Norm attaining Operators on some Classical Banach Spaces

Walter Schachermayer, Linz.

We construct an operator from $L^1[0,1]$ to $C[0,1]$ which may not be approximated by norm attaining operators with respect to the operator norm. This solves a question raised by J. Johnson and J. Wolfe in [1] (see also [3], [7]) and furnishes the first example of a pair of classical Banach spaces such that the norm attaining operators are not dense. $C[0,1]$ is the first example of a classical Banach space which does not have property B (c.f. [5]).

On the other hand, we show that a weakly compact operator from $C(K)$ into a Banach space X may be approximated in norm by norm attaining operators. This shows in particular that the norm attaining operators are dense in $B(C(K),L^1[0,1])$ and $B(C(K),\ell^2)$, thus solving two other questions raised by Johnson and Wolfe. of [4].

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Let X,Y be Banach spaces, $B(X,Y)$ the Banach space of bounded linear operators A from X to Y and $NA(X,Y)$ the subset of norm attaining operators A, i.e. there is $x \in X$, $\|x\| = 1$ with $\|Ax\| = \|A\|$. The question of density of $NA(X,Y)$ in $B(X,Y)$ was studied in [1],[3],[4],[5],[7] and [8]. As regards classical Banach spaces it was shown in [3] that $NA(L^1(\mu),L^1(\nu))$ is dense in $B(L^1(\mu),L^1(\nu))$ and it was shown in [4] that $NA(C(K),C(L))$ is dense in $B(C(K), C(K))$, but the cases treated in the present paper were left open.

After a preparatory part 1 we shall construct in part 2 an example from which the subsequent result follows

Theorem A: $NA(L^1[0,1], C[0,1])$ is not dense in $B(L^1[0,1], C[0,1])$.

Finally, in part 3, which may be read independently of part 1 and 2 we prove

Theorem B: Let K be a compact Hausdorff space, X a Banach space and $A : C(K) \to X$ a weakly compact operator. Then A may be approximated in the operator norm by elements of $NA(C(K),X)$.

Hence, if X does not contain c_o isomorphically (in particular if $X = L^1(\mu)$ or $X = \ell^2$), $NA(C(K),X)$ is dense in $B(C(K),X)$.

Our notation will be standard. L^1 will denote the usual Lebesgue space over $[0,1]$ equipped with Lebesgue measure, T the compact space $[0,1]$ and C the Banach space of continuous functions on T. Our Banach spaces will be real.

1. Denote $\sigma^*C(T,L^\infty)$ the Banach space of functions $F : T \to L^\infty$ which are continuous with respect to the σ^*-topology of L^∞, equipped with the norm

$$\|F\| = \sup \{\|F_t\|_{L^\infty} : t \in T\}.$$

We shall need the following trivial but useful representation:

1.1. Proposition:
The Banach spaces $B(L^1,C)$ and $\sigma^*C(T,L^\infty)$ are naturally isometrically isomorphic. The correspondence between $A \in B(L^1,C)$ and $F \in \sigma^*C(T,L^\infty)$ is given by $F_t = A^*(\delta_t)$, where A^* denotes the adjoint of A and δ_t the Dirac measure at T.

A is norm attaining iff there is $t \in T$ such that the function $|F_t|$ equals $\|F\|$ on a set of positive measure.

□

1.2. Hence the problem of whether we may approximate $A \in B(L^1,C)$, $\|A\| = 1$, by $\tilde{A} \in NA(L^1,C)$ (with $\|\tilde{A}\| = 1$ say) up to ε is equivalent to the following question: Given $F \in \sigma^*C(T,L^\infty)$, $\|F\| = 1$, does there exist $\tilde{F} \in \sigma^*C(T,L^\infty)$, $\|\tilde{F}\| = 1$ and $\|F - \tilde{F}\| < \varepsilon$ such that $\tilde{F}$ hits at some point t a norm attaining element of the unit sphere of L^∞ (i.e. $|F_t|$ equals 1 on a set of positive measure).

There is a natural guess for the construction of such a $\tilde{F}$ for a given F with $\|F\| = 1$: By changing sign if necessary we may suppose that there is $t_o \in T$ such that F_{t_o} is greater than $1-\varepsilon$ on a set of positive measure. Then

$$\tilde{F}_t = (F_t + \varepsilon) \wedge 1$$

defines a function $\tilde{F}$ into the unit ball of L^∞ with $\|F_t - \tilde{F}_t\| \leq \varepsilon$ for $t \in T$ and such that $\tilde{F}_{t_o}$ equals 1 on a set of positive measure. The crux is, however, that it may happen that $\tilde{F}$ is not σ^*-continuous. Suppose for example that there is a sequence $(s_n)_{n=1}^\infty$ in T tending to s_o

such that F_{s_n} equals the n-th Rademacher function R_n (and therefore by σ^*-continuity $F_{s_o} = 0$). Then for the $\tilde{F}$ constructed above $(\tilde{F}_{s_n})$ is a sequence of independent random variables taking the values 1 and $-1+\varepsilon$ with probability 1/2 and so converges σ^* to the constant function $\varepsilon/2$. However, $\tilde{F}_{s_o}$ equals the constant function ε, i.e. $\tilde{F}$ is not σ^*-continuous. It will be this phenomenon on which our example is essentially based.

But it will be instructive to investigate this idea a little further in order to get some information and motivation, how to construct our example. On the unit ball $\mathbf{O}L^\infty$ of L^∞ we also have the topology of the L^1-norm, which is finer than the σ^*-topology and for which $\mathbf{O}L^\infty$ is a complete metric space. Suppose that $F \in \sigma^*C(T,L^\infty)$, $\|F\| = 1$ is continuous with respect to the L^1-topology on $\mathbf{O}L^\infty$. Then it is easy to check that $\tilde{F} = (F + \varepsilon) \wedge 1$ is also continuous with respect to the L^1-norm, so a fortiori $F \in \sigma^*C(T,L^\infty)$.
Hence $A \in B(L^1,C)$ is approximable by norm attaining operators if the corresponding F is continuous with respect to the L^1-norm on L^∞.
Let us still note that this latter class of operators is strictly larger than the class of Riesz-representable operators from L^1 to C (for definition see [2]). Indeed, it is easy to check that A is Riesz-representable iff for $t_n \to t_o$ we have that F_{t_n} converges almost everywhere to F_{t_o}. Note that for bounded sequences in L^∞ the almost sure convergence is strictly stronger than the convergence with respect to the L^1-norm. Hence the class of operators A such that the corresponding F is $\|.\|_1$-continuous includes the class of Riesz-representable operators, and - after the above remarks - it is not difficult to give examples showing that the inclusion is strict. Hence the above observation gives, for the special situation of operators from L^1 to C, a slightly more general result than ([7], th. 2).

To finish the introductory part of the paper, we shall show that for arbitrary $F \in \sigma^*C(T,L^\infty)$ there are "many" points of T at which F is $\|.\|_1$-continuous. This shows

the limitations for the subsequent construction of a counter-example: The above described phenomenon can only occur at "few" points.

1.3. Proposition: Let $F \in \sigma^* C(T, L^\infty)$. There is a dense G_δ-set $M \subseteq T$ such that, at every point of M, F is continuous with respect to the L^1-norm.

Proof: Let

$$K = \{f \in L^1 : \|f\|_\infty \leq 1\},$$

which is compact if equipped with the weak topology of L^1. Definie

$$\begin{aligned} G : T \times K &\longrightarrow \mathbb{R} \\ (t,f) &\longrightarrow \langle F_t, f \rangle \end{aligned}$$

where $\langle .,. \rangle$ denotes the scalar product between L^∞ and L^1. Clearly G is separably continuous. It follows from ([6], th. 1.2) that there is a dense G_δ-set M in T such that, for $t \in M$ and $f \in K$, G is jointly continuous at (t,f).

In view of the compactness of K this implies that if $(t_n)_{n=1}^\infty \subseteq T$ tends to some $t \in M$, then F_{t_n} tends to F_t uniformly on K, i.e. F_{t_n} tends to F_t in the L^1-norm. □

2. We now proceed to the proof of theorem A. It will be convenient to introduce an ad-hoc-concept.

2.1. Definition: Let $F \in \sigma^* C(T, L^\infty)$, $\|F\| \leq 1$ and S be a subset of T. For $r \in T$, F is called "hopeless at r" (resp. "S-hopeless at r") if

a) $F_r = \sum_{i=1}^n (1-\lambda_i)\chi_{A_i}$

for some partition $(A_i)_{i=1}^n$ of Ω and scalars $0 \leq \lambda_i \leq 1$, and

b) for every A contained in some A_i and $\varepsilon > 0$ there is $s \in T$ (resp. $s \in S$) with $|r-s| < \varepsilon$ and $\mu\{\omega \in A : |F_s(\omega)| \leq \varepsilon\} \geq \mu(A).\lambda_i/2$

2.2. Proposition:
Let $F \in \sigma^* C(T,L^\infty)$, $\|F\| \leq 1$ be hopeless at some $r \in T$ and $\tilde{F} \in \sigma^* C(T,L^\infty)$ with $\|\tilde{F}\| \leq 1$ and $\|F - \tilde{F}\| \leq 1/2$. Then for $i = 1,\ldots,n$

$$\text{ess sup } \{\tilde{F}_r(\omega) : \omega \in A_i\} \leq 1 - \lambda_i/4 \qquad (1)$$

In particular if F satisfies in addition for every $n \in \mathbb{N}$

$$\mu\{F_r > 1 - 2^{-n}\} \leq \alpha_n \qquad (2)$$

then $\tilde{F}_r$ satisfies for every $n \in \mathbb{N}$

$$\mu\{\tilde{F}_r > 1 - 2^{-n-2}\} \leq \alpha_n \qquad (3).$$

Remark:
The intuitive meaning of 2.2 and the reason for the concept of "hopelessness": If F is hopeless at r and the distribution of F_r obeys the inequalities (2) and if $\tilde{F}$ is a perturbation of F, then the distribution of $\tilde{F}_r$ obeys (3) and $\tilde{F}_r$ therefore is a "hopeless candidate" for being a norm attaining element of the unit sphere of L^∞.

Proof:
For $A \subseteq A_i$ it follows immediately from definition 2.1, the σ^*-continuity of $\tilde{F}$ and the assumption $\|\tilde{F}\| \leq 1$ that

$$\int_A \tilde{F}_r(\omega)\, d\mu(\omega) \leq (\mu(A) \cdot \lambda_i/2) \cdot 1/2 + \mu(A)(1-\lambda_i/2).1$$
$$\leq \mu(A).(1-\lambda_i/4).$$

This implies (1). The second part of the proposition is an immediate consequence. □

In the next proposition we shall fix the program for our construction.

2.3. Proposition: There exists $F \in \sigma^* C(T,L^\infty)$ and a dense subset $D \subseteq T$ such that
(1) $\|F\| = 1$ and $F_t \geq 0$ for $t \in T$.
(2) For $t \in T$ and $n \in \mathbb{N}$
$$\mu\{F_t \geq 1 - 2^{-n}\} \leq 2^{-n+2}.$$
(3) F is hopeless at every $r \in D$.

Let us admit 2.3 for the moment and show how theorem A follows:

Proof of theorem A: Let $F \in \sigma^* C(T, L^\infty)$ be given as in 2.3 and suppose there is some norm attaining $\overline{F} \in \sigma^* C(T, L^\infty)$ with $\|F\| - \|\overline{F}\| \leq 1/4$. Let $\tilde{F} = \overline{F}/\|\overline{F}\|$; then $\|\tilde{F}\| = 1$ and $\|F - \tilde{F}\| \leq 1/2$.
For $\alpha \in]0,1[$ define the function

$$N_\alpha : L^\infty \longrightarrow \mathbb{R}$$
$$f \longrightarrow \sup \{\int_A f d\mu : \mu(A) = \alpha\},$$

which is σ^*-lower semicontinuous. Applying 2.2 and assumption (2) an easy computation shows that

$$N_{2^{-n}}(\tilde{F}_r) \leq 2^{-n}(1-2^{-n-6})$$

for $r \in D$ and so by the σ^* lower semicontinuity of N_α for $r \in T$. This implies readily that, for no $r \in T$, $\tilde{F}_r$ equals 1 on a set of positive measure. As $\tilde{F}_r \geq -1/2$, for $r \in T$, we see that no $\tilde{F}_r$ is a norm attaining element of the unit sphere of L^∞. In view of proposition 1.1 this proves theorem A. □

We still have to prove 2.3. and this will involve a rather laborious construction. For preparation we need an elementary probabilistic lemma whose proof we include for the sake of completeness.

2.4. Lemma:
Let $(g_n)_{n=1}^\infty$ be an independent sequence of random variables so that $\|g_n\|_\infty \leq 1$ and $E(g_n) = 0$. Let $(h_n)_{n=1}^\infty$ be a sequence of random variables such that $\|h_n\|_\infty \leq 1$ and each h_n is independent of the sequence $(g_k)_{k=1}^\infty$. Then $(g_n h_n)_{n=1}^\infty$ tends to zero in the σ^*-topology of L^∞.

Proof: Suppose not. After passing to a subsequence and changing signs, if necessary, we can find $A \in \Sigma$ and $\alpha > 0$ such that, for every n,

$$\int_A g_n h_n \, d\mu \geq \alpha.$$

By may be passing to a subsequence once more we may assume (by Komlos' theorem), that the Cesaro means

$$s_n = n^{-1} (g_1 h_1 + \ldots + g_n h_n)$$

converge almost surely to a random variable s. Of course

$$\int_A s \, d\mu \geq \alpha .$$

Hence there is $\beta > 0$ such that the set $B = \{s > \beta\}$ has positive measure. B is independent of every g_n hence for every n

$$\int_B g_n h_n \, d\mu = E(g_n) . E(h_n \cdot \chi_B) = 0,$$

which implies that

$$\int_B s \, d\mu = 0,$$

the required contradiction. □

Proof of proposition 2.3.:

Define for $k = 0,1,2,..$

$$S_k = \{q \in [0,1[\ : \text{there is } \ell \leq k \text{ s.t. } q = \bar{n}_1^{-1} + .. + \bar{n}_\ell^{-1}\} \cup \{0\},$$

where n_i are supposed to be natural numbers. It is plain to check that

- $S_0 \subsetneq S_1 \subsetneq S_2 \subsetneq \ldots$
- S_k is compact.
- $S_k' = S_{k-1}$, i.e. S_{k-1} consists precisely of the accumulation points of S_k.
- $\bigcup_{k=0}^{\infty} S_k = \mathbb{Q} \cap [0,1[$.

We shall also prepare for $k \in \mathbb{N}$ partitions $A^k = (A_i^k)_{i=1}^{2^k}$ of Ω into sets of μ-measure 2^{-k} and for $q \in \mathbb{Q} \cap [0,1[$, $\lambda \in \mathbb{Q} \cap]0,1]$ and $n \in \mathbb{N}$ a "biased Rademacher function" $BR_n^{q,\lambda}$ such that $BR_n^{q,\lambda}$ equals $-(1-\lambda)$ with probability $\lambda/(2-\lambda)$ and equals $\lambda/2$ with probability $(2-2\lambda)/(2-\lambda)$.

The sequence of partitions $(A^k)_{k=1}^{\infty}$ and the family of random variables $\{B_n^{q,\lambda} : q \in \mathbb{Q} \cap [0,1[,\ \lambda \in \mathbb{Q} \cap]0,1],\ n \in \mathbb{N}\}$ are supposed to be all independent.

Now we shall construct inductively a sequence $(F^k)_{k=1}^{\infty}$ in $\sigma^* C(T,L^\infty)$ satisfying the following

Induction Hypotheses:

(a) $\|F^k\| = \sup\{\|F_s^k\|_{L^\infty} : s \in S_k \setminus S_{k-1}\} = 1 - 2^{-k-1}$ and $F_t^k \geq 0$ for $t \in T$.

(b) For $t \in T$ and $0 \leq n \leq k-1$

$$\mu\{F_t^k \geq 1-2^{-n}\} \leq 2^{-n+2} - 2^{-k}.$$

(c) F^k is S_k-hopeless at every $r \in S_{k-1}$ and, for $\ell < k$, $F_r^k = F_r^\ell$ for all $r \in S_\ell$.

(d) For all $t \in \mathbb{Q} \cap [0,1[$, F_t^k is a simple function with rational coefficients and measurable with respect to the σ-algebra generated by the partitions $\{A^\ell,\ \ell = 1,..,k\}$ and the random variables $\{B_n^{q,\lambda} : q \in S_{k-1}, \lambda \in \mathbb{Q} \cap]0,1], n \in \mathbb{N}\}$.

(e) F_t^k is locally Lipschitzian with respect to $\|.\|_\infty$ on $T \setminus S_{k-1}$, i.e. for $t \in T \setminus S_{k-1}$ there is a constant $L^k(t)$ such that for $t' \in T$, $|t-t'| < L^k(t)^{-1}$ we have

$$\|F_t^k - F_{t'}^k\|_\infty \leq L^k(t) \cdot |t-t'|.$$

(f) $\mu\{F_t^k \neq F_t^{k-1}\} \leq 2^{-k}$ $\qquad \forall t \in T.$

To start the induction define F^0 by letting F_t^0 be identically the constant function 1/2. For the convenience of the reader and in order to make the idea of construction clearer we carry out the first induction step, which is technically simpler, before passing to the general one.

Step 1: A: Definition of F^1 on S_0: Let $F_0^1 = F_0^0$.

B: Definition of F^1 on $S_1 \setminus S_0$: We define

$$F_s^1 = F_s^0 + BR_n^{0,1/2} \cdot \chi_{A_1^1}$$

if $s \in S_1 \setminus S_o$ is of the form $s = n^{-1}$ for odd n and

$$F_s^1 = F_s^o + BR_n^{O,1/2} \cdot \chi_{A_2^1}$$

if $s \in S_1 \setminus S_o$ is of the form $s = n^{-1}$ for even n.

C: Definition of F^1 on $T \setminus S_1$: For every $s \in S_1 \setminus S_o$ we define $\varepsilon(s) = \text{dist}\,(s, S_1 \setminus \{s\})/2$, which is strictly positive and rational. Let $t \in T \setminus S_1$; if there is a (necessarily unique) $s \in S_1 \setminus S_o$ such that $|t-s| < \varepsilon(s)$ then we define

$$F_t^1 = F_t^o + (1 - |t-s|/\varepsilon(s))[F_s^1 - F_s^o].$$

If there is no such s we define simply

$$F_t^1 = F_t^o.$$

It is plain to check that (a),(b),(d),(e) and (f) are satisfied for F^1. Let us show (c): F^1 is S_1-hopeless at O. Write F_O^1 as

$$F_o^1 = \sum_{i=1}^{2} (1-1/2) \cdot \chi_{A_i^1} .$$

Fix $A \subsetneq A_i^1$, say $A \subsetneq A_1^1$. Then

$$\lim_{n\to\infty} \mu\{A \cap \{BR_n^{O,1/2} = -1/2\}\} = \mu(A).\mu\{BR_n^{O,1/2} = -1/2\}$$
$$= \mu(A) \cdot 1/3.$$

Hence if $s = n^{-1}$ where n is large and odd

$$\mu\{\omega \in A : F_s^1 = O\} > 1/4 . \mu(A),$$

which readily shows that F^1 is S_1-hopeless at O.

Finally we have to show that F^1 is σ^*-continuous. In view of (e) we have to check the σ^*-continuity only at O. Let $(t_j)_{j=1}^{\infty} \subsetneq T$ tend to O. By passing to a subsequence we may suppose that for $s \in S_1 \setminus S_o$, there is at most one t_j which lies in $]s - \varepsilon(s), s + \varepsilon(s)[$. Then the sequence $F_{t_j}^1 - F_o^1$ is of the form described in lemma 2.4: It is the product of some $B_{n_i}^{O,1/2}$ with a scalar multiple (less or equal to one) of either $\chi_{A_1^1}$ or $\chi_{A_2^1}$; so we may conclude that $F_{t_j}^1 - F_o^1$ tends σ^* to zero, i.e. F^1 is σ^*-continuous at $t = O$.

Step k: Suppose $F^1,\dots,F^{k-1}$ and $L^{k-1}(t)$ defined.

A: Definition of F^k on S_{k-1} : Let $F^k_r = F^{k-1}_r$ for $r \in S_{k-1}$.

B: Definition of F^k on $S_k \setminus S_{k-1}$: First fix $r \in S_{k-1}\setminus S_{k-2}$ and write

$$F^{k-1}_r = \sum_{i=1}^{m(r)} (1-\lambda^r_i)\,\chi_{B^r_i}$$

where $\{B^r_i\}_{i=1}^{m(r)}$ is a partition of Ω, belonging to the σ-algebra generated by $\{A^\ell : \ell < k\}$ and $\{B^{q,\lambda}_n : q \in S_{k-2}\}$ and where the λ^r_i are rational coefficients with $2^{-k} \leq \lambda^r_i \leq 1$. Now find a strictly positive rational number $\delta(r)$, such that

$$\delta(r) \leq \min\{\mathrm{dist}\,(r, S_{k-1}\setminus\{r\}), L^{k-1}(r)^{-1}.\min\{\lambda^r_i, 1-\lambda^r_i : \lambda^r_i<1\}\}.$$

Note that the intervals $\{]r, r+\delta(r)[: r \in S_{k-1}\setminus S_{k-2}\}$ are mutually disjoint.

Now fix $s \in S_k\setminus S_{k-1}$: if s is of the form $s = r+n^{-1}$ for some $r \in S_{k-1}\setminus S_{k-2}$ with $n^{-1} < \delta(r)$ (in which case r is unique), define

$$F^k_s = F^{k-1}_s + [1-|r-s|/\delta(r)] \cdot \chi_{B^r_i} \cdot \chi_{A^k_j} \cdot BR^{r,\lambda^r_i}_n$$

if n equals $i + (j-1)\,m(r)$ modulo $2^k m(r)$, where $1 \leq j \leq 2^k$ and $1 \leq i \leq m(r)$.
If $s \in S_k\setminus S_{k-1}$ is not of the above form, define simply $F^k_s = F^{k-1}_s$.

C: Definition of F^k on $T\setminus S_k$: First find for each $s \in S_k\setminus S_{k-1}$ a strictly positive number $\varepsilon(s)$ which is smaller than $\mathrm{dist}(s, S_k\{s\})/2$ and $L^{k-1}(s)^{-1}$ multiplied by the minimum of the λ's and $(1-\lambda)$'s, where λ runs through the (finitely many) values different form zero, which F^{k-1}_s assumes.
Note again that the intervals $\{]s-\varepsilon(s), s+\varepsilon(s)[: s \in S_k\setminus S_{k-1}\}$ are mutually disjoint.

Now fix $t \in T\setminus S_k$; if t is such that there is a (necessarily unique) $s \in S_k\setminus S_{k-1}$ with $t \in]s-\varepsilon(s), s+\varepsilon(s)[$ we define

$$F^k_t = F^{k-1}_t + [1 - |s-t|/\varepsilon(s)].[F^k_s - F^{k-1}_s].$$

If t is not of this form, define simply $F_t^k = F_t^{k-1}$.

We have defined F^k on all of T and we now have to check properties (a) to (f):

(a) Let $r \in S_{k-1} \setminus S_{k-2}$; then it follows from the definition of B_n^{r,λ_i^r} and the hypothesis

$$0 \leq F_r^{k-1} \leq 1 - 2^{-k}$$

that for $n \in \mathbb{N}$, $1 \leq i \leq m(r)$, $1 \leq j \leq 2^k$

$$0 \leq F_r^{k-1} + \chi_{B_i^r} \cdot \chi_{A_j^k} \cdot BR_n^{r,\lambda_i^r} \leq 1 - 2^{-k-1} .$$

The factor $[1-|r-s|/\delta(r)]$ is chosen small enough to get for $s \in S_k \setminus S_{k-1}$

$$0 \leq F_s^k \leq 1 - 2^{-k-1}$$

and the factor $[1-|s-t|/\varepsilon(s)]$ is small enough to get for $t \in T \setminus S_k$

$$0 \leq F_t^k \leq 1 - 2^{-k-1}.$$

Finally, in order to show that

$$\|F^k\| = \sup \{F_s^k : s \in S_k \setminus S_{k-1}\} = 1 - 2^{-k-1}$$

find, for $\varepsilon > 0$, $r \in S_{k-1} \setminus S_{k-2}$ and λ_i^r such that $\lambda_i^r < 2^{-k}+\varepsilon$. If n is sufficiently large and n equals i modulo 2^k. then, for $s = r+n^{-1}$

$$\|F_s^k\|_\infty > 1 - 2^{-k-1} + \varepsilon.$$

(b) and (f): Since $\mu(A_j^k) = 2^{-k}$, we see that F_t^k differs from F_t^{k-1} at most on a set of measure 2^{-k}, from which (f) and (b) follow .

(c): F^k is S_{k-1}-hopeless at every $r \in S_{k-2}$ by induction hypothesis. For $r \in S_{k-1} \setminus S_{k-2}$ write

$$F_k^{k-1} = F_r^k = \sum_{j=1}^{2^k} \sum_{i=1}^{m(r)} \lambda_i^r \chi_{(B_i^r \cap A_j^k)} .$$

For every A, contained in $B_i^r \cap A_j^k$ for some i and j, we have

$$\lim_{n\to\infty} \mu\{\omega \in A : BR_n^{r,\lambda_i^r} = 1 - \lambda_i^r\} = \mu(A).\lambda_i^r/(2-\lambda_i^r)$$

It follows that given $\varepsilon > 0$ for n large enough and equal to $i + (j-1)m(r)$ modulo $2^k m(r)$ we have for $s = r+n^{-1}$

$$\mu\{\omega \in A : |F_s(\omega)| \leqq \varepsilon\} > \mu(A) \cdot \lambda_i^r/2.$$

(d) and (e): obvious.

Finally we have to show that F^k is σ^*-continuous. In view of (e) we have to check the σ^*-continuity only at S_{k-1}.

Let $(t_j)_{j=1}^\infty \in T$ tend to $s_o \in S_{k-1}$. By passing to a subsequence we may assume that, for each $s \in S_k \setminus S_{k-1}$ there is at most one t_j which lies in $]s-\varepsilon(s), s+\varepsilon(s)[$. Hence

$$F_{t_j}^k - F_{t_j}^{k-1} = c_j \cdot \chi_{C_j} \cdot BR_{k_j}^{s_j,\lambda_j},$$

where c_j is a constant between 0 and 1, C_j is a set depending only on $\{BR_k^{q,\lambda} : q \in S_{k-2}\}$ and $\{A^\ell\}_{\ell \leqq k}$ while s_j is some element of $S_{k-1} \setminus S_{k-2}$ (and λ_j and k_j are some elements of $\mathbb{Q} \cap]0,1]$ and $\mathbb{N}$ respectively). If $j \neq i$ then either $s_j \neq s_i$ or $k_j \neq k_i$; so we may apply lemma 2.4 and conclude that $F_{t_j}^k - F_{t_j}^{k-1}$ tends σ^* to zero. Since

$$\sigma^*\text{-}\lim_j F_{t_j}^{k-1} = F_{s_o}^{k-1} = F_{s_o}^k$$

we see that F^k is indeed σ^*-continuous. This finishes the induction step.

It follows from (f) that the sequence $(F_t^k)_{k=1}^\infty$ is $\|.\|_1$-convergent uniformly in $t \in T$, hence

$$F_t = \|.\|_1\text{-}\lim_{k\to\infty} F_t^k$$

is a welldefined element of $\sigma^* C(T,L^\infty)$. It follows from (a), (b) and (c) of the induction hypothesis that F satisfies the corresponding assumptions of 2.3 if we let $D = \mathbb{Q} \cap [0,1[$. This finishes the proof of prop. 2.3 and therefore of theorem A.

□

One may ask for which compact spaces K the norm attaining operators are dense in $B(L^1,C(K))$. It is easily deduced from the representation 1.1 that this is the case if K has a dense set of isolated points (oberved in [5]). On the other hand the above construction may be applied to some other concrete examples of compact spaces. The author has checked this, apart from $[0,1]$, for the torus, the Cantor set, a countable product of the one point compactification of $\mathbb{N}$ and for products of these compact spaces with an arbitrary compact space. This leaves of course open the question of characterising the class of compact spaces K for which the norm attaining operators are dense in $B(L^1, C(K))$.

3. We now pass to the proof of theorem B. It is based on a simple lemma. In the sequel K will denote a compact Hausdorff space.

3.1. Lemma: Let W be a weakly compact subset of the space $M(K)$ of Radon measures on K and let $\mu_o \in M(K)$.

For $\varepsilon > 0$ there is an operator $S : M(K) \to M(K), \|S\| = 1$ such that

(i) there is $f_o \in C(K)$, $\|f_o\| = 1$ with
$$\|\mu_o\| = \langle f_o, S\mu_o \rangle$$

(ii) $\|S\mu - \mu\| < \varepsilon$ for all $\mu \in W$.

Proof: Take the Hahn-decomposition $\mu_o = \mu_o^+ - \mu_o^-$ of μ_o into its positive and negative part and find a partition of K into two Borel sets B^+ and B^- such that

$$\mu_o^+(B^+) + \mu_o^-(B^-) = \|\mu_o\|.$$

Since W is weakly compact, it is uniformly inner regular ([2], lemma VI.2.13) and we can therefore find compact subsets K^+ and K^- of B^+ and B^- respectively such that

$$|\mu|(B^+\setminus K^+) + |\mu|(B^-\setminus K^-) < \varepsilon/2,$$

for all $\mu \in W$.

Fix points $x^+ \in K^+$ and $x^- \in K^-$ and let $F : K \to K$ be the map

$$F(x) = \begin{cases} x & \text{for } x \in K^+ \cup K^- \\ x^+ & \text{for } x \in B^+ \setminus K^+ \\ x^- & \text{for } x \in B^- \setminus K^- \end{cases}$$

Clearly F is Borel measurable. Define $S:M(K) \to M(K)$ by

$$S(\mu) = F(\mu),$$

$F(\mu)$ denoting the image measure of μ under F. Evidently $\|S\| = 1$ and $\|S\mu - \mu\| < \varepsilon$, for $\mu \in W$. Find $f_o \in C(K)$, $\|f_o\| = 1$ such that f_o equals 1 on K^+ and -1 on K^-; then

$$\langle f_o, S\mu_o \rangle = \|\mu_o\|. \qquad \square$$

<u>Proof of theorem B:</u> Let $A^* : X^* \to M(K)$ denote the adjoint of A and W the image of the closed unit ball of X^* under A^*. In view of the weak compactness of W we may apply Stegall's theorem ([8], p. 176) to find an operator $T : M(K) \to M(K)$ (of rank 1) such that $\|T\| < \varepsilon/(2\|A\|)$ and such that $W_1 = (\mathrm{Id} + T)W$ has an element, say μ_o, of maximal norm, i.e.

$$\|\mu_o\| = \sup\{\|\mu\| : \mu \in W_1\} = \|(\mathrm{Id} + T)A^*\|.$$

Now we apply lemma 3.1. to find $S : M(K) \to M(K)$, $\|S\| = 1$, such that

(i) $\sup\{\|S\mu - \mu\| : \mu \in W\} \leq \varepsilon/2$

(ii) there is $f_o \in C(K)$, $\|f_o\| = 1$ with

$$\|\mu_o\| = \langle f_o, S\mu_o \rangle.$$

Denote $\tilde{A}^*$ the operator $S(\mathrm{Id} + T)A^*$.

$$\|\tilde{A}^* - A^*\| \leq \|SA^* - A^*\| + \|STA^*\|$$

$$\leq \varepsilon/2 + \|S\| . \|T\| . \|A^*\| < \varepsilon .$$

The adjoint of $\tilde{A}^*$, namely $\tilde{A}^{**} : C(K)^{**} \to X^{**}$ given by $\tilde{A}^{**} = A^{**}(Id + T^*)S^*$, takes $C(K)^{**}$ into X, as A^{**} does so. Finally let $\tilde{A} : C(K) \to X$ be the restriction of $\tilde{A}^{**}$ to $C(K)$. It is plain to justify the above introduced notation, i.e. that $\tilde{A}^*$ is the adjoint of $\tilde{A}$. Also $\|\tilde{A} - A\| < \varepsilon$ and

$$\|\tilde{A}\| = \|\tilde{A}^*\| \leq \|(Id + T)A^*\| .$$

On the other hand

$$\|\tilde{A}(f_o)\| = \sup \{ \langle f_o, S(Id + T)A^*x^* \rangle : \|x^*\| \leq 1 \}$$

$$= \sup \{ \langle f_o, S\mu \rangle : \mu \in W_1 \}$$

$$\geq \langle f_o, S\mu_o \rangle = \|(Id + T)A^*\|$$

which shows readily that

$$\|\tilde{A} f_o\| = \|\tilde{A}\| .$$

□