A result of J. Bourgain: $C(L^1)$ has the Dunford-Pettis property

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This note which proves the anounced theorem is entirely based on [1]. We hope that we have succeeded to make the proof more accessible from the pedagogical point of view, but we want to stress out that there is no idea used here that has not been used in [1]. Also more general results are obtained in [1].

Let (S,Σ,μ) be a probability space and T be the Cantor set (this is only for convenience; the result carries over to arbitrary compact Hausdorff spaces by some obvious modifications of the proof). Denote by $C(T; L^1(\mu))$ or short $C(L^1)$ the space of continuous functions from T to $L^1(\mu)$.

<u>Theorem:</u> $C(L^1)$ has the Dunford-Pettis property.

<u>Proof:</u> If $C(L^1)$ does not have D-P, then there exist a sequence $\{\xi^i\}_{i=1}^{\infty}$ in $C(L^1)$ and a sequence $\{\mu^i\}_{i=1}^{\infty}$ in $C(L^1)^*$, both tending weakly to zero, and $\alpha > 0$ such that $\langle \xi^i,\mu^i \rangle \geq \alpha$ (c.f. [3]). Clearly we may assume that $\|\xi^i\|, \|\mu^i\| \leq 1$ and that ξ^i are finitely valued functions (as the Cantor set is totally disconnected). So let us work towards a contradiction!

Clearly for each $t \in T$, the sequence $\{\xi_t^i\}_{i=1}^\infty$ is a weak nullsequence in $L^1(\mu)$. Indeed, for $t \in T$ and $g \in L^\infty(\mu)$ the function $\xi \to \langle \xi_t, g \rangle$ is an element of $(C_{L^1})^*$.

Now we give the following intuitive hint: a sequence in $L^1(\mu)$ may only converge in two ways weakly to zero: it either converges strongly (this case is trivial for our purposes), or - if it does not converge strongly - it behaves "essentially like the Rademacherfunctions". This hint is made precise in the following technical arguments (which are standard), where we show that we may reduce to the case stated below ((A)). At a fist reading the reader is advised to skip the following reduction steps and to continue at (A), as it is only then, that the essence of Bourgain's argument (namely the use of Riesz-products) comes up.

Let $|\mu^i|$ denote the variation measure of μ^i, i.e. $|\mu^i|$ is the linear functional on $C(T)$ defined for $x \in C(T)_+$ by

$$\langle x, |\mu^i| \rangle = \sup \{\langle \xi, \mu^i \rangle : \|\xi_t\| \leq x(t) \qquad \forall t \in T\}$$

Let $\nu = \sum_{i=1}^\infty 2^{-i}|\mu^i|$; clearly the sequence $\{|\mu^i|\}_{i=1}^\infty$ is equi-absolutely continuous with respect to ν. Indeed, if this were not so, we could find by Rosenthal's lemma (c.f. [2], IV.2.1) a disjoint sequence $\{A_k\}_{k=1}^\infty$ of Borel subsets of T, an increasing seuquence $\{i_k\}_{k=1}^\infty$ and $\varepsilon > 0$ such that $|\mu^{i_k}|(A_k) \geq \varepsilon$ and $|\mu^{i_k}|(\bigcup_{j=1, j\neq k}^\infty A_j) < \varepsilon/2$. This would imply that $|\mu^{i_k}|$ spans an ℓ^1 in $M(K)$ and similarly that μ^{i_k} spans a ℓ^1 in $(C_{L^1})^*$, which contradicts the weak convergence of μ^i.

So there is $\delta > 0$ such that for a Borel-set $A \subseteq T$, $\nu(A) \leq \delta$ implies $|\mu^i|(A) \leq \alpha/3$ for every $i \in \mathbb{N}$.

As has already been observed, for $t \in T$, the sequence $\{\xi_t^i\}_{i=1}^\infty$ converges weakly in $L^1(\mu)$, hence there exists a constant M_t such that for every $i \in \mathbb{N}$ $\|\xi_t^i \cdot \chi_{\{|\xi_t^i| > M_t\}}\| < \alpha/9$. Clearly we may choose the map $t \to M_t$ to be ν-measurable so we may find a constant M such that for every t in some compact subset T_o of T, with $\nu(T \setminus T_o) < \delta$, we have $\|\xi_t^i \cdot \chi_{\{|\xi_t^i| > M\}}\| < \alpha/9$.

We now start an induction: We shall construct a decreasing sequence T_k of compact subsets of T, with $\nu(T \setminus T_k) < \delta$, an increasing sequence $\{i_k\}_{k=1}^\infty$ of integers and for every $t \in T_k$ we shall define an element Ψ_t^k of $L^1(\mu)$ and a finite sub-σ-algebra $\mathcal{B}_t^k$ of Σ, containing $\mathcal{B}_t^{k-1}$, such that

(i) $\qquad \|\Psi_t^k\|_{L^\infty(\mu)} \leq 2M$

(ii) $\qquad \|\Psi_t^k - \xi_t^{i_k}\|_{L^1(\mu)} < \alpha/3$

(iii) $\qquad \Psi_t^k$ is $\mathcal{B}_t^k$ - measurable and $E(\Psi_t^k | \mathcal{B}_t^{k-1}) = 0$.

Let $\mathcal{B}_t^0 = \{\Phi, S\}$.

For $k=1$ note that for $t \in T_o$, $E(\xi_t^k)$ tends to zero; hence we may find $T_1 \subseteq T_o$, $\mu(T \setminus T_1) < \delta$ and an integer i_1 such that for $t \in T_1$ $|E(\xi_t^{i_1})| < \alpha/9$. We may find a simple function φ_t^1, $\|\varphi_t^1\|_{L^\infty(\mu)} \leq M$, such that $\|\varphi_t^1 - \xi_t^{i_1}\|_{L^1(\mu)} < \alpha/9$. Note that this implies that $|E(\varphi_t^1)| \leq |E(\xi_t^{i_1})| + |E(\varphi_t^1 - \xi_t^{i_1})| < \frac{2\alpha}{9}$.

Let $\psi_t^1 = \varphi_t^1 - E(\varphi_t^1)$ and $\mathcal{B}_t^1$ a finite sub-σ-algebra of Σ such that ψ_t^1 is $\mathcal{B}_t^1$ measurable, and check that (i),(ii) and (iii) are satisfied.

For the induction step suppose T_k, i_k and ψ_t^k and $\mathcal{B}_t^k$ for $t \in T_k$ defined. Fix $t \in T_k$. As $\mathcal{B}_t^k$ is a finite σ-algebra $E(\xi_t^i | \mathcal{B}_t^k)$ tends to zero (in the L^1-norm, say). So we may find $T_{k+1} \subseteq T_k$, $\mu(T \setminus T_{k+1}) < \delta$, and an integer $i_{k+1} > i_k$ such that, for $t \in T_{k+1}$, $\|E(\xi_t^{i_{k+1}} | \mathcal{B}_t^k)\|_{L^1(\mu)} < \alpha/9$. Again we may find a simple function φ_t^{k+1}, $\|\varphi_t^{k+1}\|_{L^\infty(\mu)} \leq M$, such that $\|\varphi_t^{k+1} - \xi_t^{i_{k+1}}\|_{L^1(\mu)} < \alpha/9$. As above this implies $\|E(\varphi_t^{k+1} | \mathcal{B}_t^k)\|_{L^1(\mu)} < \frac{2\alpha}{9}$. Putting $\psi_t^{k+1} = \varphi_t^{k+1} - E(\varphi_t^{k+1} | \mathcal{B}_t^k)$ and $\mathcal{B}_t^{k+1}$ a finite σ-algebra, containing $\mathcal{B}_t^k$, such that ψ_t^{k+1} is $\mathcal{B}_t^{k+1}$-measurable, it is immediate to check that these satisfy (i),(ii) and (iii).

Finally let us observe the following: We have assumed the ξ_t^i to be simple functions and we may clearly assume that for $t_1, t_2 \in T_k$ such that $\xi_{t_1}^{i_k} = \xi_{t_2}^{i_k}$ we have applied the same construction to it, i.e. $\varphi_{t_1}^k = \varphi_{t_2}^k$. This implies that the functions $t \to \varphi_t^k$ are (simple) continuous functions on T_k, a fact which is not really essential for the following arguments, but has some cosmetic convenience for our formulation of the assumption (A).

Let $\bar{T} = \bigcap_{k=1}^{\infty} T_k$ and note that $\nu(T \setminus \bar{T}) \leq \delta$.
Let $\bar{\xi}_t^k \in C(\bar{T}; L^1(\mu))$ be defined by $\bar{\xi}_t^k = (2M)^{-1} \psi_t^k$ for $t \in \bar{T}$ and let $\bar{\mu}^k \in C(\bar{T}; L^1(\mu))^*$ the restriction of μ^{i_k} to $\bar{T}$.

Then

$$\left\langle 2M\bar{\xi}^{k},\bar{\mu}^{k}\right\rangle \geqq \left\langle \xi^{i_k},\mu^{i_k}\right\rangle - \left|\left\langle \xi_t^{i_k},\mu^{i_k}\cdot\chi_{T\setminus\bar{T}}\right\rangle\right| - \left|\left\langle \xi_t^{i_k}-\psi_t^{k},\mu^{i_k}\chi_{\bar{T}}\right\rangle\right|$$

$$\geqq \alpha - \alpha/3 - \alpha/3 = \alpha/3 ,$$

i.e. $\left\langle \bar{\xi}^{k},\bar{\mu}^{k}\right\rangle \geqq \frac{\alpha}{6M} = \bar{\alpha}.$

Writing ξ^i instead of $\bar{\xi}^k$ and μ^i instead of $\bar{\mu}^k$ for $i=k$ and α instead of $\bar{\alpha}$, we arrive at our desired assumtion:

(A) If $C(T;L^1)$ does not have the Dunford-Pettis property, then there is a compact subspace $\bar{T}$ of T, and sequences $\{\xi^i\}_{i=1}^{\infty} \in C(\bar{T},L^1)$ and $\{\mu^i\}_{i=1}^{\infty} \in C(\bar{T};L^1)^*$, $\|\xi^i\|\,\|\mu^i\| \leq 1$, both tending weakly to zero and an $\alpha > 0$ such that $\left\langle \xi^i,\mu^i\right\rangle \geqq \alpha$ and

(1) For each $t \in \bar{T}$ and $i \in \mathbb{N}$ ξ_t^i belongs to $L^{\infty}(\mu)$ and $\|\xi_t^i\|_{L^{\infty}(\mu)} \leqq 1$

(2) For each $t \in \bar{T}$ the sequence $\{\xi_t^i\}_{i=1}^{\infty}$ is a martingale difference sequence, i.e. there exists an increasing sequence of (finite) σ-algebras $\{\mathcal{B}_t^i\}_{i=0}^{\infty}$ on S, $\mathcal{B}_t^o = \{\Phi,S\}$, such that ξ_t^i is $\mathcal{B}_t^i$-measurable and $E(\xi_t^i|\mathcal{B}_t^{i-1}) = 0$ for $i \in \mathbb{N}$.

Note that (2) implies clearly

(2') for each $t\in\bar{T}$ and $i_1 < i_2 < \ldots < i_k$

$$E(\xi_t^{i_1}.\xi_t^{i_2}. \ldots .\xi_t^{i_k}) = 0,$$

and it is this latter property of $\{\xi_t^i\}_{i=1}^{\infty}$, which will be used in the sequel.

By the subsequent lemma we may find positive scalars $a_1, \dots, a_n$, $\sum_{i=1}^{n} a_i = 1$ such that $\|\sum_{i=1}^{n} a_i \varepsilon_i \mu^i\| < \alpha/2$ for every choice of sigus $\varepsilon_i = \pm 1$. Denoting by ε_i the i-th Rademacher-function on $[0,1]$, this is the same as saying $\|\sum_{i=1}^{n} a_i \varepsilon_i(\omega) \mu^i\| < \alpha/2$ for every $\omega \in [0,1]$.

Now define for $\omega \in [0,1]$ the (Riesz)-products $R_\omega \in C_{L^1}$ by letting for $t \in \bar{T}$

$$R_{\omega,t} = \prod_{i=1}^{n} (1 + \varepsilon_i(\omega) \xi_t^i).$$

Clearly for each $\omega \in [0,1]$ and $t \in \bar{T}$, $R_{\omega,t} \geq 0$ and $\|R_{\omega,t}\|_{L^1} = E(\prod_{i=1}^{n} (1 + \varepsilon_i(\omega) \xi_t^i)) = 1$ as (by(2')) for every $i_1 < i_2 < \dots < i_k$, $E(\xi_t^{i_1} \cdot \dots \cdot \xi_t^{i_k}) = 0$.
Also note that for $1 \leqq j \leqq n$ and $t \in T$

$$\int_{[0,1]} \varepsilon_j(\omega) R_{\omega,t} \, d\omega = \int_{[0,1]} \varepsilon_j(\omega) \cdot \prod_{i=1}^{n} (1 + \varepsilon_i(\omega) \xi_t^i) \, d\omega = \xi_t^j .$$

Recapitulating, what we have done so far: Starting from $\mu^1, \dots, \mu^n$ we have formed the 2^n convex combinations $\sum_{i=1}^{n} a_i \varepsilon_i \mu^n$ $(\varepsilon_i = \pm 1)$. On the other hand we have formed, starting form $\xi^1, \dots, \xi^n$, the Rieszproduct R_ω (which are also in reality 2^n elements).

We may obtain the μ^i back from the $\sum_{i=1}^{n} a_i \varepsilon_i \mu^i$ as their "Rademacher-averages" just in the same way as we may get the ξ^i back from the R_ω. This and the fact that we could make the norms of the $\sum_{i=1}^{n} a_i \varepsilon_i \mu^i$ small while the norms of the R_ω's stay bounded, leads to the following contradiction which finishes the proof of the theorem:

$$\frac{\alpha}{2} > \int_{[0,1]} \|R_\omega\| \cdot \|\sum_{i=1}^{n} a_i \, \varepsilon_i(\omega) \, \mu^i\| \, d\omega$$

$$\geq \int_{[0,1]} \left\langle R_\omega, \sum_{i=1}^{n} a_i \, \varepsilon_i(\omega) \, \mu^i \right\rangle d\omega$$

$$= \sum_{i=1}^{n} a_i \left\langle \int_{[0,1]} \varepsilon_i(\omega) \, R_\omega \, d\omega, \mu^i \right\rangle$$

$$= \sum_{i=1}^{n} a_i \left\langle \xi^i, \mu^i \right\rangle \geq \alpha .$$

q.e.d.

We still have to prove a lemma, which is folklore (as is its proof also).

<u>Lemma:</u> Let $\{\eta^i\}_{i=1}^{\infty}$ be a weak null sequence in a Banach space Y. For $\varepsilon > 0$ there is a finite sequence $a_1, \dots, a_n$ of positive scalars, $\sum_{i=1}^{n} a_i = 1$, such that for all choices of signs $\varepsilon_i = \pm 1$ the norm $\|\sum_{i=1}^{n} \varepsilon_i \, a_i \, \eta^i\|$ is less than ε.

<u>Proof:</u> By the Banach - Mazur and the Hahn - Banach theorem there is no loss in assuming $Y = C_{[0,1]}$, i.e. the η^i are continuous functions on $[0,1]$. It follows easily from Grothendieck's characterisation of weak convergence in $C_{[0,1]}$ (pointwise convergence + uniform boundedness) that η^i converges weakly to zero iff $|\eta^i|$ does so. It is wellknown that we may choose a convex combination $\sum_{i=1}^{n} a_i |\eta^i|$ of norm less than ε. But this clearly implies that for each $\varepsilon_i = \pm 1$

$$\|\sum_{i=1}^{n} \varepsilon_i \, a_i \, \eta^i\| < \varepsilon .$$

I want to note, that we may proof by the same techniques that $L^1(\mu;C(T))$ has the Dunford-Pettis property. But at this stage, the reader is advised to consult Bourgain's original paper, where using local results about Banach spaces, more general results are obtained. Here I only wanted to make the essence of Bourgain's proof more accesible.

Finally let us note that the problem, if the Dunford-Pettis property for a Banach space X implies the same property for $C(T;X)$ or $L^1(\mu;X)$ remains open.

R E F E R E N C E S

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