Addendum to Integral Operators on L^p-spaces

by

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In this note we give solutions to two problems raised in [1].

1) We shall give a negative solution to problem 11.12 of [1]. ("Is the adjoint of every normal integral operator an integral operator?"):

Proposition 1.1: There is a normal integral operator $A = \mathrm{Int}(k)$ on $L^2_{(0,1)}$ such that the adjoint A^* is not integral.

The proof will be based on the following lemma, which is inspired by Olevskii's matrix ([3], lemma 1).

Denote by $\{h_n\}_{n=1}^{\infty}$ (resp. $\{w_n\}_{n=1}^{\infty}$) the normalized Haar-basis (resp. Walsh-basis) in $L^2_{(0,1)}$ as defined in [2], Vol. I, 1.a.4. (resp. [2], Vol. II, 1.g.3).

If $\{e_n\}_{n=1}^{N}$ and $\{f_n\}_{n=1}^{N}$ are two orthonormal sets in $L^2_{(0,1)}$ we shall call the operator $\sum_{n=1}^{N} f_n \otimes e_n$ "the operator that takes $\{e_n\}_{n=1}^{N}$ onto $\{f_n\}_{n=1}^{N}$".

Lemma 1.2: Fix integers n,m,p such that $1 \leq n < m$ and $0 \leq p \leq 2^m - 2^n$.

Let T be the operator that takes the Walsh functions $\{w_{2^n+j}\}_{j=1}^{2^n}$ onto the Haar functions $\{h_{2^m+p+j}\}_{j=1}^{2^n}$ and S be the operator that takes the Haar functions $\{h_{2^m+p+j}\}_{j=1}^{2^n}$ onto the Haar functions $\{h_{2^n+j}\}_{j=1}^{2^n}$. Let t (resp. s) denote the kernel function of T (resp. S). Then

(1) $S + T$ is normal. In fact, $(S+T)^* (S+T) = (S+T)(S+T)^*$ is the orthogonal projection onto the space spanned by

$\{h_{2^n+j}\}_{j=1}^{2^n} \cup \{h_{2^m+p+j}\}_{j=1}^{2^n}$.

(2) t is supported by $(p/2^m, (p+2^n)/2^m) \times (0,1)$ while s is supported by $(0,1) \times (p/2^m, (p+2^n)/2^m)$.

(3) For $y \in (0,1)$, $\|t(\cdot,y)\|_{L^1} = 2^{n-m/2}$.

(4) The operator $\mathrm{Int}(|s|)$ is a partial isometry and so, in particular, $\|\mathrm{Int}(|s|)\| = 1$.

Proof: (1): Make the following trivial (but crucial) observation. The orthonormal sets $\{w_{2^n+j}\}_{j=1}^{2^n}$ and $\{h_{2^n+j}\}_{j=1}^{2^n}$ span the same subspace of $L^2_{(0,1)}$. Indeed, this space consists of the functions constant on the intervals of the form $(\ell/2^{n+1}, (\ell+1)/2^{n+1})$ and such that the integral over every interval of the form $(\ell/2^n, (\ell+1)/2^n)$, vanishes.

Hence, $\{w_{2^n+j}\}_{j=1}^{2^n} \cup \{h_{2^m+p+j}\}_{j=1}^{2^n}$ and $\{h_{2^m+p+j}\}_{j=1}^{2^n} \cup \{h_{2^n+1}\}_{j=1}^{2^n}$ are orthonormal bases of the same subspace of $L^2_{(0,1)}$. Clearly $S+T$ takes the former to the latter and $(S+T)^*$ conversely, form which (1) follows.

(2): $$t(x,y) = \sum_{j=1}^{2^n} w_{2^n+j}(y) \cdot h_{2^m+p+j}(x) \quad \text{and}$$
$$s(x,y) = \sum_{j=1}^{2^n} h_{2^m+p+j}(y) \cdot h_{2^n+j}(x).$$

Note that h_{2^m+p+j} is supported by $((p+j-1)/2^m, (p+j)/2^m)$ from which the assertion for t follows. The argument for s is symmetric.

(3): $|w_{2^n+j}(y)| = 1$ for all $y \in (0,1)$ and $|h_{2^m+p+j}(x)| = 2^{m/2}$ on its support and 0 elsewhere. It follows from the proof of (2) that, for every $y \in (0,1)$, $|t(x,y)| = 2^{m/2}$ on an interval of the length 2^{n-m} and 0 elsewhere. This gives (3).

(4): Note that $|s|(x,y) = \sum_{j=1}^{2^n} |h_{2^m+p+j}(y)| \cdot |h_{2^n+j}(x)|$, as the summands are disjointly supported. Hence $\mathrm{Int}(|s|)$ takes the orthonormal set $\{|h_{2^m+p+j}|\}_{j=1}^{2^n}$ onto the orthonormal set $\{|h_{2^n+j}|\}_{j=1}^{2^n}$, which gives (4).

Proof of proposition 1.1: For $i = 1,2,\ldots$ let $m(i) = (i+1)^2$, $n(i) = m(i) - i$ and $p(i) = 2^{n(i)}$. Let $T_i = \mathrm{Int}(t_i)$ and $S_i = \mathrm{Int}(s_i)$ denote the corresponding integral operators defined by lemma 1.2.

As $n(i+1) > m(i)$ we see that, for $i \neq j$, the operators $(S_i + T_i)$ and $(S_i + T_i)^*$ and the operators $(S_j + T_j)$ and $(S_j + T_j)^*$ are all mutually orthogonal. Hence the operator

$$A = \sum_{i=1}^{\infty} 2^{-i} (S_i + T_i)$$

is normal. We claim that A is integral and its kernel is given by

$$k = \sum_{i=1}^{\infty} 2^{-i} (s_i + t_i).$$

Let

$$T = \sum_{i=1}^{\infty} 2^{-i} T_i \qquad t = \sum_{i=1}^{\infty} 2^{-i} t_i$$

$$S = \sum_{i=1}^{\infty} 2^{-i} S_i \qquad s = \sum_{i=1}^{\infty} 2^{-i} s_i.$$

We shall show that $\mathrm{Int}(t)$ and $\mathrm{Int}(s)$ are integral operators and equal T and S respectively. As regards S note that $\mathrm{Int}(s)$ is an absolutely bounded integral operator. Indeed

$$\mathrm{Int}(|s|) = \mathrm{Int}\left(\sum_{i=1}^{\infty} 2^{-i}|s_i|\right) = \sum_{i=1}^{\infty} 2^{-i}\,\mathrm{Int}(|s_i|)$$

is a welldefined positive operator. It follows easily that S equals $\mathrm{Int}(s)$.

As regards T, note that t is a Carleman kernel. Indeed as the t_i are supported by the disjoint strips $(2^{-i},2^{-i+1}) \times (0,1)$, it is clear that, for every $x \in (0,1)$, $t(x,.) \in L^2_{(0,1)}$. Hence the operator $Int(t)$ makes sense (a priori with values in $L^0_{(0,1)}$). But $T = Int(t)$ is true. Indeed, if P_i denotes the multiplication operator by $\chi_{(2^{-i},2^{-i+1})}$ in $L^2_{(0,1)}$, then

$$P_i oT = 2^{-i}T_i = Int(2^{-i}t_i) = P_i oInt(t),$$

which readily shows $T = Int(t)$. So we have shown that $A = Int(k)$ is a normal integral operator.

But, for every $y \in (0,1)$, $k(.,y)$ is not a member of $L^1_{(0,1)}$. Indeed, it follows from (3) that for $y \in (0,1)$ and $i \in \mathbb{N}$

$$\|t_i(.,y)\|_{L^1} = 2^{(i^2+1)/2}$$

hence $\|t(.,y)\|_{L^1} = \sum_{i=1}^{\infty} 2^{-i}.\ 2^{(i^2+1)/2} = \infty.$

As for every $y \in (0,1)$, $s(.,y)$ is in $L^1_{(0,1)}$ (actually, it is even in $L^\infty(0,1)$) we infer that for every $y \in (0,1)$

$$\|k(.,y)\|_{L^1} = \infty.$$

This of course implies that k^* is not a bounded kernel and by ([1], th. 7.5) that A^* is not integral.

□

Remark: Note the following curiosity: We have split A into a sum $T + S$ in such a way that S is absolutely bounded and T is Carleman. (Actually such a decomposition is possible for arbitrary integral operators on L^p, as is shown in [4]).

On the other hand, for absolutely bounded integral operators and normal Carleman operators it is true that the adjoint is integral too ([1], 10.6 and 11.11). The crux is, of course, that the operator T above is not normal.

2. We now give a solution to problem 16.3 of [1]. Let (X,μ) be a finite measure space, which is not purely atomic. We recall the problem: "If T is an operator on $L^2(\mu)$ and if there is a positive number ε such that UTU^* is an integral operator whenever U is a unitary operator on $L^2(\mu)$ and $\|U - Id\| < \varepsilon$, does it follow that T is a Hilbert-Schmidt operator?"

We shall show that the answer is yes (see prop. 2.6 below). It is an immediate consequence that the answer to the second half of problem 16.3 is no. ("If T is an operator on $L^2(\mu)$ and if T is not Hilbert-Schmidt, can the set of unitary operators that transform T into an integral operator have a non-empty interior?") Despite its length, the construction is elementary. Let us sketch the basic idea: If $T : L^2_{(0,1)} \to L^2_{(0,1)}$ is of the form

$$T = \sum_{n=1}^{\infty} \lambda_n (\cdot, g_n) . r_n$$

where $\{g_n\}_{n=1}^{\infty}$ is a disjointly supported orthonormal system in $L^2_{(0,1)}$ and $\{r_n\}_{n=1}^{\infty}$ are the Rademacher-functions, then T is integral iff $\Sigma|\lambda_n|^2 < \infty$, i.e. iff T is Hilbert-Schmidt. The following lemma, which is based on this remark (and which proves this remark), gives a criterion for an operator to be not integral. In the sequel we shall show how to construct for an operator $T : L^2(\mu) \to L^2(\mu)$, which is not Hilbert-Schmidt, a unitary transformation U arbitrarily close to the identify such that UTU^* satisfies the criterion of the lemma.

<u>Lemma 2.1.:</u> Let $T : L^2(\nu) \to L^2(\mu)$ be an operator such that there is a $\|\cdot\|_2$ - bounded sequence $\{g_n\}_{n=1}^{\infty}$ in $L^2(\nu)$ with

mutually disjoint supporting sets $\{B_n\}_{n=1}^{\infty}$ and a sequence of positive scalars λ_n which is not square summable such that $Tg_n/\lambda_n = r_n$ is bounded away from zero in measure, i.e. there is α such that for all n

$$\mu\{|r_n| \geq \alpha\} \geq \alpha.$$

Then T is not integral.

Proof: By hypothesis we may find a positive sequence $\{\rho_n\}_{n=1}^{\infty}$ in ℓ^2 such that $\Sigma\lambda_n\rho_n = \infty$.

Assume now T is integral, i.e. $T = \mathrm{Int}(k)$ for some kernel k. The function

$$g = \sum_{n=1}^{\infty} \rho_n \cdot |g_n|$$

is an element of $L^2(\nu)$, hence for μ-a.e. $x \in Y$

$$\int_Y |k(x,y)|\ g(y)\ d\nu(y) < \infty.$$

Then we can find $A \subseteq X$, $\mu(X \setminus A) < \alpha/2$, such that

$$\int_A \int_Y |k(x,y)|\ g(y)\ d\nu(y) < \infty,$$

hence

$$\sum_{n=1}^{\infty} \int_A \int_{B_n} |k(x,y)| g(y)\ d\nu(y) < \infty.$$

On the other hand, for every $n \in \mathbb{N}$ there is a subset A_n of A of measure $\mu(A_n) \geq \alpha/2$ such that, for $x \in A_n$, $|r_n(x)| \geq \alpha$. Hence for $x \in A_n$

$$\int_{B_n} |k(x,y)|\ g(y)\ d\nu(y) \geq$$
$$\rho_n \cdot \left|\int_Y k(x,y)\ \ g_n(y)\ d\nu(y)\right| \geq \rho_n \cdot \lambda_n \cdot \alpha.$$

Integrating we obtain

$$\int_A \int_{B_n} |k(x,y)|\ g(y)\ d\nu(y) \geq$$

$$\int_{A_n} \int_{B_n} |k(x,y)|\ g(y)\ d\nu(y) \geq \rho_n \cdot \lambda_n \cdot \alpha^2/2.$$

As $\Sigma\mu_n \rho_n = \infty$ we arrive at a contradiction. □

What we shall actually need is the following "perturbated" form of lemma 2.1, which readily follows from lemma 2.1.

<u>Lemma 2.1a:</u> Suppose in lemma 1 that g_n is such that there is a bounded sequence $\{\tilde{g}_n\}_{n=1}^{\infty}$ with mutually disjoint supporting sets $\{B_n\}_{n=1}^{\infty}$ and such that $\sum_{n=1}^{\infty} \|g_n - \tilde{g}_n\| < \infty$. Then the conclusion of lemma 1 still holds.

Our task therefore is, given a non-Hilbert-Schmidt operator $T : L^2(\mu) \to L^2(\mu)$, to fabricate a "small" unitary transform U for which UTU^* satisfies the conditions of Lemma 2.1a.

First we need a number of technical results.

<u>Lemma 2.2:</u> Let (X,μ) be a finite measure space which is not purely atomic and $T : L^2(\mu) \to L^2(\mu)$ an integral operator. Then there is a normalized sequence $\{g_n\}_{n=1}^{\infty}$ with mutually disjoint supporting sets $\{B_n\}_{n=1}^{\infty}$ such that (Tg_n, g_n) tends to zero.

<u>Proof:</u> We shall find $\{g_n\}_{n=1}^{\infty}$ such that $\|T^* g_n\| \to 0$. It is clear that there is no operator of finite rank R such that $T + R$ is onto. (This follows, e.g., from the $\langle 2,1\rangle$ -compactness of integral operators. (c.f.[1],§13). For sharper results on $\langle 2,1\rangle$ -compactness see [4]).

It follows that T^* restricted to any space of finite codimension in $L^2(\mu)$ is not an isomorphism into.

Let B_o be the purely atomic part of X and suppose $B_o,\ldots,B_{m-1}$ chosen so that for $C_n = X \setminus (B_o \cup \ldots \cup B_{n-1})$, T^* restricted to any subspace of finite codimension in $L^2(C_n, \mu|_{C_n})$ is not an isomorphism into. Find $\tilde{g}_n \in L^2(C_n, \mu|_{C_n})$, $\|\tilde{g}_n\| = 1$ such that $\|T^*\tilde{g}_n\| < n^{-1}$. Find a partition $D_1,\ldots,D_N$ of the set C_n such that for $i=1,\ldots,N$

$$\|\tilde{g}_n \cdot \chi_{D_i}\| < n^{-1}.$$

Then there is some i_o such that T^* restricted to any subspace of finite codimension in $L^2(D_{i_o}, \mu|_{D_{i_o}})$ is not an isomorphism into. Letting $B_n = C_n \setminus D_{i_o}$ and $g_n = \tilde{g}_n - \tilde{g}_n \chi_{D_{i_o}}$ we complete the induction and prove the lemma.

$\square$

Remark: Note that there are integral operators $T : L^2_{(0,1)} \to L^2_{(0,1)}$ which are isometric. Hence in the above lemma it is not possible, in general, to have $\|Tg_n\| \to 0$.

Lemma 2.3: Let $T : H \to H$ be an operator on a Hilbert space H which is not Hilbert-Schmidt. Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of closed subspaces such that codim $(M_n) \leq 1$on. Let

$$\lambda_n = \sup\{\|Tx\| : x \in M_n, \|x\| = 1\}.$$

Then $\sum_{n=1}^{\infty} \lambda_n^2 = \infty$.

Proof: The lemma is wellknown, and follows e.g. from the "Theorem on the minimax properties of eigenvalues" (c.f. [5], p.25), applied to T^*T, noting that for $\|x\| = 1$

$(T^*Tx,x) = \|Tx\|^2$ and noting that T is Hilbert-Schmidt iff T^*T is of trace class.

□

The following lemma is rather trivial but we prove it for the sake of completeness.

<u>Lemma 2.4:</u> Let (X,μ) be a probability space and $\{r_n\}_{n=1}^{\infty}$ a sequence of Rademacher functions on X, i.e. a sequence of independent Random variables taking the values ± 1 with probability 1/2. Let F be any function in $L^0(\mu)$. Then

$$\lim_{n\to\infty} \mu\{x : |F(x) + r_n(x)| > 1 - \varepsilon\} \geq 1/2.$$

<u>Proof:</u> It is well known that for measurable $E \subseteq X$

$$\lim_{n\to\infty} \mu\{x \in E : r_n(x) = +1\} = \mu(E)/2.$$

Let $D_+ = \{\lambda \in C : |\lambda+1| \leq 1-\varepsilon\}$
and $D_- = \{\lambda \in C : |\lambda-1| \leq 1-\varepsilon\}$

Then it is clear that

$$\mu\{x : |F(x)+r_n(x)| \leq 1-\varepsilon\} = \mu\{x : r_n(x) = 1, F(x) \in D_+\} + \mu\{x : r_n(x) = -1, F(x) \in D_-\}.$$

For $n\to\infty$ the latter expression tends to $\mu\{F^{-1}(D_+ \; D_-)\}/2$.
Therefore

$$\lim_{n\to\infty} \mu\{x : |F(x)+r_n(x)| > 1-\varepsilon\} = 1-\mu\{F^{-1}(D_+ \; D_-)\}/2 \geq 1/2.$$

□

<u>Lemma 2.5:</u> For $\varepsilon > 0$ there is $\alpha > 0$ such that for given complex numbers $\sigma,\tau,|\sigma| = |\tau| = 1$ the orthonormal vectors in $\mathbb{C}^4$

$$u_1 = ((1-\alpha^2)^{1/2} , 0 , 0 , \alpha)$$

$$u_2 = (-\alpha^2\tau/(1-\alpha^2)^{1/2}, (1-(2\alpha^2-\alpha^4)/(1-\alpha^2))^{1/2}, \sigma\alpha, \tau\alpha)$$

may be completed to an orthonormal basis u_1,u_2,u_3,u_4 of $\mathbb{C}^4$

in such a way that

$$\left(\sum_{i=1}^{4} \|u_i - e_i\|^2\right)^{1/2} < \varepsilon,$$

e_i denoting the unit vectors of C^4.

Hence the matrix $U = (u_1^t, u_2^t, u_3^t, u_4^t)^t$ is unitary and the distance of U to the identity is less than ε in Hilbert-Schmidt norm (and, a fortiori, in the operator norm).

Proof: We do not calculate u_3 and u_4 explicitly (this is an elementary but horrible calculation). Rather than dirtying our hands we prefer to give "qualitative" reasoning.

Apply to the vectors u_1, u_2, e_3, e_4 the Gram-Schmidt orthogonalisation procedure to obtain an orthonormal basis u_1, u_2, u_3, u_4. It is clear that for small α the vectors u_i will be close to e_i (independent of σ and τ). □

Proposition 2.6: Let (X, μ) be a finite measure space which is not purely atomic and $T : L^2(\mu) \to L^2(\mu)$ an operator which is not Hilbert-Schmidt. Then if $\varepsilon > 0$ there is a unitary transform $U : L^2(\mu) \to L^2(\mu)$, $\|U - Id\| < \varepsilon$ such that UTU^* is not integral.

Proof: First make the following observation: If $T = S + R$, where R is Hilbert-Schmidt then UTU^* is integral iff USU^* is integral. Hence we may perturbate T by an arbitrary Hilbert-Schmidt operator R and it will be sufficient to prove for the perturbated operator $T - R$ that $U(T - R)U^*$ is not integral. We may even apply (successively) countably many Hilbert-Schmidt perturbations, if we make sure that the sum of their Hilbert-Schmidt norms converges. This remark will be very convenient in the sequel.

If T is not integral, there is nothing to prove as we may take for U the identity. Hence we may assume that T is integral.

Find an orthonormal sequence $\{\tilde{g}_n\}_{n=1}^{\infty}$ in $L^2(\mu)$ with mutually disjoint supporting sets B_n, such that $\sum_{n=1}^{\infty} |(T\tilde{g}_n,\tilde{g}_n)|^2 < \infty$ (lemma 2.2). By passing to a Hilbert-Schmidt-perturbation we may assume $(T\tilde{g}_n,\tilde{g}_n) = 0$ for all $n \in \mathbb{N}$.

We may assume that there is a divisible subset $A \subseteq X$ of measure $\mu(A) = 1$, and we find a sequence of Rademacher function $\{\tilde{r}_n\}_{n=1}^{\infty}$ on A (see lemma 2.4).

We shall now construct a sequence of mutually orthogonal 5-dimensional subspaces M_k of $L^2(\mu)$ with orthonormal bases $(g_k, r_k, f_k, h_k, Tg_k)$ and unitary transformations $U_k : M_k \to M_k$.

Suppose they are defined up to $k - 1$ (if $k = 1$ suppose nothing). Let $N_k = (M_1, \dots, M_{k-1}, T^*M_1, \dots, T^*M_{k-1})^{\perp}$. Then N_k is a subspace of $L^2(\mu)$ of codimension $\leq 10(k-1)$. Choose $f_k \in N_k$, $\|f_k\| = 1$ such that

$$\|Tf_k\| > \sup\{\|Tf\| : f \in N_k, \|f\| = 1\} - 2^{-k}.$$

It follows from lemma 2.3 that $\|Tf_k\| = \lambda_k$ is not in ℓ^2. Let h_k be orthogonal to f_k, $\|h_k\| = 1$, such that $Tf_k = \lambda_k(\varphi_k f_k + \psi_k h_k)$, with $|\varphi_k|^2 + |\psi_k|^2 = 1$. Note that h_k is orthogonal to $M_1, \dots, M_{k-1}$.

As $\{\tilde{g}_n\}_{n=1}^{\infty}$ tends to zero weakly, the projection of $\tilde{g}_n$ as well as of $T\tilde{g}_n$ onto the subspace spanned by $\{M_1, \dots, M_{k-1}, f_k, h_k\}$ tends to zero (in norm), hence we may find an integer n_k, a $g_k \in L^2(\mu)$ such that $\|g_k - \tilde{g}_{n_k}\| < 2^{-k}$ and g_k as well as Tg_k are orthogonal on the space described above, (if necessary, we perturbe T by a one dimensional operator of norms less than 2^{-k}). Applying once more a one-dimensional perturbation of norm less than 2^{-k} we may in addition assume $(Tg_k, g_k) = 0$.

Let r_k be a normalized element of $L^2(\mu)$, orthogonal to $\{M_1, \ldots, M_{k-1}, f_k, h_k, g_k, Tg_k\}$ to be specified later on. M_k will be the space spanned by $\{g_k, r_k, h_k, f_k, Tg_k\}$. Let U_k be the unitary transformation on M_k obtained by letting the matrix U from lemma 2.5 act on the basis $\{g_k, r_k, h_k, f_k\}$ of M_k and mapping Tg_k onto itself.

The α in lemma 2.5 in chosen to be a fixed positive number (depending only on ε) and the complex numbers σ and τ are chosen $\sigma_k = \psi_k/|\psi_k|$ and $\tau_k = \varphi_k/|\varphi_k|$ (where $0/0 = 1$). We extend U_k to a unitary transform on $L^2(\mu)$ (still denoted U_k) by defining it to be the identity on the orthogonal complement of M_k.

Let us calculate $U_k T U_k^* g_k$:

$$U_k^* g_k = \sqrt{(1-\alpha^2)}\, g_k + \alpha . f_k$$

$$TU_k^* g_k = \sqrt{(1-\alpha^2)}\, Tg_k + \alpha . \lambda_k (\varphi_k f_k + \Psi_k h_k)$$

$$UTU_k^* = \sqrt{(1-\alpha^2)}\, Tg_k + F_k + \alpha^2 \lambda_k (\varphi_k \sigma_k + \Psi_k \tau_k) r_k .$$

Here F_k is some function in the span of g_k, f_k and h_k. Hence the function $G_k = \sqrt{(1-\alpha^2)}\, Tg_k + F_k$ does not depend on the choice of r_k (which is still free, provided $r_k \perp \{M_1, \ldots, M_{k-1}, f_k, h_k, g_k, Tg_k\}$). Note that $\rho_k = (\varphi_k \sigma_k + \Psi_k \tau_k) \geq 1$.

It follows from lemma 2.4 that for large n,

$$\mu\{|G_k+\alpha^2.\lambda_k.\rho_k.\tilde{r}_n| > \alpha^2.\lambda_k/2\} > 1/4.$$

As the projection of $\tilde{r}_n$ onto the space spanned by $\{M_1,\dots,M_{k-1},f_k,h_k,g_k,Tg_k\}$ tends to zero we may find r_k, orthogonal to this space and close to a suitable $\tilde{r}_{n_k}$, such that we still have

$$\mu\{|G_k+\alpha^2\lambda_k\rho_k r_k| > \alpha^2.\lambda_k/2\} > 1/4.$$

Recapitulating, we have shown that

$$\mu\{|U_k T U_k^* g_k/\lambda_k| > \alpha^2/2\} > 1/4.$$

Note that if U is any operator on $L^2(\mu)$ such that U and U^* coincide with U_k and U_k^* on M_k then we still have

$$\mu\{|UTU^* g_k/\lambda_k| > \alpha^2/2\} > 1/4.$$

The induction step is completed.

Finally it is clear how to define our unitary transform U: Let U equal U_k on M_k and be the identity on $\{M_1,M_2,\dots\}^{\perp}$. Clearly $\|U-Id\| < \varepsilon$.
Then $\{UTU^* g_k/\lambda_k\}_{k=1}^{\infty}$ does not tend to zero in measure and an appeal to lemma 2.1a shows that UTU^* is not an integral operator.

□

<u>Remark:</u> Let us note that it is easy to refine the above construction in such a way that UTU^* has the following stronger property: For every subset $A \subseteq X, \mu(A) > 0$, which does not consist only of atoms $P_A o UTU^*$ is not an integral operator.

REFERENCES

[1] P.R. Halmos, V.S. Sunder: Bounded Integral Operators on L^2 spaces, Erg. d. Math. 96, Springer 1978.

[2] J. Lindenstrauss, L. Tzafriri: Classical Banach spaces, vol. I and II, Erg. d. Math. 92 and 97, Springer 1977 and 1979.

[3] R.I. Ovsepian, A.Pelczyński: On the existence of a fundamental total and bounded biorthogonal sequences in every separable Banach space and related constructions on uniformly bounded orthonormal systems in L^2, Studia Math. 54(1975), p. 149-159.

[4] W. Schachermayer, L. Weiss: Almost Compactness and Decomposability of Integral Operators, to appear in Proc. A.M.S.

[5] I.C.Gohberg, M.G. Krein: Introduction to the Theory of Linear Nonselfadjoint operators, Translations of Math. Monographs Vol. 18, A.M.S., Rhode Island 1969.