THE STRONG LAW OF LARGE NUMBERS IN LOCALLY CONVEX SUSLIN SPACES

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Abstract: G.E.F. Thomas raised the question, whether the strong law of large numbers is valid for i.i.d. totally summable sequences $\{X_n\}_{n=1}^{\infty}$ of random variables with values in a quasi-complete locally convex Suslin-space E. We show by a simple truncation-argument that the answer is yes and - using an idea of Chatterji - that for Saksspaces E the assumption of total summability is also necessary for the strong law of large numbers to hold. A final example shows that the assumption that E is Suslin, is essential.

1) Denote by (Ω,Σ,μ) an (abstract) probability space, by (E,τ) a vector space E with a locally convex Hausdorff topology τ. Throughout this paper - except in the example 10) - we shall assume that E is a Suslin space. The letter X denotes a random variable defined on Ω with values in E (measurable with respect to the Borel-σ-algebra of E).

2) Definition (c.f. [6], § 5): X is called totally summable, if there is a closed, absolutely convex, bounded subset B of E, such that if $\|\cdot\|_B$ denotes the gauge-function of B,

$$\int_\Omega \|X(\omega)\|_B \, d\mu(\omega) < \infty.$$

If E is assumed to be quasicomplete, then one may define for

$A \in \Sigma$ the Pettis-integral

$$\int_A X(\omega) \quad d\mu(\omega),$$

which is an element of E ([6]).

3) Let $\{X_n\}_{n=1}$ be a sequence of independent identically distributed (i.i.d.) E-valued random-variables and denote by S_n the partial sums $X_1+\ldots+X_n$.

4) Lemma: Assume there is a compact, convex, metrisable subset K of E such that X_1 takes its values almost surely in K. Then the strong law of large numbers holds, i.e.

$$\lim_{n\to\infty} n^{-1} S_n(\omega) = E(X_1) \qquad \mu\text{-a.s.}$$

Proof: K being metrisable, there is a sequence $\{f_k\}_{k=1}^{\infty}$ in E' which induces the $\mathcal{T}$-topology on K (c.f.[2] for example). For each of the sequences of i.i.d. realvalued, bounded random variables $\{f_k \circ X_n\}_{n=1}^{\infty}$ we may apply the strong law of large numbers, i.e. for each $k \in \mathbb{N}$

$$\lim_{n\to\infty} n^{-1} f_k \circ S_n(\omega) = E(f_k \circ X_1) \qquad \mu\text{-a.s}$$

Whence, as $n^{-1} S_n$ lies almost surely in the convex set K,

$$\lim_{n\to\infty} n^{-1} S_n(\omega) = E(X_1) \qquad \mu\text{-a.s},$$

where $E(X_1)$ denotes the expectation of X_1, which is an element of K.

q.e.d.

5) <u>Theorem:</u> Let $\{X_n\}_{n=1}^{\infty}$ be an i.i.d. sequence of totally summable random variables with values in a quasicomplete locally convex Suslin space E. Then the strong law of large numbers holds, i.e.

$$\lim_{n\to\infty} n^{-1} S_n(\omega) = E(X_1) \qquad \mu\text{-a.s.}$$

Proof:

Replacing if necessary Ω by $E^{\mathbb{N}}$ and μ by its image under the map $\omega \to \{X_n(\omega)\}_{n\in\mathbb{N}}$, we may assume without loss of generality that the underlying probability space is a product space $(\Omega^{\mathbb{N}}, \Sigma^{\mathbb{N}}, \mu^{\mathbb{N}})$ and there is one random-variable $X : \Omega \to E$ such that $X_n = X \circ p_n$, p_n denoting the projection onto the n-th coordinate in $\Omega^{\mathbb{N}}$.

Let B be a closed, absolutely convex, bounded subset of E such that

$$\int \|X(\omega)\|_B \, d\mu(\omega) < \infty.$$

Fix $\varepsilon > 0$. There is $\delta > 0$ such that $A \subseteq \Omega$, $\mu(A) < \delta$ implies $\int_A \|X(\omega)\|_B \, d\mu(\omega) \leq \varepsilon$.

E being a Suslin space, every probability measure on the Borel sets of E is tight ([4], p.122 th. 10). Applying this to the image of μ under X we can find a compact set K_1 in E such that $\mu\{\omega : X(\omega) \in K_1\} > 1 - \delta$.

By the quasicompleteness of E, the closed convex hull K of $K_1 \cup \{0\}$ is still compact and - using again the fact that E is Suslin - metrisable (c.f. [4], p.106, Cor. 2).

Let $\Omega^1 = \{\omega \in \Omega : X(\omega) \in K\}$ and $\Omega^2 = \Omega \setminus \Omega^1$.

Let $X^1 = X.\chi_{\Omega^1}$ and $X^2 = X.\chi_{\Omega^2}$. Note that $\{X^1 \circ p_n\}_{n=1}^{\infty}$ and $\{X^2 \circ p_n\}_{n=1}^{\infty}$ are both sequences of i.i.d. random variables the former satisfiing the hypothesis of the preceding lemma. Whence

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} X^1 \circ p_i((\omega_m)_{m=1}^{\infty}) = E(X^1)$$

for $\mu^{\mathbb{N}}$ - allmost all $(\omega_m)_{m=1}^{\infty}$ in $\Omega^{\mathbb{N}}$.

Note that $\|E(X) - E(X^1)\|_B = \|E(X^2)\|_B \leq E(\|X^2\|_B) \leq \varepsilon$.

Of the remaining part $\{X^2 \circ p_n\}_{n=1}^{\infty}$ the following estimate takes care:

$$\limsup_{n\to\infty} \|n^{-1} \sum_{i=1}^{n} X^2 \circ p_i((\omega_m)_{m=1}^{\infty})\|_B$$

$$\leq \limsup_{n\to\infty} n^{-1} \sum_{i=1}^{n} \|X^2 \circ p_i((\omega_m)_{m=1}^{\infty})\| \qquad \mu\text{-a.s.}$$

$$\leq \varepsilon.$$

The last inequality holds almost surely, as $\{\|X^2 \circ p_n\|_B\}_{n=1}^{\infty}$ is a sequence of positive i.i.d. random variables. By the scalar strong law of large numbers we know that their means converge a.s. to $E(\|X^2\|_B)$ which is less than ε.

Noting that

$$n^{-1} \sum_{i=1}^{n} X \circ p_n = n^{-1} \sum_{i=1}^{n} X^1 \circ p_n + n^{-1} \sum_{i=1}^{n} X^2 \circ p_n$$

we see that for each $\varepsilon > 0$ the sequence $\{n^{-1} S_n\}_{n=1}^{\infty}$ may almost surely be written as a sum of a sequence, τ-converging to a value which is ε-close to $E(X)$ in the $\|\cdot\|_B$ - gauge and a sequence with the lim sup of the $\|\cdot\|_B$ - gauge bounded by ε. Letting $\varepsilon = k^{-1}$, $k = 1,2,\ldots$, one concludes that on a set of measure 1

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} X \circ p_i((\omega_m)_{m=1}^{\infty}) = \lim_{n\to\infty} n^{-1} S_n((\omega_m)_{m=1}^{\infty}) = E(X).$$

q.e.d.

6) If on a locally convex space E there is one closed bounded absolutely convex set such that its scalar multiples form a fundamental system for the bounded sets in E, E is called

a Saks space. For definitions and notations we refer to [2].

The following result was proved by Chatterji for the case of Banach-valued Pettis-integrable functions. ([1])
His argument carries over to the following more general case, establishing a converse to proposition 5 for the case of Saks-space.

7) <u>Proposition:</u> Let $(E, \|\cdot\|, \tau)$ be a Saks-space and $\{X_n\}_{n=1}^{\infty}$ an i.i.d. sequence of τ-measurable E-valued random variables. If

$$\lim_{n\to\infty} n^{-1} S_n(\omega)$$

converges almost surely with respect to the mixed topology $\gamma(\|\cdot\|, \tau)$, then

$$\int_\Omega \|X_1\| \, d\mu < \infty,$$

i.e., X_1 is totally summable.

<u>Proof:</u> If $\{n^{-1} S_n(\omega)\}_{n=1}^{\infty}$ γ-converges, it is γ-bounded and therefore norm-bounded ([2]). As

$$\begin{aligned} n^{-1} X_n &= n^{-1} (S_n - S_{n-1}) \\ &= n^{-1} S_n - (1-1/n).(n-1)^{-1} S_{n-1}, \end{aligned}$$

we infer that $\{ n^{-1} X_n(\omega) \}_{n=1}^{\infty}$ is almost surely bounded. Hence there is $M > 0$ such that

$$\mu\{\omega : \limsup \|n^{-1} X_n(\omega)\| \le M\} > 0.$$

By Kolmogoroff's 0-1-law the probability of the above event is actually 1.

The Borel-Cantelli-lemma implies that

$$\sum_{n=1}^{\infty} \mu\{\|X_n\| \leq n.M\} < \infty.$$

As the sequence $\{X_n\}_{n=1}^{\infty}$ is identically distributed

$$\sum_{n=1}^{\infty} \mu\{\|X_1\| \leq n.M\} < \infty$$

or equivalently

$$\int_{\Omega} \|X_1(\omega)\| \, d\mu(\omega) < \infty.$$

q.e.d.

8) Corollary: Let F be a separable Banach space and let $E = F'$ with $\tau = \sigma(F', F)$. (Then E is quasicomplete and Suslin.) If $(X_n)_{n\geq 1}$ is a sequence of i.i.d. Pettis summable random variables with values in (E,τ) and $\frac{1}{n} S_n$ converges almost surely in (E,τ) the X_n are totally summable, i.e. $\int \| n \| d\mu < +\infty$.

Proof: By the theorem of Banach-Steinhaus $\{\frac{1}{n} S_n(\omega)\}$ is bounded almost surely, hence converges a.s. with respect to γ([2],p.9, proposition 1-10).

9) In particular we can construct the following example:

<u>Example:</u> Of a case where E is Suslin, quasi complete, but where for Pettis summable i.i.d. random variables the strong law of large numbers fails:

It suffices to take $E = l^2$, τ the weak topology, $\Omega = [0,1]^{\mathbb{N}}$, μ product Lebesgue measure, and $X_n(\omega) = X(\omega_n)$ where $X : [0,1] \to l^2$ is Pettis integrable but not Bochner integrable (e.g. if $[0,1] = \sum_{n=1}^{\infty} A_n$ with $|A_n| = c/n^2 X(t) = n\, e_n$ for $t \in A_n$, $(e_n)_{n\geq 1}$ being the canonical basis of l^2).

10) Example: We now give an example of a locally convex space E that fails to be Suslin and an i.i.d. sequence $\{X_n\}_{n=1}^{\infty}$ of Borel-measurable, uniformly bounded Pettis-integrable E-valued random variables such that

$$\lim_{n\to\infty} n^{-1} S_n(\omega)$$

does not exist almost surely.

Denote by $[0,\omega_1]$ (resp. $[0,\omega_1[$) the compact (resp. locally compact) space of ordinals less than or equal to (resp. less than) ω_1, the first uncountable one. Let $C([0,\omega_1])$ be the Banach space of continuous functions on $[0,\omega_1]$ and $(M([0,\omega_1]),\sigma^*)$ the dual space, the Radon-measures on $[0,\omega_1]$, equipped with the weak*-topology.

Let $(\Omega,\Sigma,\mu) = ([0,\omega_1[^{\mathbb{N}}$, Borel $([0,\omega_1[)^{\mathbb{N}}, \nu^{\mathbb{N}})$, where ν denotes the σ-additive Borel measure on $[0,\omega_1[$ that gives measure 1 or 0 to each Borel set in $[0,\omega_1[$, according to whether it contains an uncountable closed set or not. (This famous example, due to J. Diendonné , may be found in [3] for example). Let $\delta : [0,\omega_1[\to M([0,\omega_1])$ denote the Dirac transform, i.e., the map associating to each $\alpha \in [0,\omega_1[$ the Dirac measure δ_α. Define a sequence $\{X_n\}_{n=1}^{\infty}$ of i.i.d. $M([0,\omega_1])$-valued Borel-measurable (w.r. to the σ^*-topology) random variables on Ω by putting $X_n = \delta \circ p_n$, p_n denoting the projection onto the n-th coordinate of $[0,\omega_1[^{\mathbb{N}}$.

It is easily seen that a Σ-measurable subset of Ω has measure 1 iff it contains a set of the form $F^{\mathbb{N}}$ for some uncountable closed subset F of $[0,\omega_1[$.

But as it is evidently absurd that for some closed uncountable F of $[0,\omega_1[$ we have, that for each sequence $\{\alpha_n\}_{n=1}^{\infty}$ in F the limit

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} \delta_{\alpha_i}$$

converges in the weak-*-topology of $M[0,\omega_1]$, we arrive at a contradiction, showing that the strong law of large numbers does not hold for $\{X_n\}_{n=1}^{\infty}$.

[1]

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