SOME REMARKS CONCERNING THE KREIN-MILMAN AND THE RADON-NIKODYM PROPERTY OF BANACH SPACES

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Abstract:

We present an example, simplifying an earlier one due to R. C. James, of a 1-separated tree with empty wedge intersections such that its closed convex hull has continuum many extreme points.

1. Introduction: The present paper deals with the (still open) question, whether the two properties of Banach spaces mentioned in the title (and abbreviated RNP and KMP) are equivalent. It is based on [S] and we have tried to emphazise the importance of the notion of a "complemented bush". This concept was defined and studied in [H] and implicitly used in [S].

We give a simplified version of an example due to R. C. James [J]. In the language of trees and bushes we construct a tree such that its closed convex hull - as well as the closed convex hull of many of its asymptotic subtrees - has continuum many extreme points.

2. Definitions and notations: There are different ways of looking at the aspects relevant for RNP in Banach spaces: One may formulate these in terms of operators from L^1 to X or in terms of trees and bushes (which again are just special martingales). These are only different sides of the same coin. As we shall use the identification of a tree or a bush and an operator from L^1 to X often throughout this paper we shall recall this here in some detail.

Δ will denote the Cantor set $\{-1,+1\}^{\mathbb{N}}$ and m the normalized Haar-measure on Δ. The Rademacher-functions r_n on Δ are the projections onto the n'th coordinate. β will denote the Borel-σ-algebra of Δ $\mathcal{B}$
and β_n the sub-σ-algebra generated by $r_1,\ldots,r_n$. $\mathcal{B}_n$

We define a tree in a Banach space X to be a collection

$$T = \{x_{n,i} : 1 \leq i \leq 2^{n-1}, n \in \mathbb{N}\}$$

in the unit-ball of X such that for each (n,i)

$$x_{n,i} = 1/2(x_{n+1,2i-1} + x_{n+1,2i})$$

and such that there is a positive separation constant $\varepsilon > 0$ such that for each (n,i)

$$\|x_{n,i} - x_{n+1,2i-1}\| \geqq \varepsilon.$$

To a tree T we associate an operator T from $L^1(\Delta,m)$ to X in the following way: For (n,i), $n \in \mathbb{N}$, $1 \leqq i \leqq 2^{n-1}$ let $(\delta_1,\ldots,\delta_{n-1})$ be the unique element of $\{-1,1\}^{n-1}$ such that

$$\sum_{k=1}^{n-1} ((\delta_k+1)/2)2^{n-k-1} = i-1$$

and let $I_{n,i}$ be the atom of β_n consisting of all $(\varepsilon_k)_{k=1}^{\infty}$ such $\mathcal{B}_n$ that $\varepsilon_k = \delta_k$ for $k = 1,\ldots,n-1$. Define

$$T(2^{n-1}\chi_{I_{n,i}}) = x_{n,i}.$$

Then T extends by linearity and continuity to a bounded operator from $L^1(\Delta,m)$ to X.

We have treated the case of trees instead of the more general case of bushes (see [H] for a definition) for notational convenience only; but the above identification carries over to this case analogously except that we have to use in an obvious way a more tedious index set, a different Cantor-set and bigger finite σ-algebras Σ_n.

Hence a tree T gives an operator T from $L^1(\Delta,m)$ to X which is easily (almost by definition) seen to be not representable. Conversely, given an operator from $L^1(\Omega,\Sigma,\mu)$ to X which is not representable one may find a subset $A \subseteq \Omega$, $\mu(A) > 0$ and an increasing sequence of finite σ-algebras Σ_n such that by the identification sketched above the traces of Σ_n on A define a bush with strictly positive separation constant (see [ST] for a detailed exposition).

In the sequel we restrict us to the notationally more convenient case of trees and operators from $L^1(\Delta,m)$ corresponding to a tree.

There is a natural ordering among the indices (n,i) of a tree, which is obtained from $(n+1,j) > (n,i)$ if $j = 2i-1$ or $j = 2i$. A <u>branch</u> of T is a sequence $(x_{n,i_n})_{n=1}^{\infty}$ such that $((n,i_n))_{n=1}^{\infty}$ is totally ordered. A <u>wedge</u> [H] of the tree T is a set of the type

$$W_{n,i} = \{x_{(m,j)} : (m,j) > (n,i)\}.$$

Recall the following notation from [S] and [ST] for an operator $T : L^1(\Delta,m) \to X$ and $A \in \beta$, $m(A) > 0$: L $\mathcal{B}$ X

$$L_A = \{Tf : f = f\chi_A \geqq 0 \text{ and } \int f dm = 1\}.$$

The connection between wedges of a tree T and these sets L_A for the corresponding operator T is given - in the case where A equals an atom $I_{n,i}$ - by the easily verified formula

$$\bar{L}_{I_{n,i}} = \overline{\text{co}}\, W_{n,i}$$

where $\overline{\text{co}}$ denotes the closed convex hull.

<u>2.1. Definition:</u> a) We say that a tree has empty wedge <u>intersections</u> if for every branch $((n,i_n))_{n=1}^{\infty}$

$$\bigcap_{n=1}^{\infty} \overline{\text{co}}\, W_{n,i_n} = \emptyset$$

b) We say that a tree is <u>complemented</u> ([H]) if there is $\Theta > 0$ such that for each (n,i)

$$\|u-v\| \geqq \Theta \min(\|u\|, \|v\|)$$

for every $u \in$ linear span $(W_{n+1,2i-1})$ and $v \in$ linear span $(W_{n+1,2i})$.

It was shown in [H] and [S] that for a tree T satisfying a) and b)

$$\overline{\text{co}}\,(T)$$

has no extreme points. In fact, it was noticed in [S] that one may even allow $\Theta > 0$ to depend on (n,i) and still arrive at the same conclusion.

It was shown in [ST] that, starting from any non-representable operator $T : L^1(\Sigma,\Omega,\mu) \to X$, one may associate (as sketched above) a bush, which even has empty wedge intersections (with the obvious extension of the above definition to bushes). Hence the problem of showing that KMP implies RNP is solved if we can achieve, in addition, the complementation property of b).

The aim of section 3 of the present paper is to give an example of a "badly uncomplemented tree" with empty wedge intersections but such that

$$\overline{co}\ (\mathcal{T})$$

(and in fact $\overline{\mathcal{L}}_A$ for each $A \in \Sigma$, $m(A) > 0$) has continuum many extreme points and equals the convex hull of its extreme points.

<u>3. A tree whose closed convex hull has many extreme points:</u> We define an operator $T : L^1(\Delta) \to c_o$ coordinatewise. For the odd coordinates it will be the well-known "Rademacher operator" which is one of the arch-examples of a non-RNP-operator:

$$(Tf)_{2n-1} = \langle f,r_n\rangle \qquad n = 1,2,\ldots$$

On the even coordinates we want T to be a compact operator, which is chosen in such a way that $T^* : \ell^1 \to L^\infty(\Delta)$ maps ℓ^1 into a dense subset of $C(\Delta)$. E.g., let $(x_n)_{n=1}^\infty$ be a dense sequence in the unit-ball of $C(\Delta)$ and let

$$(Tf)_{2n} = 2^{-n} \langle f,x_n\rangle \qquad n = 1,2,\ldots$$

Clearly T is not a RNP-operator.

In fact the tree $\mathcal{T}$ associated to T has separation constant 1 and empty wedge intersections: Indeed for each (n,i) each element of

$$\overline{co}\ (W_{n,i})$$

has entries of absolute value 1 in the coordinates $1,3,\ldots,2n-1$.

Proposition 3.1: For any measurable set $A \subseteq \Delta$, $m(A) > 0$, the set

$$K_A = \overline{\mathcal{L}_A}$$

has continuum many extreme points and equals the closed convex hull of its extreme points.

Proof: Note that

$$T^* : \ell^1 \to L^\infty(\Delta)$$

takes its values in $C(\Delta)$. Hence we may define (with an abuse of notation)

$$T^{**} : \mathcal{M}(\Delta) \to \ell^\infty,$$

which is one to one by the density of the range of T^* in $C(\Delta)$.

Now let us first assume that $A = \Delta$. Let

$$\mathcal{K}_\Delta = \{T^{**}(\mu) : \mu \text{ probability measure on } \Delta\}$$

$\mathcal{K}_\Delta$ is a σ^*-compact subset of ℓ^∞ and it is easily verified that

$$K_\Delta = \mathcal{K}_\Delta \cap c_o.$$

Now let $t = (\varepsilon_k)_{k=1}^\infty$ and $t' = (\varepsilon_k')_{k=1}^\infty$ be elements of Δ satisfying

$$\varepsilon_k = -\varepsilon_k' \text{ for all but finitely many } k. \qquad (*)$$

We claim that

$$e(t,t') = T^{**}(1/2(\delta_t+\delta_{t'}))$$

is an extreme point of K_Δ, where δ_t denotes the Dirac-measure at t.

Indeed $e(t,t')$ belongs to c_o because the odd coordinates are eventually zero while the even coordinates tend to zero (without any problem). On the other hand the segment

$$E(t,t') = \{\lambda\delta_t+(1-\lambda)\delta_{t'} : 0 \leq \lambda \leq 1\}$$

is an extremal set in the simplex of probability measures on Δ. From the injectivity of T^{**} we get that $T^{**}(E(t,t'))$ is an extremal set in K_Δ. As $T^{**}(\delta_t)$ evidently is not in K_Δ we infer that $e(t,t')$ is the only point of $T^{**}(E(t,t')) \cap K_\Delta$ and therefore an extreme point of K_Δ.

Let us finally show that K_Δ is the closed convex hull of the extreme points $e(t,t')$. Indeed let $s = (\delta_k)_{k=1}^{\infty}$ in Δ be given and define $s^n = (\delta_k^n)_{k=1}^{\infty}$ by

$$\delta_k^n = \delta_k \qquad k = 1,\dots,n$$

$$\delta_k^n = -\delta_k \qquad k > n$$

Clearly the pairs (s,s^n) satisfy (*) and

$$\lim_{n\to\infty} (1/2(\delta_s + \delta_{s^n})) = \delta_s$$

the limit taken in the σ^*-topology of $M(\Delta)$. Hence the σ^*-closed convex hull of all $(1/2(\delta_t + \delta_{t'}))$ with (t,t') satisfying (*) are all the probability measures on Δ. It follows from the $\sigma(M(\Delta), C(\Delta)) - \sigma(\ell^\infty, \ell^1)$ - continuity that K_Δ is the σ^*-clossed convex hull of the corresponding $e(t,t')$, hence K_Δ the σ-closed (and therefore norm-closed) convex hull of the extreme points $e(t,t')$.

Let us now pass to the case of $A \in \beta$, $m(A) > 0$ instead of the whole set Δ. $\mathcal{B}$ Let $\tilde{A}$ be the closure of all points of Lebesgue-density one of A. It follows easily that

$$K_{\tilde{A}} := \{T^{**}\mu : \mu \text{ probability measure supported by } \tilde{A}\}$$

equals the σ^*-closure of

$$K_A := \bar{L}_A.$$

Again the points $e(t,t')$ such that $t,t' \in \tilde{A}$ and t,t' satisfy (*) are extreme points of K_A and an argument similar to ~~be~~ the above and using the Lebesgue density shows that the closed convex hull of these $e(t,t')$ equals K_A.

□

Remarks: (1) The example is astonishingly simple; it consists only of a compact perturbation of the well-known "Rademacher-operator".

However, it is entirely based on the previous very technical one of James [J], which displays essentially the same phenomena.

It was only by gradually understanding and simplifying James' example that the author finally arrived at this version.

Our example is a tree-version (this corresponds to $c_i = 2$ for $i \geqq 1$ in [J], which implies that the function g is just constant on Δ_{11}) of James' example, the functions $\Phi_{n,k}$ in [J] correspond to the "Rademacher-part" of our T while the functions ω_p correspond to the "compact part" of T.

Finally let us illustrate why the tree $\mathcal{T}$ corresponding to our T is "badly uncomplemented" i.e. badly fails condition b) of definition 2.1.

Of course we know from [H] and [S] that $\mathcal{T}$ cannot be complemented (otherwise K_Δ could not have extreme points). But it seems worth while to see this explicitly.

Fix any tree index (n,i). We shall show that the linear spans of the wedges $W_{n+1,2i-1}$ and $W_{n+1,2i}$ are not complemented: Let $m > n+1$ and find j such that $I_{m,2j-1} \subseteq I_{n+1,2i-1}$. Then

$$x_{m,2j-1} - x_{m,2j} \in \text{lin sp } W_{n+1,2i-1}$$

and

$$\|x_{m,2j-1} - x_{m,2j}\| = 2^{m-1}\|T(X_{I_{m,2j-1}} - X_{I_{m,2j}})\| \geqq$$

$$\geqq 2^{m-1}|\langle X_{I_{m,2j-1}} - X_{I_{m,2j}}, r_m\rangle| = 2$$

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Similarly,

$$x_{m,2j+2^{m-n-1}-1} - x_{m,2j+2^{m-n-1}} \in \text{lin sp } W_{n+1,2i}$$

and

$$\|x_{m,2j+2^{m-n-1}-1} - x_{m,2j+2^{m-n-1}}\| \geqq 2.$$

On the other hand

$$\lim \|x_{m,2j-1} - x_{m,2j} - (x_{m,2j+2^{m-n-1}-1} - x_{m,2j+2^{m-n-1}})\| = 0.$$

Indeed, all the odd coefficients of the above element of c_o are zero while the even coordinates tend to zero uniformly. This shows that the tree T is badly uncomplemented.

Finally let us observe why the present example cannot furnish a counter-example to the general questions of whether RNP and KMP are equivalent: If X denotes the closure of the space spanned by T then X contains a subspace isomorphic to c_o (as does any infinite-dimensional subspace of c_o). Hence X has no chance to have KMP.

However, this shows one interesting fact: So far, all the pathologies arising in the absense of RNP could be shown to happen in some set of the form K_A (see [ST] for a convincing presentation of this fact). But the present example shows that if one tries to prove the equivalence of RNP and KMP one may ont restrict oneself to such sets but has to adopt other methods of constructing "bad" sets.

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