZ-PETTIS type theorems and to the spectral theory for unbounded operators on Hilbert space. Firstly, suppose that E is a locally convex space and that D is a total subset of E'. Let $T : \ell^\infty(\mathbb{N}) \longrightarrow E$ be a linear operator and put $x_n = Te_n$.

5.9. <u>Lemma</u>: T is β-$\sigma(E,D)$ continuous if and only if for every sequence $(\lambda_n) \in \ell^\infty(\mathbb{N})$, $\Sigma\, \lambda_n x_n$ is $\sigma(E,D)$-convergent.

<u>Proof</u>: $\Longrightarrow$ follows from the fact that $\Sigma\, \lambda_n e_n$ is β-convergent. $\Longleftarrow$: it suffices to show that if $f \in D$, then $T^*f \in \ell^1(\mathbb{N})$ for then T is $\sigma(\ell^\infty,\ell^1)$-$\sigma(E,D)$ continuous and so β-continuous (since

β is stronger than $\sigma(l^\infty, l')$. But if $(\lambda_n) \in \ell^\infty(\mathbb{N})$, $f \in D$, then

$$T^* f\left(\sum_{n=1}^{N} \lambda_n e_n\right) = f\left(\sum_{n=1}^{N} \lambda_n x_n\right) \longrightarrow f\left(\sum_{n=1}^{\infty} \lambda_n x_n\right)$$

and so T^*f is β-continuous i.e. $T^*f \in \ell^1(\mathbb{N})$.

5.10. <u>Proposition</u>: If (x_n) is a sequence in E so that $\sum_n \lambda_n x_n$ converges weakly in E for each $(\lambda_n) \in \ell^\infty(\mathbb{N})$ (i.e. if (x_n) is weakly unconditionally convergent) then $\sum_n x_n$ (in fact $\sum_n \lambda_n x_n$ for each $(\lambda_n) \in \ell^\infty(\mathbb{N})$) is unconditionally convergent in E.

<u>Proof</u>: By the above Lemma, the hypothesis implies that

$$T : (\lambda_n) \longmapsto \lim \sum_{n=1}^{N} \lambda_n x_n$$

is β-$\sigma(E,E')$ continuous and so β-continuous (for the original topology of E) since β is the Mackey topology.

5.11. <u>Proposition</u>: If E is separable, D a total subset of E' and (x_n) is such that $\sum_n \lambda_n x_n$ is $\sigma(E,D)$-convergent for each $(\lambda_n) \in \ell^\infty(\mathbb{N})$ then $\sum_n \lambda_n x_n$ is convergent (in the original topology of E) for each (λ_n).

<u>Proof</u>: As above, except that we now use the closed graph theorem for Saks spaces (cf. COOPER [2], I.4.24) to deduce the continuity of T.

Using standard techniques (cf. THOMAS [17], Prop. 0.1) we can deduce the following ORLICZ-PETTIS theorems for measures.

5.12. Proposition: Let D be a total subset of the dual E' of the locally convex space E and suppose that

1) D = E'

or 2) E is separable.

Then any measure $\mu : \Sigma \longrightarrow E$ (Σ a σ-field) which is σ-additive for $\sigma(E,D)$ is σ-additive for the original topology of E.

5.13. Proposition: Let K be an Eberlein compactum, D a dense subset of K. Then a measure $\mu : \Sigma \longrightarrow C(K)$ (Σ a σ-field) is norm σ-additive if it is σ-additive for the topology τ_D of pointwise convergence on D.

This generalises a well-known result for metrisable compacta. (cf. BUCHWALTER-BUCCHIONI [1], 1.6.2). The proof uses the fact that if K is an Eberlein compactum, then C(K) is weakly compactly generated and so we can apply a closed graph theorem for operators from $\ell^\infty(\mathbb{N})$ into C(K) (cf. COOPER [2], I.4.33, p.6).

5.14. Proposition: Let (x_n) be a sequence in the locally convex space E so that each $(\lambda_n) \in \ell^\infty(\mathbb{N})$, $\sum_n \lambda_n x_n$ is $\sigma(E,D)$-convergent where D is a total subset of E' with the property that every $\sigma(E,D)$-bounded set is bounded. Then either Σx_n converges (unconditionally) or there are disjoint blocks

$$y_k := x_{n_k} + \ldots\ldots + x_{n'_k}$$

(where $n_1 < n'_1 < n_2 < n'_2 < \ldots$)

so that the mapping $(\lambda_k) \longrightarrow \sum_k \lambda_k y_k$ is an isomorphism from $(\ell^\infty, \|\ \|)$ into E.

<u>Proof</u>: Define the mapping $T : \ell^\infty(\mathbb{N}) \longrightarrow E$ by

$$(\lambda_n) \longrightarrow \sum_n \lambda_n x_n \quad (\sigma(E,D)\text{-limit}).$$

By the assumption, $T(B(\ell^\infty))$ is bounded in E i.e. T is $\|\ \|$-continuous. If $\sum_n x_n$ does not converge unconditionally, there is a continuous seminorm p on E and blocks (y_k) so that $(p(y_k))$ is bounded away from zero. A standard application of ROSENTHAL's Lemma (cf. DIESTEL-UHL [4], Ch. 1) concludes the proof.

Now suppose that T is a self-adjoint, densely defined but not necessarily continuous operator on a Hilbert space H. We shall show how the theory developed in this section can be applied to deduce the spectral theorem for T (cf. BUCHWALTER and BUCCHIONI [1] for a corresponding treatment for bounded operators). We recall the following facts which can be proved by elementary means, resp. by using the spectral theorem for bounded, self-adjoint operators (for proofs, see RIESZ and SZ.-NAGY [13]):

I. The operators $B := (I + T^2)^{-1}$ and $C := T(I + T^2)^{-1}$ are bounded, continuous and everywhere defined.

II. If $B = \int_0^1 \lambda \, dF_\lambda$ is the spectral representation of B and $P_n := F_{1/n} - F_{1/n+1}$, $H_n := P_n H$, then $H = \ell^2(H_n)$, the ℓ^2 - or

Hilbert sum of the H_n and each H_n reduces T to a bounded, self-adjoint operator T_n.

Let $\sigma(T)$ denote the spectrum of T (so that $\sigma(T)$ need not be a compact subset of $\mathbb{R}$). Now denote by y_1 the function $t \longmapsto t(1+t^2)^{-1}$ and by y_2 the function $t \longmapsto (1+t^2)^{-1}$. Let A be the subalgebra of $C^b_{\mathbb{R}}(\sigma(T))$ (the subscript denotes real-valued functions) generated by y_1, y_2 and the constants. Then if we define $y_1(T)$ and $y_2(T)$ to be $T(I+T^2)^{-1}$ and $(I+T^2)^{-1}$ respectively, it is clear how we can define an operator $\Phi : x \longmapsto x(T)$ from A into L(H). We shall sketch briefly a proof of the fact that Φ is a Saks space algebra morphism from $(A, \|\ \|, \tau_K)$ into $(L(H), \|\ \|, \tau_W)$.
First of all it is easy to see that Φ is norm-decreasing (for example, by applying the corresponding result for bounded operators componentwise to the decomposition $H = \ell^2(H_n)$).
Now suppose that $x_\alpha \longrightarrow 0$ in (A, β) and assume that (x_α) lies in the unit ball of A. Let $y = (y_n)$ and $z = (z_n) \in H = \ell^2(H_n)$. Then we can choose N so that

$\sum_{n=N+1}^{\infty} \|y_n\|^2 + \sum_{n=N+1}^{\infty} \|z_n\|^2 < \varepsilon$. Then we can show that

$$(x_\alpha(T)y|z) \longrightarrow 0$$

by estimating the tails and using the fact that $x_\alpha \longrightarrow 0$ uniformly on $\sigma(\bigoplus_{n=1}^{N} T_n)$ (the restriction of T to $\bigoplus_{n=1}^{N} H_n$) so that $x_\alpha(T)\Big|_{\sum_{n=1}^{N} H_n} \longrightarrow 0$ uniformly.

Now A is β-dense in $C^b_{\mathbb{R}}(\sigma(T))$ by the Stone-Weierstraß theorem and so we can extend Φ to a β-continuous algebra morphism from $C^b_{\mathbb{R}}(\sigma(T))$ to L(H). We continue to denote the image of $x \in C^b_{\mathbb{R}}(\sigma(T))$ by x(T). Hence we have proved (rather sketchily) the following result:

5.15. <u>Proposition</u>: Let T and H be as above. Then there is a Saks space algebra morphism $\Phi : x \longrightarrow x(T)$ from $(C^b_{\mathbb{R}}(\sigma(T)), \|\ \|, \tau_K)$ into $(L(H), \|\ \|, \tau_W)$ which extends the "substitution operator" on A.

If we now apply 5.1 we obtain the result that there is a Radon measure $\mu : Bo(\sigma(T)) \longrightarrow L(H)$ so that

$$x(T) = \int_{\sigma(T)} x \, d\mu. \quad (x \in C^b(\sigma(T)).$$

Now Φ lifts (by integration) to a mapping from $L^\infty(\sigma(T))$ into L(H) which is easily seen to be multiplicative.

Since $\mu(A) = \Phi(\chi_A)$, μ takes its values in the set of self-adjoint projections and is multiplicative (i.e. $\mu(A \cap B) = \mu(A)\mu(B)$ for $A \in Bo(\sigma(T))$). Hence the above result is just the classical spectral theorem for unbounded, self-adjoint operators. Similar methods can be applied to obtain the spectral theorem for unbounded normal operators.

§ 6. ABSOLUTELY SUMMING OPERATORS ON $C^b(S)$.

In this section, we characterise absolutely summing operators on $C^b(S)$. We begin with some preliminaries:

6.1. Definition: Let (E,τ) be a locally convex space. A family $(x_i)_{i\varepsilon I}$ in E is

weakly summable if $\sum_{i\varepsilon I} |f(x_i)| < \infty$ for each $f \varepsilon E'$;

summable if the finite partial sums form so τ-Cauchy set;

absolutely summable if $\sum_{i\varepsilon I} p(x_i) < \infty$ for each τ-continuous seminorm p on E.

Note that (x_i) is weakly summable if and only if

$$\{ \sum_{i\varepsilon J} x_i : J \varepsilon \mathcal{J}(I)\}$$

is weakly bounded (and so bounded) in E (where $\mathcal{J}(I)$ denotes the family of finite subsets of I).

A family (x_i) in a Saks space is absolutely summable if its partial sums are norm-bounded and it is τ-absolutely summable. Note that this is not the same as γ-absolute summability (cf. the sequence (e_n) in $(\ell^\infty(\mathbb{N}), \| \|, \tau_p)$ where τ_p is the topology of coordinatewise convergence).

If E is a locally convex space with defining family S of seminorms, $\ell^1\{E\}$ denotes the space of absolutely summable sequences in E, regarded as a locally convex space with the family of seminorms $\{\tilde{p} : p \varepsilon S\}$ where

$$\tilde{p} : (x_n) \longmapsto \sum_{n\varepsilon \mathbb{N}} p(x_n).$$

(E,τ) satisfies property (B) (cf. PIETSCH [12], § 1.5.5) if each bounded subset of $\ell^1\{E\}$ is contained and bounded in some $\ell^1\{E_B\}$ where B is absolutely convex and bounded in E. Examples of spaces with this property are metrisable and dualmetric spaces (cf. Th. 1.5.8 of the above reference). Of particular interest to us is the following result, which is originally due to NOUREDDINE ([11]):

6.2. Proposition: Let $(E,\|\ \|,\tau)$ be a Saks space. Then (E,γ) has property (B).

Proof: Let B be a subset of $\ell^1\{E\}$ which is not bounded for the norm i.e. is such that

$$\sup\{\Sigma \|x_n\| : (x_n) \in B\} = \infty.$$

We shall show that B is not bounded in the topology induced on $\ell^1\{E\}$ by γ. For each $i \in \mathbb{N}$, we can find $x^i = (x_n^i) \in B$ and $n(i) \in \mathbb{N}$ so that $\sum_{n=1}^{n(i)} \|x_n^i\| > (i+1)^2$.

Choose $f_n^i \in (E,\tau)'$ so that $\|f_n^i\| < 1/i$ and

$$|f_n^i(x_n^i)| > \frac{1}{i+1}\|x_n^i\|$$

(note that $(E,\tau)'$ is $\sigma(E'_{\|\ \|},E)$-dense in $(E,\|\ \|)'$).

Now $C := \{f_n^i : n=1,\dots,n(i),\ i=1,\dots\}$ is a norm-precompact subset of $(E,\tau)'$. Hence its polar U^o in E is a γ-neighbourhood of zero (for γ is finer then the topology $\tau_{pc}(E,E'_\tau)$ of uniform convergence on the norm precompact subsets of E').

Now we have

$$\sum_{n=1}^{\infty} p_U(x_n^i) \geq \sum_{n=1}^{n(i)} p_U(x_n^i) \geq \sum_{n=1}^{n(i)} |f_n^i(x_n^i)|$$

$$\geq \frac{1}{i+1} \sum_{n=1}^{n(i)} \|x_n^i\| > i+1$$

where p_U is the Minkowski functional of U and so is a γ-continuous seminorm on E. Hence B is not bounded in $\ell^1\{E\}$ which was to be demonstrated.

6.3. <u>Corollary</u>: Let (x_i) be a family in a Saks space $(E, \| \|, \tau)$.
Then 1) (x_i) is γ-weakly summable if and only if it is norm-weakly summable;

2) (x_i) is γ-absolutely summable if and only if it is norm absolutely summable.

<u>Proof</u>: 1) follows from the remarks in Def. 6.1 and the fact that the norm and γ-bounded sets are the same.
2) follows from Proposition 6.2.

Note that the corresponding result for summability is not true (once again the sequence (e_n) in $(\ell^\infty(\mathbb{N}), \| \|, \tau_p)$ provides a counterexample).

6.4. <u>Corollary</u>: A family (x_i) in $C^b(S)$ is β-weakly summable if and only if there is an $M > 0$ so that

$$\sum_{i \in I} |x_i(t)| < M$$

for each $t \in S$.

6.5. Definition: A continuous linear operator $T : E \longrightarrow F$ between locally convex spaces is absolutely summing if it maps weakly summable families into absolutely summable ones. Note that if E and F are normed spaces, then this is equivalent to the existence of a constant $\rho > 0$ so that for each finite family $\{x_1, \ldots, x_n\}$ in E

$$\sum_{j=1}^{n} \|Tx_j\| \leq \rho \sup \{ \sum_{j=1}^{n} |f(x_j)| : f \in B(F')\}.$$

The infimum of all such ρ is denoted by $\|T\|_{AS}$, the absolutely summing norm of T. It is a norm on AS(E,F), the vector space of absolutely summing operators from E into F.

6.6. Proposition: If T is an absolutely summing operator from $(C^b(X), \beta)$ into a quasicomplete locally convex space, then it maps the unit ball of $C^b(X)$ into a relatively weakly compact subset of E and so is represented by an E-valued Radon measure.

Proof: If T does not satisfy the conclusion of the Proposition, then by Prop. 5.6 there is a bounded sequence (x_n) of functions in $C^b(X)$ with disjoint support so that $p(T(x_n))$ is bounded away from zero for some continuous seminorm p on E. This obviously implies that T is not absolutely summing.

Now let E be a normed space. Then a β-continuous linear operator $T : C^b(X) \longrightarrow E$ is absolutely summing if and only if it is absolutely summing from $(C^b(X), \| \|)$ into E (this follows from 6.3). In particular, we can define $\|T\|_{AS}$ for such a T and we have that $||T||_{AS}$ is the infimum of those ρ for which

$$\sum_{j=1}^{n} ||Tx_j|| \leq \rho \sup_{t \in S} \sum_{j=1}^{n} |x_j(t)|$$

for each $\{x_1,\dots,x_n\}$ in $C(X)$.

6.7. Lemma: A β-continuous linear operator $T : C^b(X) \longrightarrow E$ (E a Banach space) is absolutely summing if and only if its representing measure μ has bounded variation $|\mu|$. Then we have $||T||_{AS} = |\mu|(X)$.

Proof: Suppose that μ has finite variation $|\mu|$, where (by definition),

$|\mu|(A) = \sup\{\Sigma\, ||\mu(A_i)|| : \{A_i\}$ a finite partition of $A\}$.

Now if $\{x_j\}_1^n$ is a finite family in $C^b(X)$ we have

$$\sum_{j=1}^{n} ||Tx_j|| = \sum_j || \int x_j d\mu || \leq \sum_j \int |x_j| d|\mu|$$

$$\leq \sup_{t \in X} \sum_j |x_j(t)| . |\mu|(X).$$

Hence T is absolutely summing and $||T||_{AS} \leq |\mu|(X)$.

Now let T be absolutely summing. Suppose that $\{A_i\}_1^n$ is a partition of S. We show that $\Sigma\, ||\mu(A_i)|| \leq ||T||_{AS}$. Choose $\{f_1,\dots,f_n\}$ in the ball of E' so that

$$||\mu(A_i)|| = f_i(\mu(A_i)).$$

Since $f_i \circ \mu$ is a bounded Radon measure on X, we can find, for $\varepsilon > 0$, compact sets $K_i \subseteq A_i$ so that

$$|f_j \circ \mu|(\bigcup_{i=1}^{n} A_i \setminus K_i) < \varepsilon/n$$

for each j. Now choose $x_i : X \longrightarrow [0,1]$ so that $x_i = 1$ on K_i, $x_i = 0$ on K_j $(j \neq i)$ and $\Sigma x_i \leq 1$ on X.

$$\text{Then} \quad \|Tx_i\| \geq |f_i(Tx_i)| = |\int x_i d(f_i \circ \mu)|$$

$$\geq |\int_{K_i} x_i d(f_i \circ \mu)| - |\int_{X \setminus K_i} x_i d(f_i \circ \mu)|$$

$$\geq |f_i \circ \mu(A_i)| - |f_i \circ \mu|(\bigcup A_j \setminus K_j)$$

$$\geq \|\mu(A_i)\| - \varepsilon/n.$$

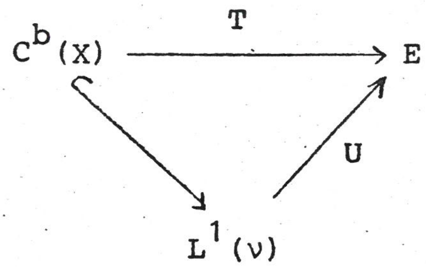
Hence $\Sigma \|\mu(A_i)\| \leq \Sigma \|Tx_i\| + \varepsilon$.

This implies that $|\mu| \leq \|T\|_{AS}$ as was to be shown.

Using this result it is now easy to show:

6.8. Proposition: Let T be a continuous linear operator from $C^b(X)$ into a quasicomplete locally convex space. Then T is absolutely summing if and only if its representing measure μ has finite variation with respect to each continuous seminorm.

6.9. Corollary: Let T be a continuous linear operator from $(C^b(X), \beta)$ into a Banach space E. Then T is absolutely summing if and only if there is a positive Radon measure ν on X so that T factors over the natural injection $C^b(X) \longrightarrow L^1(\nu)$ as follows:



In this case, then ν and U can be so chosen that

$\nu(X) = \|T\|_{AS}$ and $\|U\| \leq 1$.

<u>Proof</u>: Let T be absolutely summing and let μ be its representing measure. Put $\nu = |\mu|$, the variation of μ. Then if $x = \Sigma\, \alpha_i \chi_{A_i} \in S(\nu)$, the space of ν-measurable step functions, we define

$$U(x) := \Sigma\, \alpha_i \mu(A_i).$$

U is well-defined and $\|U\| \leq 1$ - so it extends to an operator from $L^1(\nu)$ with the required properties. Also

$$\|T\|_{AS} = |\mu|(X) = \nu(X).$$

The converse follows from the well-known fact that $C^b(X) \longrightarrow L^1(\nu)$ is absolutely summing (for example, because its representing measure is $A \longmapsto \chi_A$ which is clearly of bounded variation).

Note that the above result is no longer true for arbitrary locally convex spaces E as the following example shows:

6.10. <u>Example</u>: Let I be an uncountable set and consider the natural inclusion $T : \ell^\infty(I) \longrightarrow \mathbb{C}^I$ which is clearly absolutely summing. But any positive, finite Radon measure ν on I is supported by a countable subset of I and so the inclusion $\ell^\infty(I) \subseteq L^1(\nu)$ cannot be injective. Hence, T, being injective, cannot factorise over this mapping.

However, we can generalise the above Proposition as follows:

6.11. <u>Proposition</u>: The statement of Corollary 6.9 holds if E is replaced by a quasicomplete locally convex space with property (B).

<u>Proof</u>: We recall the notation $\ell^1[F]$ (F a locally convex space) for the space of weakly summable sequences in F with locally convex topology defined by the seminorms:

$$\varepsilon_p : (x_n) \longrightarrow \sup\{\sum_n |f(x_n)| : f \in U_p^o\} \quad (p \in S)$$

where U_p is the unit ball of p.

Then if $T : C^b(X) \longrightarrow E$ is absolutely summing, it induces a continuous mapping from $\ell^1[C^b(X)]$ into $\ell^1\{E\}$. Now put

$$D := \{(x_n) \in \ell^1[C^b(X)] : \sum_n |x_n| \le 1\}.$$

Then D is bounded in $\ell^1[C^b(X)]$ and so T(B) is bounded in $\ell^1\{E\}$ and so in some $\ell^1\{E_B\}$, where B is a Banach ball in E. But this means exactly that T is in fact an absolutely summing operator from $C^b(X)$ into the Banach space E_B and so the result follows easily from 6.9.

6.12. Corollary: If X is such that for every bounded, positive Radon measure ν on X, $L^1(\nu)$ is separable (e.g. if the compact subsets of X are metrisable), then any absolutely summing operator T from $C^b(X)$ into a quasicomplete locally convex space E with property (B) has separable range.

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