THE BANACH–SAKS PROPERTY IS NOT L^2-HEREDITARY

BY

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ABSTRACT

We construct a Banach space E, which has the Banach–Saks property and such that $L^2(E)$ does not have the Banach–Saks property. The construction is a somewhat tree-like modification of Baernstein's space.

1. Introduction

Recall that a Banach space E has the Banach–Saks property (abbreviated (BS)) if for every bounded sequence $\{x_n\}_{n=1}^{\infty}$ in E there is a subsequence $\{x'_n\}_{n=1}^{\infty}$ converging in Cesaro-mean (i.e., $\|n^{-1}\Sigma_1^n x'_k - x\| \to 0$ for some $x \in E$).

We construct an example of a Banach space E having (BS) such that $L^2_{[0,1]}(E)$ does not have (BS).

After constructing this example I have been informed that J. Bourgain has already constructed a Banach space with this property ([3], [6]). However, our construction is quite different and — as we believe — simpler and there might be some interest in the technique of the construction.

Our space E will be a somewhat tree-like modification of Baernstein's space [1] and is based on a very elementary probabilistic lemma. It will be convenient to use interpolation theory (following an idea of B. Beauzamy [2]) to avoid certain technical difficulties arising in Baernstein's construction. Let us note however that it is possible to construct our example following exactly the lines of [1].

2. An elementary probabilistic result

LEMMA 1. *Let $N \in \mathbf{N}$ and let $X_1, \cdots, X_N$ be independent random variables taking their values in the set $\{1, \cdots, N\}$ in a uniformly distributed way (i.e. for $1 \le i, j \le N$, $P\{X_i = j\} = N^{-1}$). Let*

$$Y(\omega) = \text{card}\{j : \textit{there is an } i, 1 \leq i \leq N, \textit{such that } X_i(\omega) = j\}.$$

Then

(1) $P\{Y \geq N/4\} \geq 1/4.$

PROOF.   For $0 \leq n \leq N$ let $Y_n(\omega) = \text{card}\{j : \text{there is an } i, 1 \leq i \leq n, \text{such that}$ $X_i(\omega) = j\}.$ Clearly

$$0 = Y_0 \leq Y_1 \leq \cdots \leq Y_N = Y.$$

Assume that (1) does not hold; then it fails for each Y_n. Fix $1 \leq n < N$ and let A be an atom in the σ-algebra generated by $X_1, \cdots, X_n$ such that, for $\omega \in A$, $Y_n(\omega) < N/4$. As X_{n+1} is independent of Y_n and the law of X_{n+1} is uniformly distributed,

$$P\{\omega \in A : Y_{n+1}(\omega) = Y_n(\omega) + 1\} \geq \tfrac{3}{4} \cdot P(A).$$

Summing over the atoms on which Y_n is less than $N/4$ we see that on a set of probability greater than or equal to $\tfrac{3}{4} \cdot \tfrac{3}{4}$ we have $Y_{n+1} = Y_n + 1$, hence

$$E(Y_{n+1}) \geq E(Y_n) + 9/16 \geq E(Y_n) + 1/2.$$

It follows that $E(Y) = \sum_{n=0}^{N-1} E(Y_{n+1} - Y_n) \geq N/2.$ On the other hand $Y \leq N$; hence if (1) does not hold, then

$$E(Y) \leq \tfrac{3}{4} \cdot N/4 + \tfrac{1}{4} \cdot N < N/2,$$

a contradiction. $\square$

3. Construction of the space

Let Φ_0 be the space of finite sequences and let $\{e_n\}_{n=1}^{\infty}$ be its natural base. We write n (uniquely) as $2^u + v$ $(0 \leq v \leq 2^u - 1)$, and associate to n the number $t(n) = v/2^u \in [0, 1[.$

A finite subset $\gamma = \{n_1, \cdots, n_l\}$ of $\mathbf{N}$, where $n_1 < n_2 < \cdots < n_l$, will be called *admissible*, if:

(1) $l \leq n_1$ (Baernstein's condition).

(2) Let p be defined by $2^{p-1} < n_1 \leq 2^p$. For every $0 \leq j < 2^p$ there is only one i so that $t(n_i) \in [j/2^p, (j+1)/2^p[.$

For example, for $u \in \mathbf{N}$ the set $\gamma = \{2^u, 2^u + 1, \cdots, 2^{u+1} - 1\}$ is admissible.

Let Δ be the set of admissible γ's. For $\gamma \in \Delta$ and $x = \{x_i\}_{i=1}^{\infty} \in \Phi_0$ define $\sigma(x, \gamma) = \sum_{i \in \gamma} |x_i|$ and $\|x\|_F = \sup\{\sigma(x, \gamma) : \gamma \in \Delta\}.$

Clearly $\| \ \|_F$ defines a norm on Φ_0, $\| \ \|_1 \geqq \| \ \|_F \geqq \| \ \|_{c_0}$. Let F denote the completion of $(\Phi_0, \| \ \|_F)$ and $i : l^1 \to F$ the canonical injection.

Recall [2] that an operator T from a Banach space A to a Banach space A_1 is said to have (BS) if any bounded sequence $\{x_n\}_{n=1}^\infty$ in A contains a subsequence $\{x_n'\}_{n=1}^\infty$ such that the Cesaro averages $n^{-1} \Sigma_1^n Tx_n'$ converge in A_1. It is shown in [2] that an operator T has (BS) iff the Lions–Peetre interpolation spaces $(A / \text{Ker } T, A_1)_{q,p}$, $1 < p < \infty$, $0 < q < 1$ (or equivalently the Davis–Figiel–Johnson–Pelczynski factorisation space [4]) have (BS).

PROPOSITION 2.   *The map $i : l^1 \to F$ has* (BS).

PROOF.   Let $\{x_n\}_{n=1}^\infty$ be a bounded sequence in l^1. We may suppose that x_n is bounded in norm by 1 and converges coordinatewise to zero and by a standard perturbation argument we may assume that

$$x_n = \sum_{i=r(n-1)+1}^{r(n)} \lambda_i^{(n)} e_i$$

where $r(n)$ is an increasing sequence.

We now choose inductively a subsequence $\{n_k\}_{k=1}^\infty$ and infinite subsets M_k of N. Let $M_0 = \mathbf{N}$ and $n_1 = 1$ and suppose M_{k-1} and n_k are defined. Let p be such that $2^{p-1} < r(n_k) \leqq 2^p$ and consider the partition of $[0, 1[$ into $[j/2^p, (j+1)/2^p[$, $j = 0, \cdots, 2^p - 1$. For $n \geqq n_k$ define

$$\mu_j^{(n)} = \max\{ |\lambda_i^{(n)}| : t(i) \in [j/2^p, (j+1)/2^p[\}.$$

Note that, for every n, $\Sigma_{j=0}^{2^p-1} \mu_j^{(n)} \leqq 1$ as the x_n are bounded by 1 in the l^1-norm. Find a subsequence $\bar{M}_k$ of $M_{k-1} \cap [n_k + 1, \cdots, \infty[$ such that, for every $j = 0, \cdots, 2^p - 1$, the sequence $\{\mu_j^{(n)}\}_{n \in \bar{M}_k}$ converges, to μ_j say. Clearly $\Sigma_{j=0}^{2^p-1} \mu_j \leqq 1$. Finally let M_k be the subset of $\bar{M}_k$ consisting of those n for which, for every $j = 0, \cdots, 2^p - 1$, $\mu_j^{(n)} \leqq \mu_j + 2^{-p}$ and define n_{k+1} by picking an arbitrary member of M_k. This completes the induction.

Note that for $\gamma \in \Delta$ and k such that $\inf(\gamma) \leqq r(n_k)$ and for every $l \in \mathbf{N}$

(2) $\sigma(x_{n_{k+1}} + \cdots + x_{n_{k+l}}, \gamma) \leqq 2.$

Indeed, γ may pick for every $j = 0, \cdots, 2^p - 1$ at most one index i with $t(i) \in [j/2^p, (j+1)/2^p[$ (p defined as above), hence the contribution of this index is at most $\mu_j + 2^{-p}$. Summing over j we obtain (2).

Hence for n, l as above and γ such that $r(n_{k-1}) < \inf(\gamma) \leqq r(n_k)$

$$\sigma(x_{n_1} + \cdots + x_{n_{k+l}}, \gamma) \leqq \sigma(x_{n_k}, \gamma) + \sigma(x_{n_{k+1}} + \cdots + x_{n_{k+l}}, \gamma) \leqq 3.$$

It follows readily that for every $K \in \mathbf{N}$

$$\|K^{-1}(x_{n_1} + \cdots + x_{n_k})\|_F \leq 3K^{-1}$$

from which the proposition follows. $\square$

Let $(E, \| \ \|_E)$ be the Davis–Figiel–Johnson–Pelczynski factorisation space of the injection $i : l^1 \to F$. As mentioned above, Proposition 2 implies that E has (BS). We may and do consider E as a space of sequences, containing l^1 and contained in F.

PROPOSITION 3. $L^2([0, 1]; E)$ *does not have* (BS).

PROOF.   Let $\{f_u\}_{u=1}^\infty$ be a sequence of independent random variables such that f_u takes the value e_{2^u+v} (i.e. the $2^u + v$th unit vector) with probability 2^{-u} ($v = 0, \cdots, 2^u - 1$).

$\{f_u\}_{u=1}^\infty$ is a bounded sequence in $L^\infty([0, 1]; l^1)$ hence in particular it is bounded in $L^2([0, 1]; E)$.

Also for almost every $\omega \in [0, 1]$, $\{f_u(\omega)\}_{u=1}^\infty$ converges weakly to zero in F. (Indeed, it is shown in the proof of Proposition 2 that for any sequence of unit vectors there is a subsequence converging strongly to zero in Cesaro-mean.) It follows from [4], that $\{f_u(\omega)\}_{u=1}^\infty$ converges weakly to zero in E. By [5], theorem IV.1.1, we conclude that $\{f_u\}_{u=1}^\infty$ tends weakly to zero in $L^2([0, 1]; E)$.

Now fix any subsequence $\{f_{u_k}\}_{k=1}^\infty$. As the norm of $L^2(E)$ is stronger than that of $L^1(F)$ the following assertion will prove Proposition 3:

(3) $K^{-1}\|f_{u_1} + \cdots + f_{u_K}\|_{L^1(F)} \geq 1/32, \qquad K = 2, 4, \cdots, 2^p, \cdots.$

Indeed, assume $K = 2^{p+1}$ and for $k = 2^p + 1, \cdots, 2^{p+1}$ define the random variables X_k with values in $\{0, \cdots, 2^p - 1\}$ by

$$X_k(\omega) = j \qquad \text{if } f_{u_k}(\omega) = e_n \quad \text{and} \quad t(n) \in [j/2^p, (j+1)/2^p[.$$

The random variables $\{X_k\}_{k=2^p+1}^{2^{p+1}}$ satisfy the assumptions of Lemma 1, hence on a set B of probability $\geq 1/4$ the sequence $\{X_k(\omega)\}_{k=2^p+1}^{2^{p+1}}$ hits at least 2^{p-2} different j's. Fix such an ω and find a set $\gamma = \{n_1, \cdots, n_l\} = \{u^{u_{k_1}} + v_1, \cdots, 2^{u_{k_l}} + v_l\}$ such that

   (i) $n_1 < \cdots < n_l$,
   (ii) $l = 2^{p-2}$,
   (iii) $f_{u_{k_i}}(\omega) = e_{2^{u_{k_i}+v_i}}$ for some $k_i \in \{2^p + 1, \cdots, 2^{p+1}\}$,
   (iv) the $v_i/2^{u_{k_i}}$ lie in different $[j/2^p, (j+1)/2^p[$.
Then it is easy to check that γ is admissible and therefore

$$K^{-1}\|f_{u_1}(\omega) + \cdots + f_{u_K}(\omega)\|_F \geqq 2^{-(p+1)}\sigma(f_{u_1}(\omega) + \cdots + f_{u_K}(\omega), \gamma)$$

$$\geqq 2^{-(p+1)} \cdot 2^{p-2} = 1/8.$$

Integrating over B we obtain (3). $\square$

REMARK.   The proof actually shows that for $1 < p < \infty$, $L^p(E)$ does not have (BS) and that $L^1(E)$ does not have the weak Banach–Saks property (called (BSR) in [2]).

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