BOUNDED OPERATORS ON L^p SPACES

PART II

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ABSTRACT:

As in the first part of the paper, we deal with some problems posed in [1]. In the present paper we give solutions to the problems 3.12, 8.2, 17.6 and B.4.

1. INTRODUCTION:

For definitions and notations we refer to the first part of the paper [3]. But we shall not use the concept of a Halmos-function in this second part.

2. SOLUTION TO PROBLEM [1], 3.12:

The solution is a relatively straightforward application of the Banach-Steinhaus theorem and the closed graph theorem. We first need, however, a definition and a preliminary result.

2.1. Definition: Given a Banach space $(G, \|.\|)$ we shall call a Banach space $(E, |||.|||)$ together with a continuous injection $j : E \mapsto G$, a Banach subspace of G. Similary, given an F-space G (i.e. a completely metrisable topological vector space) we shall call an F-space E together with a continuous injection $j : E \mapsto G$ an F-subspace of G.

2.2. Proposition: Let E be a Banach subspace of $L^1(\nu)$ and let $k(x,y)$ be a measurable function on $X \times Y$ such that

(1) $\forall g \in E \quad k(x,.).g(.) \in L^1(\nu)$ for μ-a.e. $x \in X$.

Then the operator

$$\begin{aligned} \mathrm{Int}(k) : E &\longmapsto L^0(\mu) \\ g &\longmapsto f(x) = \int_Y k(x,y)g(y)d\nu(y) \end{aligned}$$

is well defined and continuous.

Proof: Let k_n be the truncation of k at n, i.e.

$$k_n(x,y) = k(x,y) \cdot \chi_{\{|k(x,y)| \leq n\}}(x,y).$$

The operator

$$\begin{aligned} \mathrm{Int}(k_n) : L^1(\nu) &\longmapsto L^\infty(\mu) \\ g &\longmapsto f(x) = \int_Y k_n(x,y)g(y)d\nu(y) \end{aligned}$$

is continuous from $L^1(\nu)$ to $L^\infty(\mu)$ (its norm is at most n).

In particular $\mathrm{Int}(k_n)$ restricts to a continuous operator from E to $L^0(\mu)$, as the injection from E to $L^1(\nu)$ as well as the injection from $L^\infty(\mu)$ to $L^0(\mu)$ are continuous.

Given $g \in E$, note that $\mathrm{Int}(k_n)(g)$ converges μ-almost everywhere to $\mathrm{Int}(k)(g)$ because of the integrability condition (1). Hence $\mathrm{Int}(k_n)(g)$ converges to $\mathrm{Int}(k)(g)$ in measure, i.e. with respect to the topology of $L^0(\mu)$. Therefore the map $\mathrm{Int}(k)$ is the pointwise limit of the sequence $\mathrm{Int}(k_n)$ of continuous operators from E to $L^0(\mu)$. We may apply the Banach-Steinhaus theorem in its form for F-spaces ([4], th. III, 4.6) to infer that $\mathrm{Int}(k)$ is continuous.

□

2.3. Remark: The idea of cutting k down to k_n and to apply the Banach-Steinhaus theorem in the above proof is due to J.B. Cooper, who thus replaced a cumbersome gliding-hump-argument, that I had applied previously.

2.4. Corollary: Let E be a Banach subspace of $L^1(\nu)$, F an F-subspace of $L^o(\mu)$ and $k(x,y)$ a measurable function such that

(1) for $g \in E$ $k(x,.)g(.) \in L^1(\nu)$ for μ-a.e. $x \in X$

and

(2) for $g \in E$ $f(x) = \int_Y k(x,y)g(y)d\nu(y) \in F$.

Then Int(k) induces a continuous operator from E to F.

2.5. Remark: The corollary applies in particular to the case, where E is a closed subspace of $L^2(\nu)$ and $F = L^2(\mu)$, thus answering problem 3.12 of [1] in the positive.

Proof of 2.4: By 2.2 and condition (1) the graph of Int(k) is a closed subspace of $E \times L^o(\mu)$. By condition (2) the graph of Int(k) is contained in $E \times F$ and, as F injects continuously into $L^o(\mu)$, it is closed in $E \times F$.

The closed graph theorem ([4], th. III, 2.3) implies that Int(k) is a continuous operator from E to F.

□

3. SOLUTION TO PROBLEM [1], 8.2:

We have already indicated in [3] one possible way to recapture the kernel k from the values of the operator Int(k) "effectively".

We now present a different "effective procedure", which uses only the (scalar-valued) Radon-Nikodym theorem. Of course, by the vagueness of the term "effective" it will depend on the taste of the reader if he accepts the following construction as a satisfactory answer to problem 8.2.

Let k be a kernel inducing an operator Int(k) from $L^2(\nu)$ to $L^2(\mu)$ and suppose for the moment $k \in L^1(\mu \times \nu)$. Given measurable sets $A \subseteq X$, $B \subseteq Y$,

$$\int_A \int_B k(x,y)\,d\nu(y)\,d\mu(x) = (\mathrm{Int}(k)\chi_B, \chi_A)$$

The right hand side depends only on the values of Int(k). If we denote by λ the measure $k(x,y).(\mu \times \nu)$ on $X \times Y$, the above expression equals $\lambda(A \times B)$. By the integrability of k the measure λ is finite and absolutely continuous with respect to $\mu \times \nu$.

The above formula gives the values of λ on the rectangles, the usual Caratheodory procedure extends λ to the product σ-algebra $\mathcal{X} \times \mathcal{Y}$ and the Radon-Nikodym theorem gives $k(x,y) = \dfrac{d\lambda}{d(\mu \times \nu)}$.

Unfortunately a kernel k may induce a continuous (even compact) operator from $L^2(\nu)$ to $L^2(\mu)$, without k being integrable (although the measure spaces (X,μ) and (Y,ν) are assumed to be finite).
In this case we may not apply brutally the above construction. But, given a kernel k, observe that for each $\varepsilon > 0$ there is $X_\varepsilon \subseteq X$, $\mu(X \setminus X_\varepsilon) < \varepsilon$ such that k restricted to $X_\varepsilon \times Y$ is integrable. Indeed the function $x \longmapsto \|k(x,.)\|_{L^1(\nu)}$ is μ-measurable and μ-almost everywhere finite, hence we only have to take $X_\varepsilon = \{x : \|k(x,.)\|_{L^1(\nu)} \leq M\}$ for M large enough. Having this in mind we may present our construction.

<u>3.1. Proposition</u>: Let $T : L^2(\nu) \mapsto L^2(\mu)$ be an operator of the form $T = \mathrm{Int}(k)$. Then there is an "effective procedure" to recapture k from the values of T.

<u>Proof</u>: For measurable sets $A \subseteq X$, $B \subseteq Y$ define

$$\lambda(A \times B) = (T\chi_B, \chi_A).$$

Clearly λ may be extended to a finitely additive set function on the algebra generated by the rectangles. Let

$$|\lambda|(A \times B) = \sup\{\sum_{i=1}^{n} |\lambda(A_i \times B_i)|,\ A_i \times B_i \text{ disjoint subrectangles of } A \times B\}.$$

It is easily seen, that

$$|\lambda|(A\times B) = \int_A\int_B |k(x,y)|\, d\nu(y)\, d\mu(x).$$

This expression may be equal $+\infty$. But we know from the discussion preceding the proposition that for $\varepsilon > 0$ there is X_ε with $\mu(X\setminus X_\varepsilon) < \varepsilon$ and $|\lambda|(X_\varepsilon \times Y) < \infty$. Hence we may extend the restriction of λ to $X_\varepsilon \times Y$ to the product σ-algebra and by the Radon-Nikodym theorem we may find the values of k on $X_\varepsilon \times Y$ (to be exact: almost everywhere on $X_\varepsilon \times Y$).

Finally it is clear how to find k on all of $X \times Y$. Let $\varepsilon = n^{-1}$ and find successively k on $X_{n^{-1}} \times Y$.

□

4. SOLUTION to [1], 17.6:

The answer is no: There exists an integral operator $\mathrm{Int}(k) : L^2(\nu) \mapsto L^2(\mu)$, an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ in $L^2(\nu)$ and a square summable sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive scalars such that $\sum_{n=1}^{\infty} |\alpha_n . \mathrm{Int}(k)(e_n)|$ is infinite on a set of positive measure.

Actually it is easy to provide such an example in view of the remark in [1], 17.6 that a positive solution to problem 17.6 would solve positively problem [1], 11.8. As we have seen in [3], the answer to [1], 11.8 is negative and a close look at the counterexamples [3], 6.6, 6.8 and 6.9 shows that they also provide counterexamples to [1], 17.6.

However we prefer not to repeat these examples but rather to give a very easy counterexample, taylormade for problem [1], 17.6.

Fix a sequence $\{\beta_n\}_{n=-\infty}^{+\infty}$ of scalars, which is not square summable but such that

$$c(t) = \sum_{n=-\infty}^{+\infty} \beta_n e^{2\pi i n t}$$

converges in $L^1(T)$, where T denotes the onedimensional torus equipped with

Lebesgue measure. For example $\{n^p\}_{n=-\infty}^{+\infty}$ for $-\frac{1}{2} \leq p < 0$ is such a sequence (c.f. [2], ex. II, 1.3).
Denote by C the convolution operator on $L^2(T)$ induced by c (c.f. [1], th. 12.2). Clearly C is an absolutely bounded integral operator and the kernel corresponding to C is given by

$$k(t,s) = c(t-s) = \sum_{n=-\infty}^{+\infty} \beta_n e^{2\pi in(t-s)}.$$

Note that the operator C maps $e^{2\pi in}$ to $\beta_n . e^{2\pi in}$. Indeed, as $e^{2\pi in}$ is an element of $L^\infty(T)$, it defines a continuous linear functional on $L^1(T)$, hence the following equations hold true.

$$\int_T k(t,s) e^{2\pi ins} ds = \sum_{m=-\infty}^{+\infty} \int_T e^{2\pi ins} . \beta_m . e^{2\pi im(t-s)} ds$$

$$= \sum_{m=-\infty}^{+\infty} \beta_m . e^{2\pi imt} . \int_T e^{2\pi i(n-m)s} ds$$

$$= \beta_n . e^{2\pi int}.$$

Find a square-summable sequence $\{\alpha_n\}_{n=-\infty}^{+\infty}$ such that $\sum_{n=-\infty}^{+\infty} |\alpha_n . \beta_n| = \infty$.
Then

$$\sum_{n=-\infty}^{+\infty} |\alpha_n . C(e^{2\pi in})(t)| = \sum_{n=-\infty}^{+\infty} |\alpha_n . \beta_n| = \infty$$

for almost every $t \in T$.

□

5. SOLUTION TO PROBLEM [1], B.4:

5.1. Proposition: There is an $\mathbb{R}_+$-valued Lebesgue measurable function $k(x,y)$ on $[0,1] \times [0,1]$ such that for every Lebesgue measurable $\mathbb{R}_+$-valued function g on $[0,1]$, whcih is different from 0 on a set of positive measure,

$$\int k(x,y)g(y)dy = \infty$$

for almost each $x \in [0,1]$.

Whence, in the language of [1], all nontrivial subkernels of k have domain $\{0\}$.

<u>Proof</u>: Let h be the function on $[0,1] \times [0,1]$,

$$h(x,y) = |x - y|^{-1} \quad \text{if } x \neq y$$
$$h(x,y) = 0 \quad \text{if } x = y.$$

It is shown in [1], ex. 3.2, that for any positive, measurable function g on $[0,1]$, not vanishing almost everywhere, the set

$$A_h = \{x : \int g(y)h(x,y)dy = \infty\}$$

has strictly positive measure. Our task is to replace h by some k such that this set is always of measure 1.

Let $\{r_n\}_{n=1}^{\infty}$ be an ennumeration of the rationals in $[0,1]$ and let $h_n(x,y)$ be the r_n-th translate of h, i.e.

$$h_n(x,y) = h(x \dot{-} r_n, y),$$

where $\dot{-}$ denotes subtraction modulo 1. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of strictly positive numbers, such that

$$m_2\{(x,y) : p_n h_n(x,y) \geq 2^{-n}\} \leq 2^{-n},$$

where m_2 denotes Lebesgue measure on $[0,1] \times [0,1]$.

Define

$$k(x,y) = \sum_{n=1}^{\infty} p_n . h_n(x,y).$$

By the Borel-Cantelli lemma k is m_2-almost everywhere finite.
By changing k on a set of measure zero, we may assume that k is everywhere $\mathbb{R}_+$-valued.

Let g be a positive function on Y, different from zero on a set of positive measure. The set

$$A_h = \{x : \int g(y)h(x,y)dy = \infty\}$$

is of strictly positive measure. As the set

$$A_k = \{x : \int g(y)k(x,y)dy = \infty\}$$

contains all rational translates of A_h (modulo 1), A_k has measure 1.

□

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