THE CLASS OF BANACH SPACES, WHICH DO NOT HAVE c_o AS A SPREADING MODEL, IS NOT L^2-HEREDITARY

by

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1. INTRODUCTION: In [4] the problem was raised wether the fact, that a Banach space E does not have c_o as a spreading model, implies that $L^2([0,1]; E)$ has the same property. It was conjectured that the answer is no, as the property of not having c_o as a spreading model is somewhat dual to the Banach-Saks property (see [2]) and for this latter property J. Bourgain has constructed a counterexample ([3]).

The present author has constructed independently of J. Bourgain another space E with the Banach-Saks property and $L^2(E)$ failing it ([6]) and it turns out that the dual E' gives a counterexample to the problem raised in the title.

2. THE EXAMPLE: Let $\gamma = \{n_1, n_2, \ldots, n_k\}$ an increasing finite sequence of natural numbers. Write $n_i = 2^{u_i} + v_i$ where this expression is unique, if we require that $0 \leq v_i < 2^{u_i}$. As in [6] we associate to every n_i the real number $t(n_i) = v_i/2^{u_i} \in [0,1[$ and call γ admissible if

(1) $k \leq n_1$

(2) For every $0 \leq j < 2^{u_1+1}$ there is only one i such that $t(n_i) \in [j/2^{u_1+1}, (j+1)/2^{u_1+1}[$.

For an admissible $\gamma = (n_1, \ldots, n_k)$ and $x \in \mathbb{R}^{(\mathbb{N})}$, the space of finite sequences we define

$$\|x\|_\gamma = \sum_{i=1}^{k} |x_{n_i}|.$$

For our purposes it will this time be convenient, not to use interpolation but to follow Baernstein's original definition ([1]): For $x \in \mathbb{R}^{(\mathbb{N})}$ define

$$\|x\|_E = \sup \{ (\sum_{\ell=1}^{\infty} \|x\|_{\gamma_\ell}^2)^{1/2} \}$$

where the sup is taken over all increasing sequences $\{\gamma_\ell\}_{\ell=1}^{\infty}$ of admissible sets (i.e. the last member of γ_ℓ is smaller than the first member of $\gamma_{\ell+1}$).

Let $(E, \|\ \|_E)$ be the completion of $\mathbb{R}^{(\mathbb{N})}$ with respect to this norm. In an analoqous way as in [6] one shows that E has the Banach-Saks property.

PROPOSITION 1:
E' does not have c_o as a spreading model.

PROOF: As E does not have ℓ^1 as spreading model ([2]), no quotient of E' has c_o as spreading model ([5]), hence in particular E' does not have c_o as spreading model.

□

To show that $L^2(E')$ does have c_o as spreading model we need a trivial probabilistic lemma, whose proof is left to the reader.

LEMMA: Let $k \in \mathbb{N}$ and $\varepsilon > 0$; there is $N(k,\varepsilon)$ such that for $M > N(k,\varepsilon)$ and for independent randomvariables $X_1,\ldots,X_k$ taking their values in $\{1,\ldots,M\}$ in a uniformly distributed way, we have

$$P \left\{ \begin{array}{l} \omega : \text{there is } 1 \le i < j \le k \text{ with} \\ X_i(\omega) = X_j(\omega) \end{array} \right\} < \varepsilon$$

□

PROPOSITION 2: $L^2_{(0,1)}(E)$ has c_o isometrically as spreading model.

PROOF: Similarly as in [6] we let $\{\vec{f}_u\}_{u=1}^{\infty}$ be an independent sequence in $L^2(E')$ such that $\vec{f}_u$ takes the value e_{2^u+v} with probability 2^{-u} (for $v=0,\ldots,2^u-1$). This time the e_{2^u+v} are unit-vectors in E'.

Clearly $\|\vec{f}_u\|_{L^2(E')} = 1$ and for every sequence $u_1 < u_2 < \ldots < u_k$ and $\varepsilon_i = \pm 1$

$$\left\| \sum_{i=1}^{k} \varepsilon_i \vec{f}_{u_i} \right\|_{L^2(E')} \geq 1$$

Hence the following claim will prove the proposition.

CLAIM: For every $k \in \mathbb{N}$

$$\limsup_{u\to\infty} \left\{ \left\| \sum_{i=1}^{k} \varepsilon_i \vec{f}_{u_i} \right\| : \begin{array}{l} u \leq u_1 < \ldots < u_k \\ \varepsilon_i = \pm 1 \end{array} \right\} = 1$$

To prove the claim fix k and $\varepsilon > 0$ and let u be such that $2^u > \max(k, N(k,\varepsilon))$, where the $N(k,\varepsilon)$ is defined in the preceeding lemma. Now fix $u \leq u_1 < u_2 < \ldots < u_k$ and a sequence of signs $\varepsilon_1, \ldots, \varepsilon_k$.

To apply the above lemma let $X_1, \ldots, X_k$ be the random variables with values in $\{1, \ldots, 2^{u_1+1}\}$ defined by

$$X_i(\omega) = m \quad \text{if} \quad \vec{f}_{u_i}(\omega) = e_{2^{u_i}+v} \quad \text{and}$$
$$t(2^{u_i}+v) = v/2^{u_i} \in [(m-1)/2^{u_1+1};\ m/2^{u_1+1}[$$

It follows form the above lemma and the definition of admissible sets γ that there is a subset $A \subseteq [0,1]$ of measure greater than $1-\varepsilon$ such that for $\omega \in A$ the set $\gamma_\omega = \{n_1, \ldots, n_k\}$ corresponding to the indices of the unit vectors $\{\vec{f}_{u_1}(\omega), \ldots, \vec{f}_{u_k}(\omega)\}$ is admissible. Hence for $\omega \in A$ we have

$$\left\| \sum_{i=1}^{k} \varepsilon_i \vec{f}_{u_i}(\omega) \right\|_{E'} = \sup \left\{ \left\langle \sum_{i=1}^{k} \varepsilon_i \vec{f}_{u_i}(\omega), x \right\rangle : \|x\|_E \leq 1 \right\}$$

$$\leq \sup \left\{ \left\langle \sum_{i=1}^{k} \varepsilon_i \vec{f}_{u_i}(\omega), x \right\rangle : \|x\|_{\gamma_\omega} \leq 1 \right\}$$

$$= 1.$$

Integrating we obtain

$$\|\sum_{i=1}^{k} \varepsilon_i \vec{f}_{u_i}\|^2_{L^2(E')} \leq \int_A \|\sum_{i=1}^{k} \vec{f}_{u_i}(\omega)\|^2_{E'}\, d\omega + \int_{[0,1]\setminus A} (\sum_{i=1}^{k} \|\vec{f}_{u_i}\|_{E'})^2\, d\omega$$

$$\leq 1 + k^2\varepsilon.$$

This proves the claim and therefore proposition 2.

□

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