Almost Compactness and Decomposability of Integral Operators

by

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Abstract: Let $(X,\mu),(Y,\nu)$ be finite measure spaces and $1 < q \leq \infty$, $1 \leq p \leq q$. An integral operator $\mathrm{Int}(k) : L^q(\nu) \to L^p(\mu)$ becomes compact, if we cut away a suitably chosen subset of X of arbitrarily small measure. As a consequence we prove that $\mathrm{Int}(k)$ may be written as the sum of a Carleman operator and an orderbounded integral operator, where the orderbounded part may be chosen to be compact and of arbitrarily small norm.

1. Introduction:

$(X,\mathcal{X},\mu)$ and $(Y,\mathcal{Y},\nu)$ will denote finite measure spaces. For $1 \leq p,q \leq \infty$ we call an operator $T : L^q(\nu) \to L^p(\mu)$ integral, if there is a measurable kernel-function $k(x,y)$ on $X \times Y$ such that for $g \in L^q(\nu)$

$$Tg(x) = \int_Y k(x,y)\, g(y)\, d\nu(y) \qquad \mu\text{-a.e.}$$

The integrand is required to be Lebesgue-integrable for μ-a.e. $x \in X$(c.f. [7] or [9]). In this case we write $T = \mathrm{Int}(k)$.

There are two wellbehaved subclasses of integral operators: $\mathrm{Int}(k)$ is called Carleman if, for μ-a.e. $x \in X$, $k(x,\cdot) \in L^r(\nu)$ where $r^{-1} + q^{-1} = 1$. The operator $\mathrm{Int}(k)$ is called orderbounded if it transforms orderbounded sets into orderbounded sets or equivalently if $|k|$ also defines an integral operator from $L^q(\nu)$ to $L^p(\mu)$. In this case we call $Int(|k|)$ the modulus or absolute value of $Int(k)$.

Let us specify the following notation: If $g \in L^\infty(\mu)$ we denote by P_g the multiplication operator $f \to f.g$ on $L^p(\mu)$. If $g = \chi_A$ is a characteristic function we write P_A for P_{χ_A}.

<u>2.Preliminaries:</u> In this section we recall known result for later reference.

<u>2.1. Theorem</u>(Nikishin, [11], th. 4): Let $0 \leq q \leq \infty$ and $T : L^q(\nu) \to L^0(\mu)$ be a positive, continuous operator. For $\varepsilon > 0$ there is an $A \subseteq X$, $\mu(X \setminus A) < \varepsilon$ and such that $P_A \circ T$ takes its values in $L^q(\mu)$.

<u>2.2. Theorem:</u> (Maurey, [10], prop. 9): Let $0 < p \leq q \leq \infty$ and $T : L^q(\nu) \to L^p(\mu)$ be a positive, continuous operator. For $r^{-1} = p^{-1} - q^{-1}$ there is a strictly positive function $g \in L^\infty(\mu)$ such that $g^{-1} \in L^r(\mu)$ and $P_g \circ T$ takes its values in $L^q(\mu)$.

We also need a technical result, which follows easily from ([9], th. 4.7. and th. 5.12).

<u>2.3. Lemma:</u> Let $1 \leq q \leq \infty$, $1 \leq p < \infty$ and $k(x,y) \geq 0$ be such that $Int(k)$ defines an operator from $L^q(\nu)$ to $L^p(\mu)$. Let $k_n(x,y) \geq 0$ be such that $k = \sum_{n=1}^\infty k_n$.

(a) $\sum_{n=1}^\infty Int(k_n)$ converges unconditionally to $Int(k)$ in the strong operator topology of $B(L^q(\nu), L^p(\mu))$.

(b) If $1 < q \leq \infty$ and $Int(k)$ is compact then the above sum converges unconditionally in the norm of $B(L^q(\nu), L^p(\mu))$.

3. Almost compactness of positive integral operators:

3.1. Theorem: Let $1 < q \leq \infty$ and $k(x,y) \geq 0$ be such that $Int(k)$ defines an operator from $L^q(\nu)$ to $L^q(\mu)$. Given $r < \infty$ we may find $g \in L^\infty(\mu)$ such that $g^{-1} \in L^r(\mu)$ and $P_g \circ Int(k) : L^q(\nu) \to L^q(\mu)$ is compact.

Proof: Let us start with the easy case $q = \infty$: It is an old result, dating back to Dunford's paper [4] in 1936, that a σ^*-continuous $T: L^\infty(\nu) \to L^\infty(\mu)$ is integral iff for $\varepsilon > 0$ there is $A \subseteq X$, $\mu(X \setminus A) < \varepsilon$ and such that $P_A \circ T$ is compact (see also [5] and [12]). So find a partition $(A_n)_{n=1}^\infty$ of X such that $P_{A_n} \circ Int(k)$ is compact and, given $r < \infty$, find a nullsequence $(\alpha_n)_{n=1}^\infty$ of strictly positive scalars such $g^{-1} = \sum_{n=1}^\infty \alpha_n^{-1} \chi_{A_n} \in L^r(\mu)$. It is easy to check that $P_g \circ Int(k)$ is compact.

Now assume that $1 < q < \infty$: Given $r < \infty$ find $1 < p < q$ such that $r^{-1} \geq p^{-1} - q^{-1}$. The operator $Int(k)$ is a compact operator from $L^q(\nu)$ to $L^p(\mu)$ (c.f. [1] or [9], th. 5.4.; compare also [3]). Let $k_n = k \cdot \chi_{\{n-1 \leq k < n\}}$ and deduce from 1.3.b that $\sum_{n=1}^\infty Int(k_n)$ converges to $Int(k)$ unconditionally in the norm of $B(L^q(\nu), L^p(\mu))$. So we may find a sequence $0 = n_0 < n_1 < \ldots < n_m < \ldots$ such that for $m \geq 2$

$$\left\| \sum_{n=n_{m-1}}^{n_m - 1} Int(k_n) \right\|_{B(L^q, L^p)} < 2^{-m}.$$

Let $$\bar{k}_m = m \sum_{n=n_{m-1}}^{n_m -1} k_n .$$

and $$\bar{k} = \sum_{m=1}^{\infty} \bar{k}_m .$$

Clearly $\bar{k} \geq k$ but $\mathrm{Int}(\bar{k}) = \sum_{m=1}^{\infty} \mathrm{Int}(\bar{k}_m)$ is still a continuous (even compact, but we shall not need this) operator from $L^q(\nu)$ to $L^p(\mu)$. We may apply Maurey's factorisation theorem (1.2 above) to find $g \in L^\infty(\mu)$ such that $g^{-1} \in L^r(\mu)$ and such that $P_g \circ \mathrm{Int}(\bar{k})$ takes its values in $L^q(\nu)$. From 1.3.a)

$$P_g \circ \mathrm{Int}(\bar{k}) = \sum_{m=1}^{\infty} m \Big(\sum_{n=n_{m-1}}^{n_m -1} \mathrm{Int}(g.k_n) \Big),$$

the sum converging unconditionally in the strong operator topology of $B(L^q(\nu), L^q(\mu))$. This implies that the sum

$$P_g \circ \mathrm{Int}(k) = \sum_{m=1}^{\infty} \Big(\sum_{n=n_{m-1}}^{n_m -1} \mathrm{Int}(g.k_n) \Big)$$

converges in the norm of $B(L^q(\nu), L^q(\mu))$. As each of the summands is clearly compact the operator $P_g \circ \mathrm{Int}(k) : L^q(\nu) \to L^q(\mu)$ is compact.

<u>3.2. Remark:</u> The theorem does not hold for $q = 1$: Let $T : L^1(\nu) \to L^1[0,1]$ be a positive surjective operator, where (Y,ν) is a purley atomic measure space (i.e. $L^1(\nu)$ is isometric to l^1). Then T is integral but for every positive $g \in L^\infty(\mu)$, which does not vanish identically, the operator $P_g \cdot \mathrm{Int}(k)$ is not compact.

However, we have the following result by duality:

<u>3.3. Corollary:</u> Let $1 \leq p < \infty$ and $k(y,x) \geq 0$ such that $\mathrm{Int}(k)$ defines an operator from $L^p(\mu)$ to $L^p(\nu)$. Given

$r < \infty$ we may find $g \in L^\infty(\mu)$ such that $g^{-1} \in L^r(\mu)$, and $\mathrm{Int}(k) \circ P_g : L^p(\mu) \to L^p(\nu)$ is compact.

4. Almost compactness of general integral operators:

4.1. Theorem: Let $1 < q \leq \infty$ and $1 \leq p \leq q$ and let $\mathrm{Int}(k) : L^q(\nu) \to L^p(\mu)$ be an integral operator. For $\varepsilon > 0$ there is $A \subseteq X$ with $\mu(X \setminus A) < \varepsilon$ such that both $P_A \circ \mathrm{Int}(k)$ and its modulus $P_A \circ \mathrm{Int}(|k|)$ are compact operators from $L^q(\nu)$ to $L^q(\mu)$.

Proof: Write $k = k_1 - k_2 + ik_3 - ik_4$, where $k_j \geq 0$. Each $\mathrm{Int}(k_j)$ defines a positive continuous operator from $L^q(\nu)$ to $L^0(\mu)$. By Nikishin's theorem (2.1 above) we may find $B_j \subseteq X$, $\mu(X \setminus B_j) < \varepsilon/8$ such that $P_{B_j} \circ \mathrm{Int}(k_j)$ is a positive continuous operator from $L^q(\nu)$ to $L^q(\mu)$. It is an easy consequence of theorem 3.1 that we may find $A_j \subseteq B_j$, $\mu(X \setminus A_j) < \varepsilon/4$, such that $P_{A_j} \circ \mathrm{Int}(k_j)$ is compact from $L^q(\nu)$ to $L^q(\mu)$. For $A = \bigcap_{j=1}^{4} A_j$, the operator $P_A \circ \mathrm{Int}(k)$ satisfies the requirements.

4.2 Remark: In the situation of theorem 4.1. it is not possible to find a big set B on the left hand side (i.e. from Y) so that $\mathrm{Int}(k) \circ P_B$ is compact. For example let k be the kernel on $[0,1] \times [0,1]$, $k(x,y) = 2^{n/2} \cdot r_n(y)$ if $x \in [2^{-n}, 2^{-(n-1)}]$, where r_n denote the n'th Rademacher function. Then $\mathrm{Int}(k) : L^2_{[0,1]} \to L^2_{[0,1]}$ is such an example.

Theorem 4.1. is a strengthening of the known result of "twosided cutting off", which seems to be due to Korotkov [8].

4.3. Remark: What happens in the case $p > q$?
If $Int(k) : L^q(\nu) \to L^p(\mu)$ is given, then for $q > 1$ the above theorem applies and provides a compact operator $P_A \circ Int(k)$ from $L^q(\nu)$ to $L^q(\mu)$. One would like to have the operator compact from $L^q(\nu)$ to $L^p(\mu)$ but this is only possible for few pairs of indices as is shown in the following proposition:

4.4. Proposition: a) Let $1 < q < \infty$ and $p = \infty$; for every continuous operator $T : L^q(\nu) \to L^\infty(\mu)$ and $\varepsilon > 0$ there is an $A \subseteq X$ with $\mu(X \setminus A) < \varepsilon$ such that $P_A \circ T : L^q(\nu) \to L^\infty(\mu)$ is compact.

b) On the other hand, for $1 \leq q < p < \infty$ and for $q = 1$, $p = \infty$ there are integral operators $Int(k) : L^q(\nu) \to L^p(\mu)$ such that for every $A \subseteq X$, $\mu(A) > 0$ the operator $P_A \circ T : L^q(\nu) \to L^p(\mu)$ is not compact.

Proof: a) This result was known to A. Grothendieck [6]. Let us phrase it in the terminology of [13]: $L^q(\nu)$ is Asplund for $1 < q < \infty$ hence $T(\text{ball}(L^q(\nu)))$ is equimeasurable, which is just what we have to prove.

b) For $q = 1$ and $1 \leq p \leq \infty$ let T be a positive surjective operator from l^1 (represented as $L^1(\nu)$ over a finite measure space (Y,ν)) onto $L^p_{[0,1]}$ (resp. onto the subspace $C_{[0,1]}$ of $L^\infty_{[0,1]}$, if $p = \infty$).

If $1 < q < p < \infty$ then there are operators of potential type from $L^q_{[0,1]}$ to $L^p_{[0,1]}$ that are not compact (c.f. [8], p. 147 ff.). It is clear, that an operator of potential type may not be made compact by restricting to a subset of positive measure.

5. Decomposition of integral operators:

5.1. Theorem: Let $1 < q \leq \infty$, $1 \leq p \leq q$ and $\mathrm{Int}(k) : L^q(\nu) \to L^p(\mu)$ an integral operator. Given $\varepsilon > 0$ we may write k as $k^c + k^o$ where $\mathrm{Int}(k^c)$ is a Carleman operator from $L^q(\nu)$ to $L^p(\mu)$ and $\mathrm{Int}(k^o)$ as well as its modulus $\mathrm{Int}(|k^o|)$ are compact operators from $L^q(\nu)$ to $L^q(\mu)$ of norm less than ε.

Proof: We start with the the trivial case $q = \infty$ and $1 \leq p \leq \infty$. Every $\mathrm{Int}(k) : L^\infty(\nu) \to L^p(\mu)$ is automatically Carleman, hence we may choose $k^c = k$ and $k^o = 0$.

Let now $1 < q < \infty$, $1 \leq p \leq q$. By th. 4.1. we may find a partition $(A_i)_{i=1}^\infty$ of X such that for

$k_i(x,y) = \chi_{A_i}(x).k(x,y)$ the operator $\mathrm{Int}(|k_i|)$ is compact from $L^q(\nu)$ to $L^q(\mu)$. By lemma 2.3. we may find numbers n_i such that

$$\|\mathrm{Int}(|k_i|) - \mathrm{Int}(|k_i| \cdot \chi_{\{|k_i| \leq n_i\}})\| < \varepsilon/2^i.$$

Let $k_i^c = k_i \cdot \chi_{\{|k_i| \leq n_i\}}$ and $k_i^o = k_i - k_i^c$ and define

$$k^c = \sum_{i=1}^{\infty} k_i^c \quad \text{and} \quad k^o = \sum_{i=1}^{\infty} k_i^o.$$

It is now easy to verify the asserted properties of k^c and k^o.

5.2. Remark: We do not know whether for arbitrary $1 \leq p,q \leq \infty$ an integral operator $\mathrm{Int}(k) : L^q(\nu) \to L^p(\mu)$ may be decomposed into a Carleman and an orderbounded part. We know that this is possible in some cases not covered by 5.1. For $p = q = 1$ for example, this is trivially possible as every continuous operator $T : L^1(\nu) \to L^1(\mu)$ is order-bounded. However, we do not have the full strength of 5.1. in this case: The operator from remark 3.2. may not be decomposed in such a way as to make the orderbounded part compact or arbitrarily small in norm.

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