INTEGRAL OPERATORS ON L^p SPACES, PART I.

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Abstract: We investigate the relation between the theory of integral operators (i.e. operators induced by a kernel) and the theory of Radon-Nikodym derivatives of vector measures. Our approach enables us to solve some problems posed in [12]. Some other problems from the same book will be investigated in a second part of the present paper.

1. Introduction: Our paper is based on the book [12] of P. R. Halmos and V. S. Sunder. There are 15 problems posed in this book and we shall solve some of them in this paper and some others in part II.

The basic idea of our approach is very simple: We consider a kernel $k(x,y)$ not as a scalar-valued function of two variables but rather as a function of one variable, namely x, into a space of functions on Y, which will usually be $L^1(Y,\nu)$. To make this approach precise, we introduce in section 3 the notion of a Halmos function. This concept enables us to give a very easy solution to problem 8.4 of [12]. Let us point out, however, that this problem, as well as the problem of characterising integral operators, has been solved some time ago in the russian literature (c.f. [3] and [10] and 3.7 and 4.5 below; I would like to thank M. Jerschow for pointing this out to me).

In section 4 we give a new characterisation of integral operators: " $T : L^q \longrightarrow L^p$ is integral iff it transforms order bounded sets into equimeasurable sets" (4.4 below). Similar characerisations are given for the case of absolutely bounded kernels and Carleman kernels (4.7 and 4.10 below).

In section 5, which is inspired by martingale theory, we show that an integral operator is the limit of its composition with the conditional expectations with respect to the σ-algebras generated by the countable partitions. This result gives a positive solution to problem 8.2 of [12]. A different solution to this problem will be given in part II. Finally, section 6 deals with problem 11.8 of [12]: Do the integral operators form a right ideal? We show that the composition of an integral operator with an order-bounded map is integral (6.2). But the answer to all of the questions posed in 11.8 is negative (see examples 6.6 and 6.8). A refinement of example 6.8 shows that the answer to the first question of problem 7.1 of [12] is also negative: There are two integral operators such that their kernels are multipliable, but such that the composition of the two operators is not integral.

We have not striven for maximal generality and have restricted ourselves to L^p-spaces. However, the specialisatio to L^2-case, as in [12], seemed too narrow a framework: Even if one is only interested in integral operators from $L^2(\nu)$ to $L^2(\mu)$ one is naturally led to consider the spaces $L^1(\nu)$ and $L^0(\mu)$: The reason for the appearance of $L^1(\nu)$ is that the Halmos function (see definition 3.3 below) will not in general take its values in $L^2(\nu)$ but only in $L^1(\nu)$; the reason for the appearance of $L^0(\mu)$ is that $T : L^2(\nu) \rightarrow L^2(\mu)$ is integral iff $j_{2,0} \circ T : L^2(\nu) \rightarrow L^0(\mu)$ is integral, $j_{2,0}$ denoting the canonical injection of $L^2(\mu)$ into $L^0(\mu)$. But $j_{2,0} \circ T$ is in many respects easier to handle than T. For example, it often happens that T is not order bounded, while $j_{2,0} \circ T$ is. For these reasons we have chosen the following setting: We shall consider continuous operators $T : L^q(\nu) \rightarrow L^p(\mu)$, where we usually allow q to vary over $[1,\infty[$ and p over $[0,\infty]$. The case $q = \infty$ could also have been treated along the same lines, provided an obvious σ^*-continuity - condition is added. A similar remark applies to the case $0 \leq q < 1$, where one has to take some care due to the fact that there are no continuous linear functionals on $L^q(\nu)$. But we have prefered to accept the asymmetry in our choice of p and q rather than waste a couple of lines in most of the statements.

We have divided our article into two parts: In the first we present these results which are based on the concept of a Halmos function. In the second we present those problems posed in [12], which are independent of this concept. Hence the two parts may be read (almost) independently.

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2. Definitions and notations:

2.1.:

$(X,\mathcal{X},\mu)$, $(Y,\mathcal{Y},\nu)$, $(Z,\mathcal{Z},\pi)$ will always denote measure spaces. In contrast to the setting of [12] where σ-finite measure spaces are considered, we shall always assume that the measure spaces are finite in order to have L^{p_1} imbedded into L^{p_2} for $p_1 \geq p_2$. The finiteness assumption is not really a restriction: all our results carry over - mutatis mutandis - to the σ-finite case by the usual change of measure technique (c.f. [12], §6). On the other hand, we do not assume that the measure-spaces are separable, although this assumption would simplify some proofs (see 3.8 and 4.4 below).

The letter T will usually denote a continuous, linear

operator from $L^q(\nu)$ to $L^p(\mu)$, where $1 \leq q < \infty$, $0 \leq p \leq \infty$, if no special assumptions are made.
Let us recall that $L^0(\mu)$ is the complete metrisable topological vector space of equivalence classes of $\mathbb{C}$-valued measurable functions, equipped with the topology of convergence in measure. For $1 \leq p,q \leq \infty$ the letters p', q' will denote the numbers conjugate to p and q.

A subset M of $L^p(\mu)$ will be called <u>order bounded</u> if it is contained in an order interval $I(f)$ for some $f \in L^p(\mu)_+$ where $I(f) = \{f' \in L^p(\mu) : |f'| \leq f\}$.
The operator T will be called <u>order bounded</u>, if it transforms orderbounded sets into order bounded sets.

A function γ from X into a Banach space E will be called <u>strongly measurable</u> if it is essentially separable valued and measurable with respect to the Borel-σ-algebra of E. A function $\gamma : X \to E$ will be called <u>scalarly measurable</u> if for every $g \in E^*$ the composed function $x \to \langle \gamma(x), g \rangle$ is measurable. It is wellknown that an essentially separable valued, scalarly measurable function is strongly measurable (c.f. [5]).

2.2 We shall often encounter the following situation:
Given a function $g \in L^1(Y,\nu)_+$ then $g.d\nu$ is a finite measure and we may consider the Banach space

$$L^1(Y,g.\nu) = \{h \in L^0(\nu) : \|h\|_{L^1(g.\nu)} = \int |h(y)| \cdot g(y)\, d\nu(y) < \infty\}.$$

If g is bounded away from 0 by a positive constant, then $L^1(g.\nu)$ is a subspace of $L^1(\nu)$. The dual $L^1(g.\nu)^*$ may be identified with the space spanned by the orderinterval of g, if the duality is defined in the obvious way, namely for $h \in L^1(g.\nu)$, $h^* \in L^1(g.\nu)^*$

$$\langle h, h^* \rangle = \int_Y h(y)\, h^*(y)\, d\nu(y).$$

For a subset A of X we shall denote by χ_A the characteristic function of A. If A is measurable, then multiplication by χ_A is a continuous projection on $L^p(\mu)$, which will always be denoted by P_A.

Finally we want to note that we shall, as is customary, often identify a function with its equivalence class. But at some places it will be important to be careful about this difference (e.g., 3.5 below) and then we shall have to distinguish rather pedantically between a function and its equivalence class.

3. Halmos - functions

3.1 By a kernel we shall mean a $\mu \times \nu$ - measurable function $k : X \times Y \longrightarrow \mathbb{C}$ such that for μ-a.e. x in X, $k(x,.)$ is integrable (i.e. a member of $L^1(\nu)$).

for $1 \leq q \leq \infty$ we shall say that the kernel k is *q-bounded* if

(1) given $g \in L^q(\nu)$, the product $k(x,\cdot)\, g(\cdot)$ belongs to $L^1(\nu)$ for μ a.e. $x \in X$

Note that every kernel is ∞-bounded. For $0 \leq p \leq \infty$ we shall say that a q-bounded kernel is *(p,q)-bounded*, if

(2) for every $g \in L^q(\nu)$ the function f defined by $f(x) = \int_Y g(y)\, k(x,y)\, d\nu(y)$ belongs to $L^p(\mu)$.

By condition (1) f is μ-almost everywhere defined. Note that every q-bounded kernel is (0,q)-bounded.

It is a classical result, dating back to S.Banach's book ([1], in [16] we shall discuss this in detail), that a (p,q) - bounded kernel defines a continuous operator Int (k) from $L^q(\nu)$ to $L^p(\mu)$ via the formula:

$$g(y) \longrightarrow f(x) = \int g(y)\ k(x,y)\ d\nu(y).$$

An operator $T : L^q(\nu) \rightarrow L^p(\mu)$ will be called <u>integral</u> if it is of the form $T = \mathrm{Int}(k)$ for a (p,q)-bounded kernel.

3.2 Our approach is to consider a kernel not as a scalar valued function of two variables but rather as a function of one variables, namely x, into a space of functions on Y, which will usually be $L^1(\nu)$. To make this precise we give the following definition, which will turn out to be a mere reformulation of the concept of a kernel.

3.3 <u>Definition</u> We call a <u>Halmos-function</u> a strongly measurable function

$$\gamma : X \rightarrow L^1(\nu).$$

For $1 \leqq q \leqq \infty$ we shall say that the Halmos-function γ is <u>q-bounded</u> if

(1) for every $g \in L^q(\nu)_+$ the function γ takes μ-almost everywhere its values in $L^1(g.\nu)$.

For $0 \leqq p \leqq \infty$ we shall say that a q-bounded Halmos-function is <u>(p,q)-bounded</u> if

(2) for every $g \in L^q(\nu)$ the function f defined by

$$f(x) = \langle \gamma(x), g \rangle$$

belongs to $L^p(\mu)$.

The symbol $\langle\ ,\ \rangle$ refers to the duality of $L^1(|g|.\nu)$, as g may be identified with a member of the unitball of $L^1(|g|.\nu)^*$, as was explained in 2.2. Hence by (1) the function f is μ-almost everywhere defined.

Note that every Halmos function is ∞-bounded and that every q-bounded Halmos function is (O,q)-bounded.

3.4 We are ready to establish the correspondence between kernels and Halmos-functions. Clearly this correspondence is given by the map i that takes a kernel k(x,y) to the Halmos-function

$$\gamma(x) = [k(x,\cdot)],$$

where $[k(x,\cdot)]$ denotes the ν-equivalence class of functions to which $k(x,\cdot)$ belongs (for fixed x). However, some care is needed, due to the difference between functions and their equivalence classes. Although the following result is a mere formality, 8.4 of [12] is an immediate consequence of it (see 3.6. below).

3.5 Proposition: The map i introduced above establishes a bijective correspondence between the equivalence classes (with respect to $\mu\times\nu$) of (p,q) - bounded kernels k and the equivalence classes (with respect to μ) of (p,q)-bounded Halmos functions.

<u>Proof</u>: It is wellknown ([18], 26.6), that

$$L^1(\mu\times\nu) = L^1(\mu;L^1(\nu)),$$

the right hand side denoting the Banach space of (equivalence classes of) Bochner-integrable $L^1(\nu)$-valued functions. The correspondence again is given by associating to $k \in L^1(\mu\times\nu)$ the $L^1(\nu)$-valued function $\gamma : x \to [k(x,y)]$.

As the kernels, that we consider, are not necessarily $\mu\times\nu$-integrable (even (2,2)-bounded kernels may fail to be), we cannot apply the above formula directly. One could show that the formula $L^p(\mu\times\nu) = L^p(\mu, L^p(\nu))$ is valid for $0 \leq p \leq \infty$. But as we could not trace this result in the litterature for the case $p = 0$ (the relevant case for us), we proceed differently and apply a simple trick.

Given a kernel $k(x,y)$ let $k' = \text{arctg} \circ k$. Clearly k' is uniformly bounded, hence in particular in $L^1(\mu\times\nu)$. So we may find $\gamma' \in L^1(\mu;L^1(\nu))$ such that $\gamma'(x) = [k'(x,y)]$ for μ-a.e. $x \in X$. Putting $\gamma(x) = \text{tg} \circ \gamma'(x)$, we see that γ is welldefined and $\gamma(x) = [k(x,y)]$ for μ-a.e. $x \in X$.

Conversely given a Halmos function γ let $\gamma' = \text{arctg} \circ \gamma$ and procced exactly as above to find the corresponding kernel k.

It is plain to check that this correspondence maps the equivalence classes of (p,q)-bounded kernels bijectively onto the equivalence classes of (p,q)-bounded Halmos functions.

□

3.6. As a first application of the above formalities we give a solution to problem 8.4 of [12]. This problem has been solved in full generality some time ago in the russian literature (c.f. [3] and [10]). Nevertheless we present the proof since it is such a simple consequence of our approach. Actually it was this problem that was the starting point of the author's present work.

<u>3.7. Proposition:</u>

Let $0 \leq p \leq \infty$ and $1 \leq q < \infty$ and $h : X \times Y \longrightarrow \mathbb{C}$ be a (not necessarly $\mathcal{X} \times \mathcal{Y}$-measurable) function such that

(1) given $g \in L^q(\nu)$, the product $h(x,\cdot)\ g(\cdot)$ is $\mathcal{Y}$-measurable and ν-integrable for μ-a.e. $x \in X$, and

(2) for $g \in L^q(\nu)$ the function

$$f : x \longmapsto \int h(x,y)\ g(y)\ d\nu(y)$$

is in $L^p(\mu)$ (and hence $\mathcal{X}$-measurable).

Then there is an essentially unique $\mathcal{X} \times \mathcal{Y}$-measurable (p,q)-bounded kernel k that induces the same operator, i.e. for $g \in L^q(\nu)$, $\int h(x,y) g(y) d\nu(y) = \int k(x,y) g(y) d\nu(y)$ for μ-a.e. $x \in X$.

<u>Proof:</u> Putting as above $\gamma(x) = [h(x,\cdot)]$ we obtain a μ-almost everywhere $L^1(\nu)$-valued function. By hypothesis, for each $g \in L^\infty(\nu)$ the composed function $x \longrightarrow \langle \gamma(x), g \rangle$ is $\mathcal{X}$-mesurable i.e. γ is a scalarly measurable function. In the case, where $L^1(\nu)$ is separable (as is always assumed in [12], for example), γ is strongly measurable and by 3.5 corresponds to a kernel $k(x,y)$ which clearly is (p,q)-bounded and satisfies the assertion.

In the case, where $L^1(\nu)$ is not separable, we need the following beautiful theorem, due G.Edgar [7]:"Given a scalarly measurable function γ with values in a weakly compactly generated Banach space E (e.g. $L^1(\nu)$ for σ-finite ν) there is a strongly measurable function $\tilde{\gamma}$ which is scalarly equivalent to γ." This theorem therefore furnishes a strongly measurable function $\tilde{\gamma}$, which in turn defines a kernel k by 3.5. It is evident that also in this case k is the desired (p,q)-bounded kernel.

□

4. Characterisation of integral operators

4.1. We now turn to the main theme of the paper: Given a continuous operator $T : L^q(\nu) \longrightarrow L^p(\mu)$, under what conditions is there a (p,q)-bounded kernel k which induces the operator?

In this paragraph, which is only motivational in order to avoid technical difficulties we restrict ourselves to the case $1 \leq p \leq \infty$.

The question whether T is induced by a kernel turns out to be just the question whether the adjoint $T^* : L^{p'}(\mu) \rightarrow L^{q'}(\nu)$ is - in a certain sense - Rieszrepresentable. (See [5] for a definition of Rieszrepresentability and its relation with the derivative of a vector valued measure). Indeed, if T is induced by the kernel k and γ is the Halmos-function corresponding to k, then for $f \in L^{p'}(\mu)$

$$T^*(f) = \int_X \gamma(x)\, f(x)\, d\mu(x)$$

The above integral is interpreted as a Pettis-integral in the space $L^1(\nu)$, since, by hypothesis, for every $g \in L^\infty(\nu)$ (actually for every $g \in L^q(\nu)$) and $f \in L^{p'}(\mu)$ the function $x \to f(x) . \langle\gamma(x), g\rangle$ is integrable and

$$\langle g, T^* f\rangle = \langle Tg, f\rangle = \int_X f(x) . \langle\gamma(x), g\rangle\, d\mu(x). \qquad (*)$$

However, this integral has a very particular flavour: Although $T^*(f)$ is in $L^{q'}(\nu)$ the integration takes place in the bigger space $L^1(\nu)$ (as γ takes its values not necessarily in $L^{q'}(\nu)$); but for every single $g \in L^q(\nu)$ the expression $\langle\gamma(x), g\rangle$ makes sense for μ-a.e. $x \in X$ and the line (*) makes sense also. We do not give a formal definition of the integral (which would only be a reformulation of the concept of a Halmos-function); our intention was only to point out the relationship with the theory of Radon-Nikodym derivatives of vector valued measures. The results of this theory will serve as a guide for the investigation of integral operators (e.g. the theorems that a strongly measurable function has relatively compact range on sets of "large" measure, that the indefinite integral of a Bochner integrable function has relatively compact range , the intimate relationship with martingale-convergence etc.). Of course, these results do not carry over in a direct way because of the special nature of the integral in question, but the ideas behind them do.

Let us mention however one easy direct application, which was already observed by N. Dunford in 1936,[6]: if $p = \infty$ and $1 < q < \infty$ then every continuous operator $T : L^q(\nu) \to L^p(\mu)$ is integral. Indeed the adjoint operator T^* takes $L^1(\mu)$ into the reflexive space $L^{q'}(\nu)$; it is wellknown that T^* is therefore Riesz representable ([5]) and in this case we even get an $L^{q'}(\nu)$ - valued Bochner-integrable Halmos-function γ.

Let us point out that one may also prove in the same way that for $q = 1$ and $1 < p \leq \infty$ every continuous operator $T : L^q(\nu) \to L^p(\mu)$ is integral, but this is slightly more involved and we shall not do this here.

4.2. Definition: We recall the following concept due to A. Grothendieck ([11]; see also [2],[14],[19])
A subset M of $L^0(\mu)$ is called equimeasurable if, for $\varepsilon > 0$, there is $X_\varepsilon \subsetneq X$ with $\mu(X \setminus X_\varepsilon) < \varepsilon$, such that M restricted to X_ε is relatively norm-compact in $L^\infty(X_\varepsilon, \mu|X_\varepsilon)$.

Note that an equimeasurable set is lattice - bounded in $L^0(\mu)$, while the converse is not true (for an example take the unitball of $L^\infty(\mu)$, for μ is not purely atomic).

4.3. We also recall (and prove) the very elementary fact that an operator from $L^1(\mu)$ to a Banach space E is compact iff it is Riesz - representable by Bochner-integrable function with relatively compact range: denote by $L^\infty(\mu) \times E$ the algebraic tensorproduct of $L^\infty(\mu)$ and E. We may identify

$L^\infty(\mu) \otimes E$ in obvious ways with the following vectorspaces:

1) The bounded measurable functions $\gamma : X \to E$ with the range contained in a finite-dimensional subspace of E.
2) The finite-rank operators from $L^1(\mu)$ to E.
3) The σ^*-continuous finite-rank operators from E^* to $L^\infty(\mu)$.

The correspondence between a function γ from 1) and an operator T from 3) for example, is given by

$$T(g)(x) = \langle g, \gamma(x) \rangle \qquad (*)$$

which holds for $g \in E^*$ and μ-a.e. $x \in X$. Now take in 1) the essential supremum-norm and in 2) and 3) the usual operator norm and check that these norms are the same via the obvious identifications (the corresponding norm on $L^\infty(\mu) \otimes E$ is the "injective" tensor product norm). Passing to the completions we see that we may naturally identify
1) The Banach-space of (equivalence classes of) measurable relatively compactvalued functions $\gamma : X \to E$ (equipped with the essential supremum - norm).
2) The compact operators from $L^1(\mu)$ to E
3) The compact σ^*-continuous operators from E^* to $L^\infty(\mu)$.
The correspondence between a function γ from 1) and an operator T from 3) is still given by formula (*).

__4.4 Theorem:__ Let $0 \leq p \leq \infty$, $1 \leq q < \infty$. A linear map $T : L^q(\nu) \to L^p(\mu)$ is integral iff T maps order-intervals into equimeasurable sets.

<u>4.5. Remark:</u> The theorem resembles the following beautiful characterisation of integral operators due to Bukhvalov ([3], see also [17]): " T is integral iff T maps dominated sequences that converge in measure to almost surely convergent sequences". Indeed on the left hand side (i.e. on $L^q(\nu)$) both characterisations use the order-intervals, while - as regards the right hand side - the concept of equimeasurability is very much related to almost sure convergence (see [14], for example). However, it is not at all evident, how to deduce one theorem from the other, allthough this is possible, as was shown to me by A. Schep. Still we believe, that our theorem has its own flavor and that the method of proof is interesting in its own right.

<u>Proof of 4.4:</u> $\Longrightarrow$: Suppose that $T = \mathrm{Int}(k)$ for a (p,q)-bounded kernel k. Then let γ be the Halmos function associated to k and fix an order-interval $I(g)$, $g \in L^q(\nu)_+$. Clearly we may assume that $g \geq 1$. By definition $\gamma(x)$ lies in $L^1(g.\nu)$ for μ-a.e. $x \in X$. Note that γ, viewed as an $L^1(g.\nu)$ - valued function,is strongly measurable, Hence for $\varepsilon > 0$ there is $X_\varepsilon \subseteq X$, $\mu(X \setminus X_\varepsilon) < \varepsilon$, such that γ restricted to X_ε has relatively compact range in $L^1(g.\nu)$. It follows from formula (*) in 4.3 that the operator

$$\tilde{T} : L^1(g.\nu)^* \longrightarrow L^\infty(X_\varepsilon, \mu|_{X_\varepsilon})$$
$$h \longrightarrow f(x) = \langle \gamma(x), h \rangle$$

is a compact operator.

Note that the unit ball of $L^1(g.\nu)^*$ is just the order interval $I(g)$ and that $\tilde{T}$ is just the restriction of $P_{X_\varepsilon} \circ T$ to $L^1(g.\nu)^*$ (which may be identified with the subspace of $L^q(\nu)$ spanned by $I(g)$).

Hence T transforms the order interval $I(g)$ into a set whose restriction to X_ε is relatively compact in $L^\infty(X_\varepsilon, \mu|_{X_\varepsilon})$. This is exactly what we had to show.

$\Leftarrow$: Converšly suppose T satisfies the asumption of the theorem.

First we show that the condition that T transforms order intervals into equimeasurable set implies that T is a continuous operator from $L^q(\nu)$ to $L^p(\mu)$. By the closed graph theorem it will be sufficient to show that if a sequence $\{g_n\}_{n=1}^\infty$ in $L^q(\nu)$ with $\|g_n\| \leq n^{-1} \cdot 2^{-n}$ is such that Tg_n converges to f_o in $L^p(\mu)$, then f_o equals 0.

Let $g = \sum_{n=1}^\infty n.|g_n|$, which belongs to $L^q(\nu)_+$. Hence $T(I(g))$ is equimeasurable and in particular order bounded in $L^o(\mu)$, i.e. there is an $f \in L^o(\mu)$ such that $T(I(g)) \subseteq I(f)$. It follows that $|Tg_n| \leq n^{-1}. f$, so Tg_n converges to 0 μ-a.s. As Tg_n converges to f_o in $L^p(\mu)$, we conclude that $f_o = 0$ as required.

Now apply the assumption that T maps order intervals into equimeasurable sets to the order interval $I(1)$, (i.e. the unit ball of $L^\infty(\nu)$), to find a partition $\{A_n\}_{n=1}^\infty$ of X such that $T(I(1))$, restricted to A_n, is relatively compact in $L^\infty(A_n, \mu|_{A_n})$.

This means that $P_{A_n} \circ T$ restricts to a compact operator from $L^\infty(\nu)$ to $L^\infty(A_n, \mu|_{A_n})$. This operator is σ^*-continuous as T is a continuous operator from $L^q(\nu)$ to $L^p(\mu)$.

By 4.3 there is a function $\gamma_n : A_n \to L^1(\nu)$ such that for $g \in L^\infty(\nu)$

$$P_{A_n} \circ T(g) = \langle \gamma_n(x), g \rangle \qquad \text{for } \mu\text{-a.e. } x \in A_n.$$

Hence
$$T(g) = \langle \gamma(x), g \rangle \qquad \text{for } \mu\text{-a.e. } x \in X,$$

where γ is obtained by glueing together the γ_n's, i.e. $\gamma(x) = \gamma_n(x)$ if $x \in A_n$.

The function γ, which is clearly strongly measurable, is therefore a (p,∞)-bounded kernel, inducing the restriction of T to $L^\infty(\nu)$. We still have to show that γ is actually (p,q)-bounded and induces T on all of $L^q(\nu)$.

Fix $h_0 \in L^q(\nu)$ and let $g = 1 + |h_0|$. Repeat the above argument to find a partition $\tilde{A}_n$ and functions $\tilde{\gamma}_n : \tilde{A}_n \to L^1(g.\nu)$ such that for all $h \in L^1(g.\nu)^*$ (in particular for h_0)

$$T(h)(x) = \langle \tilde{\gamma}(x), h \rangle \qquad \text{for } \mu\text{-a.e. } x \in X,$$

where $\tilde{\gamma}$ is again obtained by glueing together the $\tilde{\gamma}_n$'s. Let $\{g_m\}_{m=1}^{\infty}$ be a sequence in $L^\infty(\nu)$ separating points of a separable subspace of $L^1(\nu)$ in which γ and $\tilde{\gamma}$ take μ-almost everywhere their values.

It is easy to see that this is possible, but the reader who is content with the assumption that $L^1(\nu)$ is separable may instead take any sequence in $L^\infty(\nu)$ which separates points of $L^1(\nu)$. For each $m \in \mathbb{N}$

$$\langle \tilde{\gamma}(x), g_m \rangle = T g_m(x) = \langle \gamma(x), g_m \rangle \text{ for } \mu\text{-a.e. } x \in X.$$

Hence $\gamma(x) = \tilde{\gamma}(x)$ μ-almost everywhere, if we identify $L^1(g.\nu)$ with a subspace of $L^1(\nu)$. So γ lies almost everywhere in $L^1(g.\nu)$ and

$$T(h_o)(x) = \langle \gamma(x), h_o \rangle \qquad \text{for } \mu\text{-a.e. } x \in X.$$

□

<u>4.6. Definition:</u> For $0 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$, a kernel k will be called <u>absolutely (p,q)-bounded</u> if $|k|$ is a (p,q)-bounded kernel. An operator $Int(k) : L^q(\nu) \to L^p(\mu)$ induceed by an absolutely (p,q)-bounded kernel will be called <u>absolutely integral.</u>

<u>4.7. Proposition:</u> For $0 \leqq p \leqq \infty$, $1 \leqq q < \infty$ a linear map $T : L^q(\nu) \to L^p(\mu)$ is absolutely integral iff it is integral and order bounded. Hence T is absolutely integral iff it transforms order intervals into equimeasurable sets which are in addition orderbounded in $L^p(\mu)$.

<u>4.8. Remark:</u> The easy proof of the above proposition is left to the reader. Let us instead make a few comments: the proposition implies in particular that every (0,q)-bounded kernel is absolutely (0,q)-bounded, a fact which is also easily seen directly. In addition, note that for $0 \leqq p < \infty$ a subset M of $L^p(\mu)$ is equimeasurable and order bounded in $L^p(\mu)$ iff there is a strictly positive function h on X such that $h^{-1} \in L^p(\mu)$ and such that $h.M = \{h.f : f \in M\}$ is a relatively compact subset of $L^\infty(\mu)$. This observation (whose easy proof again is left to the reader) might be used to rephrase 4.7.

Finally let us mention that V.Sunder has recently shown in [20], using a theorem of Nikishin [15], that for an integral operator $T : L^q(\nu) \longrightarrow L^p(\mu)$ and $\varepsilon > 0$ there is $X_\varepsilon \subseteq X$, $\mu(X \setminus X_\varepsilon)$ such that $P_{X_\varepsilon} \circ T$ is absolutely integral.

<u>4.9. Definition:</u> For $0 \leq p \leq \infty$ a (p,q)-bounded kernel k will be called a <u>(p,q)-bounded Carleman-kernel</u> if $k(x,\cdot) \in L^q(\nu)$ for μ-a.e. $x \in X$. An operator $Int(k) : L^q(\nu) \to L^p(\mu)$ induced by a (p,q)-bounded Carleman kernel will be called a <u>Carleman-integral</u> operator.

The Halmos function associated to a Carleman-kernel therefore takes its values in $L^{q'}(\nu)$ and this makes things easier. In this case the Halmos function γ has been studied in ([12], § 11) under the name of Carleman-function. A reasoning analogous to the proof of 4.4 , except that it is considerably simpler, furnishes the following characterisation.

<u>4.10. Proposition:</u> A linear map $T : L^q(\nu) \to L^p(\mu)$ is Carleman - integral iff it transforms the unit ball of $L^q(\nu)$ into an equimeasurable set.

□

5. Atomic measure spaces and conditional expectations

5.1. There is an intimate relation between the validity of the Radon-Nikodym theorem and of the martingale - convergence - theorems (c.f.[4]). Hence it is no surprise that the concept of conditional expectation gives some information about integral operators. Let $A = \{A_n\}_{n=1}^{\infty}$ be a countable partition of X consisting of μ-measurable sets and let $\Sigma(A)$ be the σ-algebra generated by A. Denote by E_A the conditional expectation operator on $L^p(\mu)$. (Here again we restrict ourselves to the case $1 \leqq p \leqq \infty$ to avoid technical difficulties; but at the cost of writing some additional lines the following reasoning may be carried over to the case $0 \leqq p < 1$).

The countable partitions A of X are directed by refinement and so it makes sense to speak about "convergence along the net $\mathfrak{A}$ of countable partitions A of X".

5.2. Proposition: A subset M of $L^p(\mu)$, $1 \leqq p \leqq \infty$, is equimeasurable iff E_A converges (along $\mathfrak{A}$) to the identity uniformly on M with respect to the $L^{\infty}(\mu)$-norm.

5.3.Remark: The phrase "with respect to the $L^{\infty}(\mu)$-norm" means that, given $\varepsilon > 0$, there is a countable partition A_{ε} of X such that for every countable partition $A \geqq A_{\varepsilon}$ we have $\|E_A(f) - E_{A_{\varepsilon}}(f)\|_{\infty} < \varepsilon$, for each $f \in M$. However, $E_A(f)$ will not, in general, be a member of $L^{\infty}(\mu)$.

Proof of 5.2: First note that a subset $\tilde{M}$ of $L^\infty(\tilde{X},\tilde{\mu})$, ($(\tilde{X},\tilde{\mu})$ any measure space) is relatively compact iff $E_{\tilde{A}}$ converges to the identity uniformly on $\tilde{M}$, where $\tilde{A}$ runs through the net of finite partitions $\tilde{X}$.

If M is equimeasurable, find a countable partition $\mathcal{B} = \{B_n\}_{n=1}^\infty$ of X such that M restricted to B_n is relatively compact in $L^\infty(B_n, \mu|_{B_n})$.

Given $\varepsilon > 0$ we may find a countable partition A_ε, obtained by splitting each of the B_n into a finite number of subsets, such that

$$\|E_A(f) - E_{A_\varepsilon}(f)\|_{L^\infty(\mu)} \leq \varepsilon \qquad \text{for each } f \in M \text{ and } A \geq A_\varepsilon,$$
as required.

Conversely : Suppose M satisfies the hypothesis of the proposition. Choose partitions $A^{(k)} = \{A_n^{(k)}\}_{n=1}^\infty$, $k = 1,2,\ldots$, such that for each $A \geq A^{(k)}$, $\|E_A(f) - E_{A^{(k)}}(f)\|_{L^\infty(\mu)} \leq k^{-1}$ for all $f \in M$. Given $\varepsilon > 0$, choose for each k a number $N(k)$ such that $\mu(\bigcup_{n=N(k)}^{\infty} A_n^{(k)}) < \varepsilon/2^k$.

It is easily seen that $X_\varepsilon := \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{N(k)} A_n^{(k)}$ is such that $\mu(X \setminus X_\varepsilon) < \varepsilon$ and M restricted to X_ε is relatively compact in $L^\infty(X_\varepsilon, \mu|_{X_\varepsilon})$.

□

5.4. Corollary: For $1 \leq p \leq \infty$, $1 \leq q < \infty$ a linear map $T : L^q(\nu) \to L^p(\mu)$ is integral iff $E_A \circ T$ converges to T with respect to the L^∞-norm along the net $\mathfrak{A}$ of countable partitions uniformly on every order-interval of $L^q(\nu)$.

Proof: 4.4 and 5.2.

□

5.5. Remark: For a continuous operator $T : L^q(\nu) \to L^p(\mu)$ the composition $E_A \circ T$ is always a Carleman-integral operator, in fact a very "natural" one; namely the one induced by the by the kernel $k_A(x,y) = \mu(A_n)^{-1} . T^*(\chi_{A_n})$ if $x \in A_n$. This is because the operator T takes its values in the subspace $L^p(X,\Sigma(A),\mu_{\Sigma(A)})$ of $L^p(\mu)$, which is in reality just l^p, and in this case the question whether an operator is integral becomes trivial.(12 , th. 7.3.).

Hence corollary 5.4 essentally states that the space of integral operators from $L^q(\nu)$ to $L^p(\mu)$ is the completion of the vector space of "natural operators of the form $E_A \circ T$ with respect to a certain uniform stucture.

This uniform structure has the ugly feature that it does not define a topological vector space, as the scalar multiplication is not continuous. But such topologies have sometimes turned out to be useful in analysis, e.g. the uniform topology on $C(\mathbb{R})$ ([9], 2M.6, p. 34) or the Whitney topology on $C^\infty(\mathbb{R})$ ([13]).

The above corollary might also be regarded as a solution to problem 8.2 of [12] on the existence of an "effective" procedure for recapturing the kernel k of an integral operator $T : L^q(\nu) \to L^p(\mu)$.

If $T = \mathrm{Int}(k)$ is integral, then the kernels k_A , which were defined effectively above, converge to the kernel k, in the norm of $L^1(\mu\times\nu)$ for example; (the corollary implies convergence with respect to a considerable stronger topology). Note, however, that the convergence has to be understood in the sense of remark 5.3 and that, in general, k does not belong to $L^1(\mu\times\nu)$, even for (2,2) - bounded kernels. However, the reader might have doubts about the effectiveness of a procedure which involves convergence along the net of countable partitions of X. Hence we shall give in [16] a different solution to problem 8.2 which seems (at least to the author) to be more "effective".

6. Right ideal properties of integral operators

6.1. We now turn to problem 11.8. of [12] : Do the integral operators from a right ideal ? We shall give three counter-examples (arranged in ascending order of difficulty) which show that the answer to all of the questions posed in 11.8 (as well as to the first question of problem 7.1) is no. First we give a positive result, which is an immediate consequence of 4.4 and 4.7 .

6.2. Proposition: Let $0 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$, $1 \leqq r < \infty$ and let $S : L^r(\pi) \to L^q(\nu)$ be an order bounded operator. If $T : L^q(\nu) \to L^p(\mu)$ is integral (resp. absolutely integral) than $T \circ S$ is integral (resp. absolutely integral)

□

6.3. Remark: Note that in the case $q = \infty$, $1 \leqq r < \infty$ or $r = 1$ and $1 \leqq q \leqq \infty$ every continuous operator $S : L^r(\pi) \to L^q(\nu)$ orderbounded. Hence in those cases we have a right ideal property for integral operators. with respect to the continuous operators.

6.4. Let us now turn to the general case, where S is not necessarily order bounded. A typical example of a continuous operator that is not order bounded is the isometry on $L^2_{[0,1]}$ that takes the exponential basis into the Haar basis. This operator and variations of it will be used in the following examples. For clarity of notation we recall the definition of the Haar basis of $L^2_{[0,1]}$.

<u>6.5. Definition:</u> The sequence of functions $\{h_n\}_{n=1}^{\infty}$ defined by $h_1 = 1$ and for $u = 0,1,2,\ldots,$ $v = 1,\ldots,2^u$

$$h_{2^u+v}(x) = \begin{cases} \sqrt{2^u} & \text{if } x \in [(2v-2)\,2^{-u-1}, (2v-1)\,2^{-u-1}] \\ -\sqrt{2^u} & \text{if } x \in [(2v-1)2^{-u-1}, 2v2^{-u-1}] \\ 0 & \text{otherwise} \end{cases}$$

is called the <u>Haar basis</u> of $L^2_{[0,1]}$.

<u>6.6. Example:</u> There exists an absolutely integral operator $T : L^2(\nu) \to L^2(\mu)$ and a continuous operator $S : L^2(\tau) \to L^2(\nu)$ such that $T \circ S$ is not integral.

Let $(X,\mu) = (Y,\nu) = (\mathbb{Z},\tau) = (\mathbb{I},m)$, the unit interval equipped with Lebesgue measure, and let $k(x,y)$ be the absolutely (2,2)-bounded kernel defined in 11.1 of [12], i.e.

$$k(x,y) = \begin{cases} 0 & \text{if } x \leq y \\ (x-y)^{-1/2} & \text{if } x > y. \end{cases}$$

(Actually any index in $[-1/2, -1[$ would work instead of $-1/2$). Denote by γ the Halmos function corresponding to k.

Let $S : L^2(\tau) \to L^2(\nu)$ be the isometric operator mapping the element $e^{2\pi im}$ to the element $h_{b(m)}$, where b is any bijection between $\mathbb{Z}$ and $\mathbb{N}$.

The composition $R = \mathrm{Int}(k) \circ S$ is not integral. Indeed, suppose there is a Halmos function $\beta : X \to L^1(\tau)$ inducing R. By the Riemann - Lebesgue lemma this would imply that for μ-a.e. $x \in X$

$$\lim_{|m|\to\infty} \left\langle \beta(x), e^{2\pi im} \right\rangle = 0. \qquad (*)$$

$$\text{But } \left\langle \beta(x),\ e^{2\pi im} \right\rangle = R(e^{2\pi im})\ (x)$$
$$= \mathrm{Int}(k) \circ S(e^{2\pi im})\ (x)$$
$$= \mathrm{Int}(k)\ (h_{b(m)})\ (x)$$
$$= \left\langle \gamma(x),\ h_{b(m)} \right\rangle,$$

the above equations holding μ-almost everywhere on X. Hence we would have

$$\lim_{n\to\infty} \left\langle \gamma(x),\ h_n \right\rangle = 0. \qquad (**)$$

for μ-a.e. $x \in X$.

But this is absurd: Let $\{\varepsilon_u(x)\}_{u=1}^{\infty}$ denote the dyadic expansion of $x \in X$, i.e. $x = \sum_{u=1}^{\infty} \varepsilon_u(x) 2^{-u}$ with $\varepsilon_u = 0,1$. (For the countably many dyadic points x choose any one of the two possible dyadic expansions). It follows from the Borel-Cantelli-lemma that there exist for μ-a.e. $x \in X$ arbitrarily large u's such that $\varepsilon_{u+1}(x) = 0$ and $\varepsilon_{u+2}(x) = 1$. Fix x and u such that these two equations hold true and let $\bar{x} = \sum_{i=1}^{u} \varepsilon_i(x)\ 2^{-i}$; note that $2^{-u-2} \leq x - \bar{x} \leq 2^{-u-1}$.

Let $v = 1 + \sum_{i=1}^{u} \varepsilon_i(x)\ 2^{u-i}$ and let $n = 2^u + v$; the Haar function h_n assumes the value $\sqrt{2^u}$ on $[\bar{x}, \bar{x} + 2^{-u-1}]$, $-\sqrt{2^{-u}}$ on $[\bar{x} + 2^{-u-1}, \bar{x} + 2^{-u}]$ and 0 elsewhere. Hence

$$\left\langle \gamma(x),\ h_n \right\rangle = \int_{\bar{x}}^{x} \sqrt{2^u}\ (x-y)^{-1/2}\ dy$$
$$\geq \sqrt{2^u} \int_{x-2^{-u-2}}^{x} (x-y)^{-1/2}\ dy = 1.$$

This shows that for μ-a.e. $x \in X$ $(**)$ does not hold and this contradiction finishes the proof. □

For the next examples we need a lemma which I have been unable to find in the literature.

6.7. Lemma: Let $\{h_n\}_{n=1}^{\infty}$ be the Haar basis in $L^2_{[0,1]}$. For $\{\lambda_n\}_{n=1}^{\infty} \in l^2$ and $\{\alpha_u\}_{u=1}^{\infty} \in l^2$ the sum

$$\sum_{u=0}^{\infty} \sum_{v=1}^{2^u} \lambda_{2^u+v} \cdot \alpha_u \cdot h_{2^u+v}(x)$$

converges absolutely for μ-a.e. $x \in [0,1]$.

Proof: We shall show that

$$F(x) = \sum_{u=0}^{\infty} \sum_{v=1}^{2^u} |\lambda_{2^u+v} \; \alpha_u \; h_{2^u+v}(x)|$$

is in $L^1_{[0,1]}$, which will imply the assertion. Apply the Cauchy-Schwarz inequality to the identity

$$\sum_{v=1}^{2^u} |\lambda_{2^u+v} \, h_{2^u+v}(x)| = \Big(\sum_{v=1}^{2^u} \lambda_{2^u+v} \cdot \chi_{[\frac{v-1}{2^u}, \frac{v}{2^u}]}(x)\Big) \cdot \Big(\sum_{v=1}^{2^u} |h_{2^u+v}(x)|\Big)$$

to obtain the following estimates

$$\int_0^1 F(x)\,dx = \sum_{u=0}^{\infty}\Big(|\alpha_u| \cdot \int_0^1 \sum_{v=1}^{2^u} |\lambda_{2^u+v} \, h_{2^u+v}(x)|\,dx\Big)$$

$$\leq \sum_{u=0}^{\infty} |\alpha_u| \cdot \Big(\int_0^1 \Big(\sum_{v=1}^{2^u} \lambda_{2^u+v} \, \chi_{[\frac{v-1}{2u}, \frac{v}{2^u}]}(x)\Big)^2 dx\Big)^{1/2} \cdot \Big(\int_0^1 \sum_{v=1}^{2^u} |h_{2^u+v}(x)|^2 dx\Big)^{1/2}$$

$$\leq \sum_{u=0}^{\infty} |\alpha_u| \cdot \Big(\sum_{v=1}^{2u} |\lambda_{2^u+v}|^2 . 2^{-u}\Big)^{1/2} \; . \; (2^u)^{1/2}$$

$$= \sum_{u=0}^{\infty} |\alpha_u| \cdot \Big(\sum_{v=1}^{2^u} |\lambda_{2^u+v}|^2\Big)^{1/2}$$

$$\leq \Big(\sum_{u=0}^{\infty} |\alpha_u|^2\Big)^{1/2} \cdot \Big(\sum_{u=0}^{\infty} \sum_{v=0}^{2^u} |\lambda_{2^u+v}|^2\Big)^{1/2} < \infty \; .$$

□

6.8. Example: There are integral operators $T : L^2(\nu) \to L^2(\mu)$ and $S : L^2(\tau) \to L^2(\nu)$ such that the composition is not integral.

Again let $(X,\mu) = (Z,\tau) = (\mathbb{T},m)$ but this time let (Y,ν) be $\mathbb{N}$ equipped with counting measure. Contrary to our general assumption, (Y,ν) is not a finite measure space, but this is only for notational convenience: one could rewrite the example replacing ν by the measure that puts mass 2^{-n} on the point $\{n\}$.

Define the kernel $k(x,y)$ (y will now denote an integer!) by

$$k(x,1) = 1$$

and

$$k(x,2^u+v) = (u+1)^{-1} \cdot h_{2^u+v}(x)$$

where $u = 0,1,2,\ldots$, $v = 1,\ldots,2^u$. Given $\{g(y)\}_{y=1}^{\infty} \in L^2(\nu) = l^2$, we infer from the preceding lemma that, for μ-a.e. $x \in X$, $g(\cdot)\, k(x,\cdot) \in l^1$, i.e. k is a 2-bounded kernel. Actually k is (2,2)-bounded, as Int(k) maps the $(2^u + v)$'th unit-vector to the function $(u+1)^{-1}\, h_{2^u+v}$ and is therefore a compact operator from $L^2(\nu)$ to $L^2(\mu)$.

Let $S : L^2(\tau) \to L^2(\nu)$ be the isometric operator mapping $e^{2\pi im}$ to the unitvector $e_{b(m)}$ of $l^2 = L^2(\nu)$, where b again is any bijection between $\mathbb{Z}$ and $\mathbb{N}$. Then S is an integral operator as (Y,ν) is atomic (see remark 5.5 above or [12] , th. 7.3).

But $R = T \circ S$ is not integral. Indeed, R maps bijectively the members of the exponential basis in $L^2(\tau)$ to functions of the form $(u+1)^{-1}\, h_{2^u+v}$ in $L^2(\mu)$. So if there were a Halmos function $\beta : X \to L^1(\tau)$ inducing

R, again, by Riemann-Lebesgue,

$$\lim_{|m|\to\infty} \left\langle \beta(x),\ e^{2\pi i m} \right\rangle = \lim_{|m|\to\infty} R\,(e^{2\pi i m})(x) = 0$$

for μ-a.e. $x \in X$.

As $R(e^{2\pi i m}) = (u + 1)^{-1} h_{2^u+v}$ for appropriate indices (u,v), this would mean that

$$\lim_{u\to\infty} (u + 1)^{-1}\ h_{2^u+v}(x) = 0 \qquad \text{for } \mu\text{-a.e. } x \in X$$

which is of course absurd. □

A refinement of the above example furnishes a counterexample to the first question of problem 7.1. of [12]:

6.9. Example: There are integral operators $Int(k) : L^2(\nu) \to L^2(\mu)$ and $Int(l) : L^2(\tau) \to L^2(\nu)$ such that k and l are multipliable, i.e., for $\mu \times \tau$ -a.e.(x,z), the product $k(x,\cdot)\ l(\cdot,z)$ is ν-integrable (c.f. [12],§7), and such that $R = Int(k) \circ Int(l)$ is not integral.

Denote by $\tilde{\mathbb{I}}$ the unit interval $\mathbb{I}$ minus the dyadic points. For $u \in \mathbb{N}$ and $x \in \tilde{\mathbb{I}}$, let $v_u(x)$ be the unique number in $\{1,\dots,2^u\}$ such that $x \in [\frac{v_u(x)-1}{2^u}, \frac{v_u(x)}{2^u}]$.

If $\{\varepsilon_u(x)\}_{u=1}^{\infty}$ denotes the dyadic expansion of x, then

$$v_u(x) = 1 + \sum_{i=1}^{u} \varepsilon_i(x)\ 2^{u-i}.$$

We now define a permutation p_u of the set $\{1,\dots,2^u\}$: If $v = 1 + \sum_{i=1}^{u} \varepsilon_i\ 2^{u-i}$ for a (unique) sequence of $\varepsilon_i = 0,1$, then let $p_u(v) = 1 + \varepsilon_u + \sum_{i=1}^{u-1} (1 - \varepsilon_i)\ 2^{u-i}$. Note that $p_u^{-1} = p_u$. Now define for $u \in \mathbb{N}$ and $z \in \tilde{\mathbb{I}}$

$$\bar{v}_u(z) = p_u(v_u(z)).$$

The significance of the permutation p_u lies in the fact that it transforms nested sequences of dydadic intervals into sequences of disjoint intervals.

<u>Claim 1:</u> For $z \in \tilde{\Pi}$ the intervals $[\frac{\bar{v}_u(z)-1}{2^u}, \frac{\bar{v}_u(z)}{2^u}]$, $u = 1,2,\ldots$ are mutually disjoint in $\tilde{\Pi}$. Hence for $x \in \tilde{\Pi}$, $z \in \tilde{\Pi}$ the equality $v_u(x) = \bar{v}_u(z)$ holds for at most one $u \geq 1$.

For the first assertion note that the interval $[\frac{\bar{v}_u(z)-1}{2^u}, \frac{\bar{v}_u(z)}{2^u}]$ is formed by the $x \in \tilde{\Pi}$ with $\varepsilon_1(x) \neq \varepsilon_1(z), \ldots,$ $\varepsilon_{u-1}(x) \neq \varepsilon_{u-1}(z)$ and $\varepsilon_u(x) = \varepsilon_u(z)$.

For the second assertion observe that $v_u(x) = \bar{v}_u(z)$ iff x is in the interval $[\frac{\bar{v}_u(z)-1}{2^u}, \frac{\bar{v}_u}{2^u}]$.

After these preliminaries we turn to the construction of our kernels: Let $(X,\mu), (Y,\nu), (Z,\pi)$ and k be as in the above example 6.8, except that we now take $\tilde{\Pi}$ instead of Π to avoid the technical difficulties arising from the non-uniqueness of the dyadic expansion of a dyadic point.

Define $l(y,z)$ by

$$l(1,z) = 1$$

$$l(2,z) = h_2(z),$$

and

$$l(2^u + p_u(v)) = h_{2^u+v}(z)$$

for $u = 1,2,\ldots,$ $v = 1,\ldots,2^u$. The operator $S = \mathrm{Int}(l)$ is the isometry which maps the Haar function h_1 (resp. h_2) in $L^2(\pi)$ onto the unit vector e_1 (resp. e_2) in $L^2(\nu) = l^2$ and the Haar funciton h_{2^u+v} to the $(2^u + p_u(v))$'th unit vector for $u = 1,2,\ldots,$ $v = 1,\ldots,2^u$.

Claim 2: The kernels k and l are multipliable. Actually for every (x,z) there is at most one $y \geq 3$ such that $k(x,y)\ l(y,z) = 0$.

If $u \geq 1$, then $k(x,2^u + v)$ is non-zero (for $1 \leq v \leq 2^u$) iff $v = v_u(x)$, while $l(2^u + v, z)$ is non-zero iff $v = \bar{v}_u(z)$. Hence claim 2 follows from claim 1.

Claim 3: The composed operator $R = \mathrm{Int}(k) \circ \mathrm{Int}(l)$ is not integral.

First note that R maps h_1 (resp. h_2) onto h_1 (resp. h_2) and for $u \geq 1$, h_{2^u+v} onto $(u+1)^{-1}\ h_{2^u+\rho_u(v)}$.

For $u \geq 1$ and $x \in X$, define

$$g_{u,x}(z) = \sum_{i=1}^{u} 2^{-i/2}\ h_{2^i+\bar{v}_i(x)}(z).$$

Observe that $g_{u,x}$ belongs to the unitball of $L^{\infty}(\pi)$, as the supports of the components of the sum are disjoint. Although x varies in the uncountable set X, there are in reality, for fixed u, only 2^u different $g_{u,x}$'s hence $\mathcal{G} = \{g_{u,x} : u \in \mathbb{N},\ x \in X\}$ is a countable subset of the unitball of $L^{\infty}(\pi)$.

We shall show that

$$\sup\{R(g)(x) : g \in \mathcal{G}\} = \infty. \tag{*}$$

for μ-a.e. $x \in X$.

This will imply that $R(\mathcal{G})$ is not order bounded in $L^{0}(\mu)$, hence R does not transform the unitball of $L^{\infty}(\pi)$ into an equimeasureable set. By 4.4 the operator R is not integral.

To prove (*), simply observe that, given $x \in X$,

$$R(g_{u,x}) = \sum_{i=1}^{u} 2^{-i/2}\ (i+1)^{-1}\ h_{2^i+\bar{v}_i(x)},$$

a function on X that assumes the value $\sum_{i=1}^{u} (i + 1)^{-1}$ on x. This finishes the proof of claim 3 and example 6.9.

□

6.10. Remark: As $R = \mathrm{Int}(k) \circ \mathrm{Int}(l)$ is not integral, it is certainly not induced by the function $m(x,z)$, defined by

$$m(x,z) = \int_Y k(x,y)\, l(y,z)\, d\nu(y).$$

It is, however, conceivable, that m is a kernel (inducing a different operator than R). But this is not the case either: one may check that for -almost no x the function $m(x,\cdot)$ belongs to $L^1(\pi)$. The argument follows the lines of the proof of claim 3 above.

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