A KREIN - MILMAN SET WITHOUT THE INTEGRAL REPRESENTATION PROPERTY

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We construct a separable Banach space E and a bounded, closed, absolutely convex subset B such that B is the closed convex hull of its extreme points but such that not every point in B is representable as the barycenter of a probability measure on the extreme points of B.

Let X be a separable Banach space not having the Radon-Nikodym property and such that its unit ball U is the closed convex hull of its extreme points $E(U)$. The space of converging sequences c for example is such a space. (Note in passing that the unit ball of c has countably many extreme points and that every point in the unit-ball of x is the barycenter of probability measure on the extreme points).

Let Δ be the Cantor set and let $E = I(C(\Delta), X)$ be the space of integral operators from $C(\Delta)$ to X, i.e. the linear operators $T : C(\Delta) \to X$ such that

$\|T\|_I = \sup \{ \sum_{i=1}^{n} \|T \chi_{A_i}\| : A_i$ disjoint clopen sets in $C\} < \infty$.

Let $B = \{T : \|T\|_I \leq 1\}$ and equip E with the topology τ of pointwise convergence on $C(\Delta)$, i.e. $T_\alpha \to T$ if for each $f \in C(\Delta)$, $\|T_\alpha(f) - T(f)\| \to 0$.

There are obvious extreme points in B, namely the $\delta_t \otimes x$, $t \in \Delta$, $x \in E(U)$. It is also obvious that these are the only extreme points of B, hence we write

$E(B) = \{\delta_t \otimes x,\ t \in \Delta,\ x \in E(U)\}$. We shall show that B is the closed convex hull of $E(B)$.
By the Hahn-Banach theorem this is equivalent to say that the polars of $E(B)$ and B coincide. Let
$\sum_{i=1}^{n} f_i \otimes x_i^*$ be an element of E', that belongs to the polar of $E(B)$. Evidently this means just that for $t \in \Delta$,
$\|\sum_{i=1}^{n} f_i(t).x_i^*\|_{X^*} \leq 1$ and this latter condition implies that $\sum_{i=1}^{n} f_i \otimes x_i^*$ belongs to the polar of B, as is readily seen from the definition of B. Hence $\overline{\Gamma}(E(B)) = B$.

We shall now show that there are points in B not representable as barycenter of probability measures on the extremals. Let T_o be an integral operator in B that is not nuclear and suppose there is a probability μ on $E(B)$ such that for each $f \in C(\Delta)$ and $x^* \in X^*$

$$\langle x^*, T_o(f)\rangle = \int_{E(B)} \langle x^*, \delta_t \otimes x\rangle \, d\mu(\delta_t \otimes x).$$

Note that $E(B)$ is homeomorphic to $E(U) \times \Delta$. As $E(U)$ is always a coanalytic set (if X is c, $E(U)$ is even a countable discrete set), there exists a desintegration of μ, i.e. there are probability measures μ_t on E_U and a probability measure ν on Δ such that $\mu = \int_\Delta \mu_t \, d\nu(t)$, i.e. we get for $f \in C(\Delta)$ and $x^* \in X^*$

$$\langle x^*, T_o(f)\rangle = \int_\Delta [\int_{E_U} \langle x^*, (\delta_t \otimes x)(f)\rangle d\mu_t(x)] \, d\nu(t).$$
$$= \int_\Delta f(t) \, .[\int_{E_U} \langle x^*, x\rangle \, d\mu_t(x)] \, d\nu(t).$$

For $t \in \Delta$ write $F(t) = \int_{E_U} x \, d\mu_t(x) \in U$ (the integral taken in the weak sense) to get a Radon-Nikodym derivative of T_o, i.e. for $f \in C(\Delta)$

$$T_o(f) = \int_\Delta f(t) \, . \, F(t) \, d\nu(t).$$

This means just that T_o is nuclear, which is a contradiction.

q.e.d.

Remark: We have contructed our example in a locally convex space E which is not even a Fréchet space, but it is not difficult to make the example live in a Banach space. Let $\{f_n\}_{n=1}^{\infty}$ be any total seguence in $C(\Delta)$, tending to zero in norm, and define the norm $\|.\|_E$ in E to be

$$\|T\|_E = \sup \{ \|T f_n\| : n \in \mathbb{N} \}.$$

It is easily verified that $\|.\|_E$ is indeed a norm and defines on B the topology τ. Letting $\tilde{E}$ be the completion of $(E, \|.\|_E)$ we have imbedded our example into a separable Banach space.

An inspection of the above argument shows, that we may imbed our example into the space $c_o(X)$ or even $\ell^2(X)$.